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Newtonian Development of the Dynamical Properties of Ionized Gases of Low Density*

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The macroscopic dynamical equations of a tenuous ionized gas in a magnetic field are developed by averaging over the individual ion and electron motions, which do not necessarily possess an isotropic distribution. It is shown that the principal motion of the gas is related to the magnetic field by the usual hydromagnetic equations, as developed for conducting liquids and dense gases; the anisotropy of the individual particle motions shows up primarily as a coefficient multiplying the pondermotive force exerted by the magnetic field on the plasma. The results reduce properly to the earlier work of Schlüter, Cowling, and Spitzer for isotropic pressure, and are in agreement with the recent developments from the Boltzmann equation. It is pointed out that the magnetic lines of force are permanently connected and move in the frame of reference of the electric drift. It is shown that near static equilibrium, when the principal motions vanish, there remain small macroscopic drift motions of the gas in the field inhomogeneities.

It is also shown that the field equations, obtained by assuming that the radius of gyration of the thermal motions is small compared to the scale of the field, are valid even near neutral surfaces, on which the field density vanishes.

I. INTRODUCTION

THE usual hydromagnetic equations

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{v}\times\mathbf{B})+\frac{c^2}{4\pi\mu\sigma}\nabla^2\mathbf{B}, \qquad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{1}{\rho}\nabla p+\frac{1}{4\pi\rho}(\nabla\times\mathbf{B})\times\mathbf{B}, \qquad (2)$$

for the magnetic field $\mathbf{B}$ and the velocity field $\mathbf{v}$ in a fluid with material density ρ, hydrostatic pressure p, and electrical conductivity σ, are conventionally derived from Maxwell's equations by assuming that the conduction current density $\mathbf{i}$ is related by Ohm's law to the electric field $\mathbf{E}'$ in the frame of reference moving with the fluid,

$$\mathbf{i}=\sigma\mathbf{E}'. \qquad (3)$$

Restricting ourselves to nonrelativistic fluid velocities ($v\ll c$), it follows that

$$\mathbf{E}'=\mathbf{E}+(\mathbf{v}/c)\times\mathbf{B};$$

then from (3)

$$\mathbf{E}=-(\mathbf{v}/c)\times\mathbf{B}+\mathbf{i}/\sigma. \qquad (4)$$

Maxwell's equation,

$$4\pi\mathbf{i}+\partial\mathbf{E}/\partial t=c\nabla\times\mathbf{B},$$

upon omission of the displacement current (since[1] $v\ll c$), gives

$$\mathbf{i}=(c/4\pi)\nabla\times\mathbf{B}; \qquad (5)$$

then from (4) it follows that

$$\mathbf{E}=-(\mathbf{v}/c)\times\mathbf{B}+(c/4\pi\sigma)\nabla\times\mathbf{B}.$$

Hence the field equation

$$\partial\mathbf{B}/\partial t=-c\nabla\times\mathbf{E}$$

reduces to (1). Equation (2) is derived by adding to the classical Euler equation of motion the Lorentz force $(\mathbf{i}/c)\times\mathbf{B}$ and using (5) to express $\mathbf{i}$ in terms of $\mathbf{B}$.

In liquids, and in gases sufficiently dense that the collision rate of the free electrons is large compared to their cyclotron frequency in $\mathbf{B}$, we have no reason to doubt that (3), and hence the hydromagnetic equation (1) deduced from (3), is valid. However, in tenuous ionized gases in which the collision rate is small compared to the cyclotron frequency it is not obvious that (3) is applicable; an electric field impressed across a tenuous plasma results first of all in a drift of both electrons and ions perpendicular to the electric field with a velocity $c\mathbf{E}\times\mathbf{B}/B^2$, and there is no net transport of charge.

Schlüter[2] has shown that in the final analysis the situation, though now more complicated, may still be represented adequately by (3) in most cases. A more formal and detailed treatment of the problem based on the Boltzmann equation has been presented by Chew, Goldberger, and Low[3] and also by Watson[4] and by Brueckner and Watson.[5] They have investigated the dynamical properties of an ionized gas which is sufficiently tenuous that direct interaction between the individual ions and electrons may be entirely neglected. They restrict themselves to magnetic fields whose scale of variation is large compared to the Larmor radius of the gas particles. They have been able to show in a formal way that in the first approximation the equations of motion reduce to the hydrodynamic form given

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[5] K. A. Brueckner and K. M. Watson, Phys. Rev. **102**, 19 (1956).

by (2), and that the equation for **B** approximates to (1) with $\sigma=\infty$, provided that the system is not too greatly disturbed by large amounts of heat conduction along the lines of force, etc.

Now since the formal developments based on the Boltzmann equation neglect all interaction between individual particles, so that the collision terms vanish from the Boltzmann equation, we see that we can just as well deduce the macroscopic behavior of the gas by summing over the individual motions of the gas atoms: Because the particles do not interact we may compute their individual motions ahead of time, using Newton's equations to relate their velocities to the electromagnetic field variables; in the absence of interaction the n-body problem reduces to n one-body problems. We may then sum over all individual particle motions, obtaining the total mass motion as a function of the field variables; we may also obtain the total current density as a function of the field variables, which is then inserted into Maxwell's equations to yield the electromagnetic field equations appropriate to the tenuous ionized gas.

Thus, we shall carry out our development of the macroscopic motions of the plasma from much the same point of view as used by Schüter,[2] and by Spitzer[6]; but we shall endeavor to show explicitly the effects of anisotropy of the individual particle motions and we shall strive to express the macroscopic equations of motion in a form as closely analogous to the hydromagnetic equations as possible. We shall cover much of the same ground as Schlüter and Spitzer in the construction leading up to the final equations.[7]

The results of such a Newtonian procedure are, of course, entirely equivalent to conclusions based on the Boltzmann equation. The Newtonian point of view has the advantage that it avoids the formal mathematics associated with the appropriate solution of the Boltzmann equation; it works directly with the basic physical phenomena of the individual particle motions so that one may see immediately how the microscopic mechanics directly produce the final macroscopic fields. It has the formal disadvantage that one does not automatically obtain a general expression relating the macroscopic pressures, p_n perpendicular, and p_s parallel to **B**, to other macroscopic quantities; however, in practice the relations are usually easily supplied whenever the problem at hand is sufficiently simple that the formal relations from the Boltzmann equation are useful. Finally, it should be noted that the Newtonian procedure is limited by the fact that one is not able to go on with further developments where collisions are included as a small perturbation.

[6] L. Spitzer, Astrophys. J. **116**, 299 (1952).

[7] E. Aström [Arkiv Fysik **2**, 443 (1950)] has previously used the Newtonian point of view to establish under what circumstances it is possible to reduce the field equations in a plasma, by suitably defined dielectric and permeability coefficients, to the form assumed by the Maxwell equations in a homogeneous nonconducting medium.

Recognizing these advantages and weaknesses of a Newtonian development, we shall concentrate our attention in this paper on exhibiting the details of the macroscopic mass motions and their relation to the microscopic particle motions; in addition we shall investigate the macroscopic equations in the vicinity of a surface on which **B** vanishes, to show that though the usual approximation of small Larmor radius is not valid, the general conclusions based on that approximation are correct there.

