Approximate Solutions of the Einstein Equations for Isentropic Motions of Plane-Symmetric Distributions of Perfect Fluids

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An approximation procedure is developed and applied to the problem of determining the gravitational and hydrodynamical fields associated with a plane-symmetric distribution of a perfect fluid with arbitrary caloric equation of state in isentropic motion. Special relativistic hydrodynamics in co-moving coordinates are discussed and methods are given for solving specific problems. These solutions provide a zero-order approximation of the corresponding general relativistic problems. The linear equations describing the higher order corrections are discussed and the application of the method of characteristics is pointed out. The existence of shock waves in special relativity makes plausible their existence in general relativity, and they may be used to avoid using physically unacceptable solutions of the field equations.

1. INTRODUCTION

It is the purpose of this paper to discuss by means of an example an approximate procedure for solving the Einstein field equations of general relativity. The procedure consists of considering the coefficients of the metric tensor as a power series in the constant

$$k=8\pi G/c^2, \tag{1.1}$$

where G is Newton's constant of gravitation, and obtaining a sequence of sets of equations, each set of equations arising from the coefficient of a single power of k.

The Einstein field equations are

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R=-kT^{\mu\nu}, \tag{1.2}$$

where $T^{\mu\nu}$ represents the stress-energy tensor which "creates" the gravitational field. The equations obtained by using the zero-order terms are

$$(R^{\mu\nu})^{(0)}=0, \tag{1.3}$$

where the left-hand side is the Ricci tensor calculated from the zero-order terms in the expansion of the $g_{\mu\nu}$, which we denote by $g_{\mu\nu}^{(0)}$.

We shall assume that $g_{\mu\nu}^{(0)}$ represent a flat (Minkowski) space. This assumption will be shown to be plausible for the example considered. Equation (1.3) does not imply that space-time is flat, as has been shown by examples.[1