Fortunately, except in one case we shall not have to consider the microscopic velocity distribution: We shall find that the contribution of each individual electron and ion in the final macroscopic field equations involves the particle mass M multiplied by the square of the thermal velocity w; upon summing over all particles to obtain the macroscopic equations we obtain, then, $\sum Mw^2$, which is just the hydrostatic pressure regardless of the velocity distribution.

II. CURRENT DENSITY

A. Individual Particle Motion

Consider the nonrelativistic motion of a particle of mass M and charge q in an electric field **E** and a magnetic field **B** whose scale L and period of variation T are both large compared to the radius of gyration R and cyclotron period $2\pi/\Omega$ of the charged particle. We shall decompose the motion **w** of the particle into the usual circular motion about the guiding center plus the drift of the guiding center. We use the subscripts n and s to denote the component of a vector perpendicular and parallel to **B**, respectively.

It has been shown by Watson[4] that the principal motion of the guiding center is the drift

$$\mathbf{u}_n=c(\mathbf{E}\times\mathbf{B})/B^2+(\tfrac{1}{2}Mw_n^2c/qB^4)\mathbf{B}\times\nabla B^2/2 + (Mw_s^2c/qB^4)\mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}], \quad (6)$$

perpendicular to **B**, and the drift $\mathbf{u}_s$ parallel to **B**, where

$$Md\mathbf{u}_s/dt=-(\tfrac{1}{2}Mw_n^2/B^4)\mathbf{B}\{\mathbf{B}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}]\}. \quad (7)$$

The first term on the right-hand side of (6) represents the usual electric drift. The second term is the drift perpendicular to both **B** and ∇B^2 as a consequence of shear in **B**. The third term is the drift arising from the reaction against **B** of the centrifugal force of the motion w_s along a curved line of force; as will be shown later $\mathbf{B}\times[(\mathbf{B}\cdot\nabla)\mathbf{B}]$ represents the curvature of the line of force. The acceleration along the lines of force given by (7) arises from the reaction of the centrifugal force of the circular motion against diverging lines of force, and represents a repulsion along the lines of force away from regions of dense field.

The circular motion about the guiding center of the particle is described principally by the familiar adiabatic invariant $w_n/B^{\frac{1}{2}}=$constant.

On the basis of the above motions of the individual particle let us now compute the resulting total current density in an electrically neutral plasma composed of N electrons, with charge $-e$, and N ions, with charge $+e$ per unit volume; the generalization to ions with charge Ze and ZN electrons per unit volume will be obvious.

B. Circulating Current

We consider first the net current density resulting from the circular motion of each particle, ions and electrons, about its individual guiding center; we shall for the moment ignore the drift $\mathbf{u}_n$ and $\mathbf{u}_s$ of the guiding center. Taking advantage of the large scale of the magnetic field, $L \gg R$, we may construct at any point a locally Cartesian coordinate system with its z axis tangent to the line of force through the origin of the local system. We shall consider all the ions lying between the planes $z=0$ and $z=\delta Z$; the guiding centers of the ions lie between $z=0$ and $z=\delta Z$, and the circular motion of each ion about its guiding center carries it in a circle parallel to the $z=0$ or xy plane.

We suppose that the number of ion guiding centers per unit volume is $N(x,y)$. The cyclotron frequency $\Omega(x,y)$ is related to the field $B(x,y)$ at each guiding center in the usual way

$$\Omega(x,y)=qB(x,y)/Mc. \tag{8}$$

We denote the particle velocity about the guiding center by $w_n(x,y)$ where x and y are the coordinates of the center. Hence the radius of gyration is

$$R(x,y)=w_n(x,y)/\Omega(x,y). \tag{9}$$

To compute the x component of the ion current density in the layer δZ, we must calculate the rate at which the distribution $N(x,y)$ and circular velocity $w_n(x,y)$ causes charge to be transported across the element of area $x=X$, $Y\leq y\leq Y+\delta Y$, $0<z<\delta Z$. To determine which ions the circular motion carries through this element of area we construct the locus of all points (x_1,y_1) which lie at a distance $R(x_1,y_1)$ from (X,Y), and the locus of all points which lie at a distance $R(x_2,y_2)$ from $(X, Y+\delta Y)$; it is easily seen from Fig. 1 that all ions whose guiding centers lie between these two loci pass through the element $\delta Y\delta Z$. The two loci are given by

$$R^2(x_1,y_1)=(x_1-X)^2+(y_1-Y)^2, \tag{10}$$

$$R^2(x_2,y_2)=(x_2-X)^2+(y_2-Y-\delta Y)^2. \tag{11}$$

With the magnetic field in the positive z direction, an ion will circle in a clockwise sense as viewed from the positive z axis. Each ion with guiding center between the two loci and for which $y>Y$ will contribute a mean current $-e\Omega/2\pi$ across $\delta Y\delta Z$; each ion with guiding center at $y<Y$ contributes $+e\Omega/2\pi$. To sum over all of the ions with guiding centers between the two loci it is easiest to use the polar coordinates (r,θ), defined by

$$x-X=r\cos\theta,\quad y-Y=r\sin\theta;$$

the element of area between the loci becomes

$$dA=\tfrac{1}{2}(r_2^2-r_1^2)d\theta,$$

where from (10) we have

$$r_1=R(X,Y)[1+(\partial R/\partial X)\cos\theta+(\partial R/\partial Y)\sin\theta],$$

and from (11)

$$r_2=[R(X,Y)+\delta Y\sin\theta] \times[1+(\partial R/\partial X)\cos\theta+(\partial R/\partial Y)\sin\theta],$$

upon expanding $R(x,y)$ about (X,Y) and neglecting all terms of second and higher orders in (R/L). Then

$$dA=\delta YR(X,Y)\sin\theta \times[1+2(\partial R/\partial X)\cos\theta+2(\partial R/\partial Y)\sin\theta]d\theta. \tag{12}$$

Upon carrying out the integration from $\theta=0$ to $\theta=2\pi$, we find that the x component of the current density across the element $\delta Y\delta Z$ is

$$i_x(X,Y)=-\tfrac{1}{2}e(\partial/\partial Y)(N\Omega R^2) =-c(\partial/\partial Y)(\tfrac{1}{2}NMw_n^2/B). \tag{13}$$

In a similar way we find that

$$i_y(X,Y)=+c(\partial/\partial X)(\tfrac{1}{2}NMw_n^2/B). \tag{14}$$

Equations (13) and (14) have been deduced on the assumption that the planes of the circular motion of each ion are everywhere parallel to the xy plane; thus we have tacitly assumed that the lines of magnetic force are straight, and all parallel to the z axis. We will now generalize the expressions (13) and (14) to include current density resulting from the circular

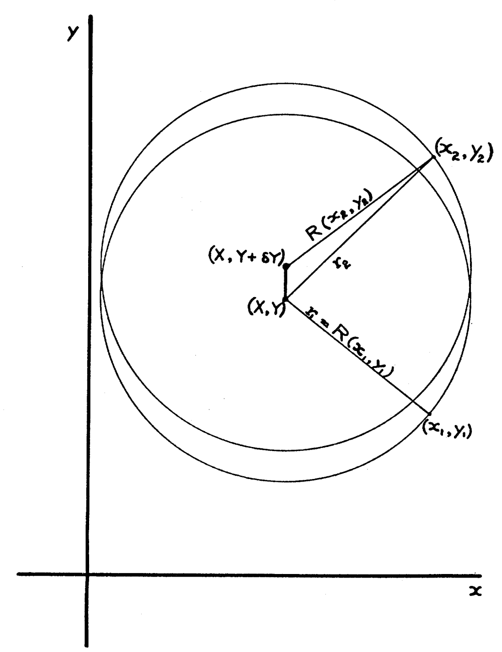

FIG. 1. Geometrical setup for computing the x component of the Larmor current density $\mathbf{i}_L$ at the segment (X,Y), $(X, Y+\delta Y)$.

motion of ions in a magnetic field whose lines of force have a radius of curvature R in a plane making an angle ϕ with the x axis. In such a case we again consider those ions with guiding centers between $z=0$ and $z=\delta z(x,y)$ where now

$$\delta z(x,y)=\delta Z[1-(x/R)\cos\phi-(y/R)\sin\phi].$$

Then the circular motion of each ion with guiding center in $0\leq z\leq\delta z(x,y)$ lies entirely between the two planes $z=0$ and $z=\delta z(x,y)$: In an element of volume $dA\delta z(x,y)$ there are $N(x,y)dA\delta z(x,y)$ guiding centers, transporting charge across the element of area $\delta Y\delta z(X,Y)$ at a rate $[q\Omega(x,y)/2\pi]N(x,y)dA\delta z(x,y)$; we proceed as before except that $\delta z(x,y)$ is now a function of x and y, and is no longer a constant. It is readily seen that $\delta z(x,y)$ will be included in the differentiation on the right-hand side of (13) and (14), yielding

$$\begin{aligned} i_x(x,y) &= -[\tfrac{1}{2}q/\delta z(x,y)](\partial/\partial y)[N\Omega R^2\delta z(x,y)] \\ &= -c\{(\partial/\partial y)(\tfrac{1}{2}NMw_n^2/B) \\ &\qquad -(\tfrac{1}{2}NMw_n^2/B)(\sin\phi/R)\}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} i_y(x,y) &= +c\{(\partial/\partial x)(\tfrac{1}{2}NMw_n^2/B) \\ &\qquad -(\tfrac{1}{2}NMw_n^2/B)(\cos\phi/R)\}. \end{aligned} \tag{16}$$

We note that the sense of the resulting current is independent of the sign of charge of the particles.

In order to express the current density in vector form, we note that we may write the identity

$$(\mathbf{B}\cdot\nabla)\mathbf{B}=(1/B^2)\{\mathbf{BB}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}]-\mathbf{B}\times[\mathbf{B}\times(\mathbf{B}\cdot\nabla)\mathbf{B}]\}.$$

The first term on the right-hand side is the component of $(\mathbf{B}\cdot\nabla)\mathbf{B}$ parallel to $\mathbf{B}$ and represents the change in the field density, as one moves along a line of force, as a consequence of the diverging of the lines of force; it affects the drift of the ion centers as given by (7). The second term is the component of $(\mathbf{B}\cdot\nabla)\mathbf{B}$ perpendicular to $\mathbf{B}$ and represents the change of direction of the lines of force as one moves along $\mathbf{B}$; it is the centrifugal term and may be expressed as

$$|-\mathbf{B}\times[\mathbf{B}\times(\mathbf{B}\cdot\nabla)\mathbf{B}]/B^2|=B^2/R. \tag{17}$$

Using (17) we may eliminate R from (15) and (16). At the same time we define the total pressure component p_n due to motions perpendicular to $\mathbf{B}$ as

$$p_n=\tfrac{1}{2}NMw_{ni}^2+\tfrac{1}{2}Nmw_{ne}^2,$$

where m and w_{ne} are the electron mass and velocity. Then for both the ions and the electrons together we have the total current

$$\mathbf{i}_L=(c/B)\mathbf{B}\times\{\nabla(p_n/B)-(p_n/B^3)(\mathbf{B}\cdot\nabla)\mathbf{B}\} \tag{18}$$

$$\begin{aligned} &=(c/8\pi p_m)\mathbf{B}\times\{\nabla p_n-\tfrac{1}{2}(p_n/p_m)\nabla p_m \\ &\qquad -(p_n/p_m)(\mathbf{B}\cdot\nabla)\mathbf{B}/8\pi\}, \end{aligned} \tag{19}$$

as a consequence of the circular motions. p_m represents the magnetic pressure $B^2/8\pi$.

C. Drift Current

As a consequence of the drift velocity $\mathbf{u}_n$, given in (6), there will be a current density, which we denote by $\mathbf{i}_D$. Noting that $\tfrac{1}{2}NMw_n^2$ is the contribution to the lateral pressure p_n, and NMw_s^2 to the longitudinal pressure p_s, we see from (6) that the total drift current, due to both ions and electrons, may be written

$$i_D=Ne[\mathbf{u}_n(\text{ions})-\mathbf{u}_n(\text{electrons})] \tag{20}$$

$$\begin{aligned} &=(c/8\pi p_m)\mathbf{B}\times\{\tfrac{1}{2}(p_n/p_m)\nabla p_m \\ &\qquad +(p_s/p_m)(\mathbf{B}\cdot\nabla)\mathbf{B}/8\pi\}. \end{aligned} \tag{21}$$

The electric drift $c\mathbf{E}\times\mathbf{B}/B^2$ contributes no net current because both ions and electrons drift in the same direction.

There will also be a drift current parallel to the field, in addition to the lateral current $\mathbf{i}_D$, as a consequence of ordinary electrical conductivity along the magnetic lines of force; with no interaction between individual particles, as we are assuming, the electrical conductivity parallel to $\mathbf{B}$ would be infinite.

Assuming no initial hydrostatic pressure gradient, from (7) we see that variation of the field density B along the lines of force results in greater acceleration along $\mathbf{B}$ for the electrons than for the protons since the electron thermal velocity is much greater than the ion thermal velocity, both represented by w_n in (7). Thus we would expect that there might arise currents along $\mathbf{B}$ as a consequence of $\mathbf{B}\cdot[(\mathbf{B}\cdot\nabla)\mathbf{B}]\neq0$. Schlüter[2] and Spitzer[6] have shown that when the distribution of individual particle velocities is isotropic no macroscopic motions and no current arises from this effect. We shall consider the problem in detail in Sec. IV for an anisotropic velocity distribution. For the present it is sufficient to note that if a current along $\mathbf{B}$ were to result, in the anisotropic case, as a consequence of the component of $(\mathbf{B}\cdot\nabla)\mathbf{B}$ parallel to $\mathbf{B}$, a space charge would quickly appear, giving rise to an electric field which would stop the current. The electric field arising in this way is

$$|\mathbf{E}|=O[(R/L)(w/c)B],$$

and so is smaller than the field given in (43) by the factor R/L; hence we shall neglect its effects.

D. Polarization Current

The final contribution to the current density which we shall consider is due to the polarization of the plasma[6,7] in the presence of an impressed electric field; it is the current resulting in the charge separation produced by inertial forces. We have just discussed the polarization along the lines of force and we shall now develop the appropriate lateral results.

We let the magnetic field lie along the z axis of a local Cartesian coordinate system. We suppose that there is an electric field in the local y direction. Then the motion of a particle of charge q and mass M is

$$M\frac{d^2x}{dt^2}=\frac{q}{c}\frac{dy}{dt}B, \tag{22}$$

$$M\frac{d^2y}{dt^2}=qE-\left(\frac{q}{c}\right)\frac{dx}{dt}B. \tag{23}$$

We eliminate x by dividing (23) by B, differentiating with respect to t, and using (22) to remove d^2x/dt^2. Then, with $\Omega=qB/Mc$, we have

$$(d/dt)[(1/\Omega)d^2y/dt^2]+\Omega dy/dt=c(d/dt)(E/B). \tag{24}$$

Remembering that the cyclotron period is small compared to the period over which E, B, and Ω change appreciably, we average over many cyclotron periods, thereby eliminating the circular motion from the equation. Hence the first term on the left-hand side of the equation drops out because it contains d^2y/dt^2, and there remains only the slowly varying portion of dy/dt,

$$\langle dy/dt\rangle=(c/\Omega)(d/dt)(E/B). \tag{25}$$

This polarization drift may be written in vector form as

$$\mathbf{u}_P=(Mc^2/qB^2)\mathbf{B}\times\{(d/dt)[\mathbf{E}\times\mathbf{B}/B^2]\}, \tag{26}$$

and results in a current

$$\mathbf{i}_P=(\rho c^2/B^2)\mathbf{B}\times\{(d/dt)[\mathbf{E}\times\mathbf{B}/B^2]\}. \tag{27}$$

The polarization velocity and current are simply expressed in terms of the electric drift $\mathbf{u}_D$ for both the ions and electrons,

$$\mathbf{u}_D=c\mathbf{E}\times\mathbf{B}/B^2, \tag{28}$$

and the usual cyclotron frequency $\Omega=qB/Mc$, according to

$$\mathbf{u}_P=(1/\Omega)(\mathbf{B}/B)\times d\mathbf{u}_D/dt, \tag{29}$$

$$\mathbf{i}_P=(\rho c^2/B^2)\mathbf{B}\times d\mathbf{u}_D/dt. \tag{30}$$

III. FIELD EQUATIONS

Having computed the current density resulting from the freely moving ions and electrons in a tenuous plasma, let us now insert the result into the Maxwell equations,

$$\partial\mathbf{E}/\partial t=+c\nabla\times\mathbf{B}-4\pi\mathbf{i}, \tag{31}$$

$$\partial\mathbf{B}/\partial t=-c\nabla\times\mathbf{E}, \tag{32}$$

so that we may determine the appropriate equations for the fields **E** and **B**.

We will find it convenient to decompose the curl of the magnetic field in (31) into components according to the vector identity

$$\nabla\times\mathbf{B}=(\nabla\times\mathbf{B})_s+(\nabla\times\mathbf{B})_n, \tag{33}$$

where

$$(\nabla\times\mathbf{B})_s=(\mathbf{B}/B^2)\mathbf{B}\cdot\nabla\times\mathbf{B}, \tag{34}$$

$$(\nabla\times\mathbf{B})_n=(\mathbf{B}/B^2)\times[-\tfrac{1}{2}\nabla B^2+(\mathbf{B}\cdot\nabla)\mathbf{B}]. \tag{35}$$

Equation (34) represents the contribution to $\nabla\times\mathbf{B}$ of the change in direction of the lines of force as one moves perpnedicular to **B**; the first term in the brackets on the right-hand side of (35) represents the contribution of the shear in **B**, the change in the magnitude of **B** as one moves perpendicular to **B**; the second term in the brackets represents the change in the direction of **B** as one moves along **B**, as discussed in (17).

A. Electric Field Parallel to B

With the aid of (34), the component of (31) parallel to **B** may be written

$$\partial\mathbf{E}_s/\partial t=(c/B^2)\mathbf{B}[\mathbf{B}\cdot(\nabla\times\mathbf{B})]-4\pi\mathbf{i}_s. \tag{36}$$

But, as noted earlier, we have the usual free conduction of electric charge along the lines of force, so that

$$\mathbf{i}_s=\sigma\mathbf{E}_s. \tag{37}$$

Substituting (37) into (35), we obtain

$$\partial\mathbf{E}_s/\partial t+4\pi\sigma\mathbf{E}_s=(c/B^2)\mathbf{B}[\mathbf{B}\cdot(\nabla\times\mathbf{B})],$$

which admits of the formal solution

$$\mathbf{E}_s(t)=\int_{-\infty}^{t}d\tau\,\exp[4\pi\sigma(\tau-t)]\mathbf{f}(\tau), \tag{38}$$

where for convenience we have written

$$\mathbf{f}(t)=(c/B^2)\mathbf{B}[\mathbf{B}\cdot(\nabla\times\mathbf{B})]. \tag{39}$$

In the limit as $\sigma\to\infty$ we have

$$\mathbf{E}_s(t)\sim f(t)/4\pi\sigma. \tag{40}$$

Thus the component of the electric field parallel to **B** vanishes in the limit of no collisions between particles, and the resulting infinite electrical conductivity along **B**; we obtain the usual result that $\mathbf{E}\cdot\mathbf{B}=0$. As a consequence of the large value of e/m no significant mass motions are involved in $\mathbf{i}_s$.

B. Electric Field Perpendicular to B

The component of the current density perpendicular to **B** may be written

$$\mathbf{i}_n=\mathbf{i}_L+\mathbf{i}_D+\mathbf{i}_P=(c/8\pi p_m)\mathbf{B}\times\{\nabla p_n+[(p_s-p_n)/p_m](\mathbf{B}\cdot\nabla)\mathbf{B}/8\pi+\rho d\mathbf{u}_D/dt\} \tag{41}$$

using (19), (21), and (30). It is to be remembered that $\mathbf{u}_D$ is the conventional electric drift velocity, given by (28). We note that the drift current due to the second term on the right-hand side of (6) has just canceled the term in (19) due to the variation of the cyclotron frequency, or magnetic field density, with position.

The component of (31) perpendicular to **B** becomes

$$\partial\mathbf{E}_n/\partial t=\tfrac{1}{2}(c\mathbf{B}/p_m)\times\{-\rho d\mathbf{u}_D/dt-\nabla(p_n+p_m)+[(\mathbf{B}\cdot\nabla)\mathbf{B}/4\pi][1+(p_n-p_s)/2p_m]\}. \tag{42}$$

Now from (28) it follows that

$$\mathbf{E}=-(\mathbf{u}_D/c)\times\mathbf{B}. \quad (43)$$

Thus, if $\partial/\partial t$ may be regarded as being of the order of $|\mathbf{u}_D/L|$, it follows that $\partial\mathbf{E}/\partial t$ is of the order of $(cB/L)(u_D{}^2/c^2)$. The individual terms on the right-hand side of (42) are all of the order of (cB/L). Hence we have the well-known result[1] that the displacement current may be neglected. But equating the right-hand side of (42) to zero gives us the equations of motion of $\mathbf{u}_D$:

$$\rho d\mathbf{u}_D/dt=-\nabla_n(p_n+p_m) +[(\mathbf{B}\cdot\nabla)\mathbf{B}/4\pi]_n[1+(p_n-p_s)/2p_m]. \quad (44)$$

If $(p_n/p_s)/p_m$ is small compared to one, as it is in a sufficiently strong magnetic field or when the thermal motions in the plasma are nearly isotropic, then

$$\rho d\mathbf{u}_D/dt=-\nabla_n p_n+(1/4\pi)(\nabla\times\mathbf{B})\times\mathbf{B} \quad (45)$$

upon using a familiar vector identity; the electric drift velocity satisfies the conventional hydrodynamic equation of motion. Using (43) to eliminate $\mathbf{E}$ from (32), we obtain the equation

$$\partial\mathbf{B}/\partial t=\nabla\times(\mathbf{u}_D\times\mathbf{B}). \quad (46)$$

Thus, we have found, as has already been shown by Low, Goldberger, and Chew,[3] and by Watson[4] and by Brueckner and Watson,[5] that the electric drift velocity in a tenuous plasma plays the same role in the equations of the motion and of the magnetic field as the mass velocity $\mathbf{v}$ in the usual hydromagnetic equations in liquids and dense gases: The magnetic lines of force are permanently connected and move with $\mathbf{u}_D$; there is no possibility for reconnection[8] of lines of force. Equation (46) is independent of the anisotropy, and so is in agreement with the work of Schlüter and Spitzer.

IV. MASS TRANSPORT VELOCITY

A. Perpendicular to B

The mass velocity $\mathbf{v}_n$ perpendicular to $\mathbf{B}$ may be computed from the expressions for the various current densities: Suppose that $\mathbf{i}_i$ is the ionic current density. Then $\mathbf{i}_i/e$ is the number of ions passing across one cm^2 each second, and $+M\mathbf{i}_i/e$ is the mass per cm^2 per second; $-m\mathbf{i}_e/e$ is the electronic contribution. The total mass flow is $N(M+m)\mathbf{v}_n$, so that

$$N(M+m)\mathbf{v}_n=(M\mathbf{i}_i/e)-(m\mathbf{i}_e/e). \quad (47)$$

With (19) and (47) we find that the mass transport velocity due to the circular motions is

$$\mathbf{v}_L=\left(\frac{c}{8\pi p_m}\right)\frac{\mathbf{B}}{eN(M+m)} \times\left\{\nabla(Mp_{ni}-mp_{ne})-\frac{1}{2}\frac{(Mp_{ni}-mp_{ne})}{p_m}\nabla p_m -\frac{(Mp_{ni}-mp_{ne})}{p_m}(\mathbf{B}\cdot\nabla)\mathbf{B}/8\pi\right\}. \quad (48)$$

From (6), or from (47) and (21), the contribution of the drift motions to the mass transport is

$$\mathbf{v}_D=\mathbf{u}_D+\left(\frac{c}{8\pi p_m}\right)\frac{\mathbf{B}}{eN(M+m)} \times\left\{\frac{\frac{1}{2}(Mp_{ni}-mp_{ne})}{p_m}\nabla p_m +\frac{(p_{si}-p_{se})}{p_m}(\mathbf{B}\cdot\nabla)\mathbf{B}/8\pi\right\}. \quad (49)$$

From (30), or from (30) and (47), the polarization mass transport is

$$\mathbf{v}_P=[c(M-m)/eB^2]\mathbf{B}\times d\mathbf{u}_D/dt. \quad (50)$$

The total mass velocity $\mathbf{v}$ is the sum of $\mathbf{v}_L$, $\mathbf{v}_D$, and $\mathbf{v}_P$. If the electron partial pressures p_{ne}, p_{se} are not immensely greater than the ion partial pressures p_{ni}, p_{si}, then, because the ion mass M is large compared to the electron mass m, if follows that $Mp_{ni}\gg mp_{ne}$ and $Mp_{si}\gg mp_{se}$. Hence the total mass velocity may be approximated as

$$\mathbf{v}\cong\mathbf{u}_D+(Mc/8\pi p_m e)\mathbf{B}\times d\mathbf{u}_D/dt+(c/8\pi p_m eN)\mathbf{B} \times\{\nabla p_{ni}+[(p_{si}-p_{ni})/p_m](\mathbf{B}\cdot\nabla)\mathbf{B}/8\pi\}. \quad (51)$$

The first term, ∇p_{ni}, in the braces results in a pressure drift, just as in (19) it resulted in the familiar pressure current.[9,10] The second term in the braces contains $(p_{si}-p_{ni})/p_m$ and represents the competition between two processes: on the one hand, thermal motion, $+p_{si}$, along curved lines of force, with the attendant centrifugal force, leads to a drift perpendicular to $\mathbf{B}$, resulting in an electric current given by the second term in the braces in (21), and in a mass transport; on the other hand, the crowding together on one side of the circular orbits of neighboring particles and the separation on the other side as neighboring particles circle curved lines of force results in an opposite electric current, and in a mass transport, proportional to $-p_{ni}$. When $p_{si}=p_{ni}$, the two effects cancel and the equation of motion (44) reduces to the usual hydromagnetic form (45); the mass transport velocity $\mathbf{v}$ simplifies to

$$\mathbf{v}=\mathbf{u}_D+(Mc/8\pi p_m e)\mathbf{B}\times d\mathbf{u}_D/dt +(c/8\pi p_m eN)\mathbf{B}\times\nabla p_{ni}. \quad (52)$$

We may use (44) to eliminate $d\mathbf{u}_D/dt$ from (51). Approximating the total material density ρ by the ion density NM, and remembering that $p_n=p_{ni}+p_{ne}$, etc., we obtain

$$\mathbf{v}=\mathbf{u}_D+(c/8\pi p_m eN)\mathbf{B}\times\{-\nabla(p_{ne}+p_m) +[(\mathbf{B}\cdot\nabla)\mathbf{B}/4\pi][1-(p_{se}-p_{ne})2p_m]\}. \quad (53)$$

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[8] E. N. Parker and M. Krook, Astrophys. J. 124, 214 (1956).

[9] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

[10] T. G. Cowling, *The Sun*, edited by G. P. Kuiper (University of Chicago Press, Chicago, 1953).

Thus, when we include the polarization mass transport explicitly, the mass transport velocity in excess of the electric drift, *viz.*, $(\mathbf{v}-\mathbf{u}_D)$, depends upon the ion partial pressure, as in (51); when we eliminate the polarization transport, the excess velocity depends upon the electron partial pressure.

From (51) or (53) we see that the mass transport velocity $\mathbf{v}$, and the electric drift velocity $\mathbf{u}_D$ are equal, neglecting terms $O(R/L)$. Thus $\mathbf{u}_D$ is, in general, the principal mass velocity. However, there are cases when $\mathbf{u}_D$ may vanish identically, and the remaining terms remain finite, as discussed in the next section; then, obviously, $\mathbf{u}_D$ is not the principal mass motion.

B. Parallel to B

The mass motions parallel to $\mathbf{B}$ arise as a consequence of pressure gradients along $\mathbf{B}$, and also as a consequence of the reaction of diverging lines of force against the centrifugal force of the circular motions of the individual particles. Unfortunately this latter effect, given by (7), depends on the anisotropy and its variation along $\mathbf{B}$; to consider it quantitatively requires explicit introduction of the velocity distribution function.

Consider, therefore, a slender static tube of magnetic flux across which the field density is uniform. We let s represent distance measured along the tube, so that the field density within the tube may be written $B(s)$. In the absence of electric fields of external origin an ion (or electron) initially circling the lines of force within the tube will always be contained within the tube, circling the same lines of force; since the field is static, the magnitude of the velocity of each particle, denoted by w, remains constant. We shall denote the number of ions per unit volume by $N(s)$. If $A(s)$ is the cross-sectional area of the tube, then since $\nabla\cdot\mathbf{B}=0$, it follows that

$$A(s)B(s)=A(0)B(0). \tag{54}$$

The number of particles per unit length of the tube is

$$n(s)=N(s)A(s). \tag{55}$$

We shall suppose for the present that all ions move with the same speed w. Then if θ is the angle of pitch of the helical trajectory of an individual particle (the angle between $\mathbf{w}$ and $\mathbf{B}$), we see that

$$w_n=w\sin\theta,\quad w_s=w\cos\theta. \tag{56}$$

We shall denote the distribution of the ion velocities by the function $f(s,\theta,t)$, so that $f(s,\theta,t)d\theta$ is the number of particles per unit length with an angle of pitch between θ and $\theta+d\theta$. It follows that

$$n(s,t)=\int_0^\pi d\theta f(s,\theta,t). \tag{57}$$

As a given particle moves along the tube its angle of pitch varies in such a way as to preserve the usual adiabatic invariant, $w_n^2/B=\text{constant}$. Hence, if $\theta(0)$ is the angle of pitch when the particle is at $s=0$, then at s $(\neq 0)$

$$\sin\theta(s)=[B(s)/B(0)]^{\frac{1}{2}}\sin\theta(0), \tag{58}$$

and

$$\frac{d\theta}{ds}=\frac{\tan\theta(s)}{2B(s)}\left(\frac{dB}{ds}\right). \tag{59}$$

We consider those particles in $(\theta,\theta+\Delta\theta)$ at s at time t. At time $t+\delta t$, where $\delta t=\delta s/w_s$ they will be at $s+\delta s$. A particle with angle of pitch $\theta(s)$ at s will have an angle of pitch of

$$\theta(s+\delta s)=\theta(s)+(d\theta/ds)\delta s,$$

at $s+\delta s$; a particle with angle of pitch $\theta(s)+\Delta\theta$ at s will have an angle of pitch of

$$\theta(s)+\Delta\theta+\left.\frac{d\theta}{ds}\right|_{\theta+\Delta\theta}\delta s=\theta(s)+\Delta\theta+\frac{d\theta}{ds}\delta s+\left(\frac{\partial}{\partial\theta}\right)_s\frac{d\theta}{ds}\delta s\Delta\theta.$$

Thus, whereas the particles occupied an interval of $\Delta\theta$ at s, they occupy the interval $\Delta\theta[1+(\partial/\partial\theta)_s(d\theta/ds)\delta s]$ at $s+\delta s$. Their velocity along the tube, $w\cos\theta$ at s, has increased to $w\cos[\theta+(d\theta/ds)\delta s]$, or $w\cos\theta-w\sin\theta\times(d\theta/ds)\delta s$, so that they spend less time at $s+\delta s$, and their spatial density is correspondingly decreased. Therefore, the distribution function, the spatial density, at (s,t) is related to the distribution function at $(s+\delta s,\ t+\delta t)$ by

$$\begin{aligned}w\cos\theta f(s,\theta,t)\Delta\theta &=[w\cos\theta-w\sin\theta(d\theta/ds)\delta s]\\ &\quad\times f[s+\delta s,\ \theta+(d\theta/ds)\delta s,\ t+\delta s/w\cos\theta]\\ &\quad\times\Delta\theta[1+(\partial/\partial\theta)_s(d\theta/ds)\delta s],\end{aligned} \tag{60}$$

from which it follows that

$$0=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial s}w\cos\theta+\frac{\partial f}{\partial\theta}\left(\frac{w\sin\theta}{2B}\right)\left(\frac{dB}{ds}\right)+f\frac{w\cos\theta}{2B}\left(\frac{dB}{ds}\right). \tag{61}$$

The equation may be somewhat simplified, and the spatial variables s and θ may be separated from t, if we let

$$f(s,\theta,t)=\psi(s,\theta)[B(0)/B(s)]^{\frac{1}{2}}\exp(wkt), \tag{62}$$

where k is a constant with dimensions of inverse length. Then

$$0=k\psi+\frac{\partial\psi}{\partial s}\cos\theta+\frac{\partial\psi}{\partial\theta}\left(\frac{\sin\theta}{2B}\right)\left(\frac{dB}{ds}\right). \tag{63}$$

We shall go no further with the time-dependent solution: we shall defer further discussion to Sec. V where we consider the static solutions.

V. STATIC EQUILIBRIUM

A. Isotropy

Complete static equilibrium obtains when the mass velocity $\mathbf{v}$ vanishes and the magnetic field is static, $\partial\mathbf{B}/\partial t=0$. Suppose that in such a case p_s and p_n eventually become equal to each other, whatever may have been their initial values. Then using a familiar vector identity, (53) requires that

$$\mathbf{u}_D=(c/8\pi p_m eN)\{-\nabla p_{ne}+(\nabla\times\mathbf{B})\times\mathbf{B}/4\pi\}\times\mathbf{B},$$

(46) becomes

$$\begin{aligned}0&=\nabla\times(\mathbf{u}_D\times\mathbf{B})\\&=\nabla\times\{(c/eN)[\nabla_n p_{ne}-(\nabla\times\mathbf{B})\times\mathbf{B}/4\pi]\}.\end{aligned}\quad(64)$$

Thus static equilibrium across the lines of force results when $[\nabla_n p_{ne}-(\nabla\times\mathbf{B})\times\mathbf{B}/4\pi]$ is expressible as the gradient of a scalar.[11,12] The simplest static condition is a force-free magnetic field, $(\nabla\times\mathbf{B})\times\mathbf{B}=0$, and uniform density N and pressure p_n. If the fields are, say, independent of the z coordinate, then $\nabla\times(\nabla_n p_{ne})=0$ for all p_{ne}, and it is not necessary that the pressure be uniform over the x and y directions.

It is interesting to note that if we require that the principal velocity $\mathbf{u}_D$ vanish, rather than the total velocity $\mathbf{v}$, then $\partial\mathbf{B}/\partial t$ is automatically zero. Again putting $p_n=p_s$, we have from (44) that

$$\nabla_n p_n=(\nabla\times\mathbf{B})\times\mathbf{B}/4\pi,\quad(65)$$

upon using a familiar vector identity. Eliminating $\mathbf{B}$ inside the braces on the right-hand side of (53) yields the mass motion

$$\mathbf{v}=(c/8\pi p_m eN)\mathbf{B}\times\nabla p_{ni}.\quad(66)$$

There is the residual mass motion, the pressure drift[9,10] given by (66).

B. Anisotropy

When $p_n\neq p_s$ the situation is more complicated. Not only are the field conditions under which $\mathbf{v}$ can be made to vanish, in (53), and $\partial\mathbf{B}/\partial t$, in (46), somewhat more involved, but the degree of anisotropy is a varying function of position; the particle velocity distribution function must satisfy (61). We shall consider time-independent solutions of (61) to show the manner in which p_n and p_s must be related in order that there be static equilibrium with $\partial/\partial t=0$.

For the velocity distribution to be independent of time, we put $k=0$ in (62). Then (63) reduces to

$$0=\frac{\partial\psi}{\partial s}\cos\theta+\frac{\partial\psi}{\partial\theta}\left(\frac{\sin\theta}{2B}\right)\left(\frac{dB}{ds}\right).\quad(67)$$

The variables are immediately separable, and we obtain

$$f(s,\theta)=c_\alpha\left[\frac{B(0)}{B(s)}\right]^{\frac{1}{2}(\alpha+1)}\sin^\alpha\theta\quad(68)$$

as the basic solution of (61); c_α is a constant.

If $F(s,\theta)$ is the distribution function per unit volume, rather than per unit length of the flux tube, then $F(s,\theta)d\theta$ represents the number of particles per unit volume with angle of pitch in $(\theta,\theta+d\theta)$, and

$$N(s)=\int^\pi d\theta F(s,\theta).\quad(69)$$

From (54) and (55) it follows that

$$F(s,\theta)=f(s,\theta)/A(s)=C_\alpha[B(0)/B(s)]^{\frac{1}{2}(\alpha-1)}\sin^\alpha\theta,\quad(70)$$

where C_α is a constant. A more general solution is

$$F(s,\theta)=\left[\frac{B(s)}{B(0)}\right]^{\frac{1}{2}}\int_0^\infty d\alpha C(\alpha)\left[\frac{B(0)}{B(s)}\sin^2\theta\right]^{\frac{1}{2}\alpha}.\quad(71)$$

For isotropy, $\alpha=1$, we have the result already shown by Schlüter[2] and by Spitzer[6] that the isotropy is preserved and the particle density is uniform, independently of how $B(s)$ may vary. We see from (70) that if $F(s,\theta)$ has the form $\sin^\gamma\theta$ at one value of s, it will depend upon θ as $\sin^\gamma\theta$ for all values of s; the form of the anisotropy does not vary with the field density. Further, it may be seen from (70) that $\gamma<1$ corresponds to $p_n<p_s$, in which case the particle density increases as $B(s)$ increases. If on the other hand $\gamma>1$, then $p_n>p_s$ and the particle density decreases with increasing $B(s)$.

If $F(s,\theta)$ is not expressible as $\sin^\gamma\theta$ for a single value of γ, but involves values of α, in (71), both greater and smaller $\alpha=1$, then we see from the presence of the factor $[B(0)/B(s)]^{\frac{1}{2}\alpha}$ that p_s/p_n will be relatively large where $B(s)$ is large and small where $B(s)$ is small.

Presumably the velocity distribution at some location, say $s=0$, serves to determine the function $C(\alpha)$,

$$F(0,\theta)=\int_0^\infty d\alpha C(\alpha)\sin^\alpha\theta.\quad(72)$$

We may compute $F(s,\theta)$ from the $C(\alpha)$ thus determined, by using (71). The particle density $N(s)$ may be reduced to

$$N(s)=\left[\frac{B(s)}{B(0)}\right]^{\frac{1}{2}}\int_0^\infty d\alpha C(\alpha)\left[\frac{4B(0)}{B(s)}\right]^{\alpha/2}\frac{\Gamma^2[\frac{1}{2}(\alpha+1)]}{\Gamma(\alpha+1)},\quad(73)$$

upon using the integral

$$\int^\pi d\theta\sin^{x-1}\theta=2^{x-1}\Gamma^2(\tfrac{1}{2}x)/\Gamma(x).\quad(74)$$

[11] S. Lundquist, Arkiv Fysik **2**, 361 (1950).
[12] S. Chandrasekhar and K. H. Prendergast, Proc. Natl. Acad. Sci. U. S. **42**, 5 (1956).

The pressures p_n and p_s may be represented as

$$p_n=\int_0^\pi d\theta F(s,\theta)\tfrac{1}{2}Mw_n^2$$

$$=2Mw^2\left[\frac{B(s)}{B(0)}\right]^{\frac{1}{2}}\int_0^\infty d\alpha C(\alpha)\times\left[\frac{4B(0)}{B(s)}\right]^{\alpha/2}\frac{\Gamma^2[\frac{1}{2}(\alpha+3)]}{\Gamma(\alpha+3)}, \quad (75)$$

$$p_s=\int_0^\pi d\theta F(s,\theta)Mw_s^2=Mw^2N(s)-2p_n. \quad (76)$$

When $F(s,\theta)$ is of the simple form $\sin^\gamma\theta$, as discussed above, then it is readily shown from (70) or (71) that

$$F(s,\theta)=\frac{N(0)\Gamma(\gamma+1)}{2^{\gamma-1}\Gamma^2[\frac{1}{2}(\gamma+1)]}\left[\frac{B(0)}{B(s)}\right]^{\frac{1}{2}(\gamma-1)}\sin^\gamma\theta. \quad (77)$$

It follows that $N(s)=N(0)[B(0)/B(s)]^{\frac{1}{2}(\gamma-1)}$, and

$$p_n=2Mw^2N(0)\frac{\Gamma(\gamma+1)\Gamma^2[\frac{1}{2}(\gamma+3)]}{\Gamma(\gamma+3)\Gamma^2[\frac{1}{2}(\gamma+1)]}\left[\frac{B(0)}{B(s)}\right]^{\frac{1}{2}(\gamma-1)}. \quad (78)$$

VI. WHEN R/L CANNOT BE SMALL

Though a development of the dynamical properties of a tenuous ionized gas in slowly varying fields ($R/L\ll 1$) is widely applicable, there is one commonly occurring, and in two dimensions unavoidable, situation where the approximation is invalid in slowly varying fields of finite extent, *viz.*, in the vicinity of a point or line on which the field density vanishes: As a particle approaches a neutral surface, $B=0$, between two regions of oppositely directed field, the Larmor radius increases without bound and no matter how slowly B may decrease, R/L becomes large.

To investigate this one essential breakdown, consider the magnetic field $\mathbf{B}$ in which the lines of force are all parallel to the z axis and point in the positive z direction. We shall suppose that the field density is a function only of x and vanishes at $x=0$; then $B=B(x)$ and $B(0)=0$. Thus the yz plane is a neutral plane.

We suppose that the scale of variation of $B(x)$ is L, and is very large. Then we may expand $B(x)$ about $x=0$. The zero order term vanishes by assumption; the second order term may be neglected provided the first order term does not vanish identically: We write

$$B(x)=B_0x/L+O(1/L^2),$$

where B_0 is a constant and is of the order of the field density far from the neutral plane. We define the radius of gyration

$$R_0=Mw_nc/qB_0 \quad (79)$$

for a particle with velocity w_n in B_0, and require that $R_0/L\ll 1$. We see, of course, that sufficiently close to $x=0$, where $B\rightarrow 0$, R/L is not small.

The equations of motion of a particle of charge q and mass M in the field $B(x)$ may be written

$$\frac{d^2x}{dt^2}=\frac{q}{Mc}B(x)v,$$

$$dv=-(q/Mc)B(x)dx,$$

where $v=dy/dt$. Besides the usual energy integral

$$(dx/dt)^2+v^2=w_n^2, \quad (80)$$

there is obviously the integral

$$v+(q/Mc)\int_{x_1}^x dxB(x)=0, \quad (81)$$

where x_1 is the value of x at which the particle is turned so that it moves parallel to the x axis, i.e., $dy/dt=0$.

The problem is now readily reduced to two quadratures. When $B(x)$ is a linear function of x, as we have supposed here, the integrals are elliptic. We define a_1 and a_2 by the relations

$$a_1^2=x_1^2-2LR_0, \quad (82)$$

$$a_2^2=x_1^2+2LR_0. \quad (83)$$

Then if the particle will be at $x=x_2$, $y=y_2$, when $t=t_2$,

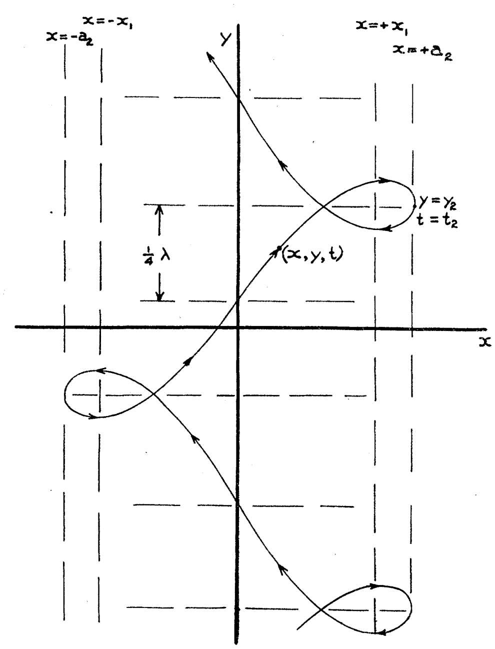

FIG. 2. The trajectory of a positively charged particle sweeping widely across the neutral plane, $x=0$. The magnetic field is in the positive z direction and varies as B_0x/L. The net drift is in the positive y direction.

we have

$$t-t_2=+\frac{2LR_0}{w_n}\int_x^{a_2}\frac{dx}{[(a_2^2-x^2)(x^2-a_1^2)]^{\frac{1}{2}}},\quad (84)$$

$$y-y_2=-\int_x^{a_2}\frac{dx(x_1^2-x^2)}{[(x^2-a_1^2)(a_2^2-x^2)]^{\frac{1}{2}}}.\quad (85)$$

We are interested in the case that $2LR_0>x_1^2$, so that a_1^2 is negative. Then $x^2-a_1^2$ has no zeros for real values of x; the orbit is symmetric about $x=0$ and is confined between $\pm a_2$, as shown in Fig. 2. We let λ represent the distance along the y axis which the particle travels in one period P of its motion. Then λ is equal to four times $y-y_2$ at $x=0$,

$$\lambda=-4\int_0^{a_2}\frac{dx(x_1^2-x^2)}{[(x^2-a_1^2)(a_2^2-x^2)]^{\frac{1}{2}}}.\quad (86)$$

The period is

$$P=(8L/R_0)\int_0^{a_2}\frac{dx}{[(a_2^2-x^2)(x^2-a_1^2)]^{\frac{1}{2}}}.\quad (87)$$

It is readily shown[13] that

$$\lambda=-4(LR_0)^{\frac{1}{2}}\{K(k)-2E(k)\},\quad (88)$$

$$P=4(LR_0/w_n^2)^{\frac{1}{2}}K(k),\quad (89)$$

where

$$k^2=a_2^2/(a_2^2-a_1^2);\quad (90)$$

$K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds, respectively. It follows that the particle possesses a drift velocity u along the y axis given by

$$u=\lambda/P$$
$$=w_n[1-2E(k)/K(k)].\quad (91)$$

The drift velocity is in the positive y direction when $k>0.9092$, zero when $k=0.9092$, and negative for $k<0.9092$; $k=0.9092$ corresponds to $x_1=1.143(LR_0)^{\frac{1}{2}}$. Thus, particles sweeping widely across the y axis, as shown in Fig. 2, drift in the positive y direction; particles confined close to the y axis, as shown in Fig. 3 drift in the negative y direction.

Given the particle distribution in the vicinity of $x=0$, we may compute the resulting current density. However, we no longer have the convenient guiding center with which to locate the individual particle trajectories, and it is not clear how to define the usual parameters, N, p, etc. Therefore, we shall content ourselves with a physical description of the dynamics in the vicinity of the neutral $x=0$ plane, based upon the above orbit calculations; we will find that stability is quickly reached and the field behaves as in classical hydromagnetic theory in a medium of infinite conductivity.

[13] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists* (Lange, Maxwell, and Springer Ltd., New York, 1954), formulas 213.00, 213.06, 312.05.

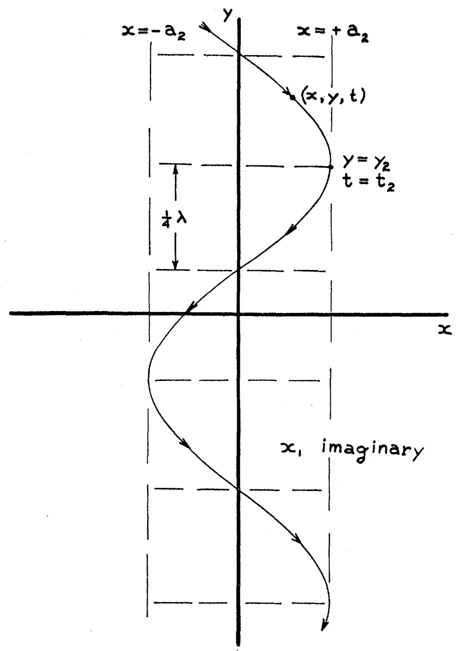

FIG. 3. The trajectory of a positively charged particle moving close to the neutral plane, $x=0$. The net drift is in the negative y direction.

Suppose that initially the only particles moving in the vicinity of the $x=0$ plane have large values of x, so that they progress in the positive y direction as shown in Fig. 2. Then Maxwell's equation $\partial\mathbf{E}/\partial t=c\nabla\times\mathbf{B}-4\pi\mathbf{i}$, in which $\nabla\times\mathbf{B}$ is in the negative y direction, shows that $\partial\mathbf{E}/\partial t$ is in the negative y direction. Now an electric field in the negative y direction results in a drift toward the y axis from both sides, according to (28), and the particle trajectory shrinks closer to the y axis, as shown in Fig. 3. Shrinking the trajectory closer to the y axis decreases x_1, until it becomes less than $1.143(LR_0)^{\frac{1}{2}}$; then $\mathbf{i}$ reverses sign, becoming more and more negative, until a balance is achieved with $c\nabla\times\mathbf{B}=4\pi\mathbf{i}$ and $\partial\mathbf{E}/\partial t=0$. Therefore, whatever the initial particle configuration in the vicinity of the neutral plane, a stable configuration will soon result, with $\mathbf{i}=(c/4\pi)\nabla\times\mathbf{B}$, just as in classical hydromagnetics; the exception to $R/L\ll 1$ in the vicinity of a neutral plane leads to no violation of our previous special conclusion that the conventional hydromagnetic equations are approximately valid in a tenuous plasma.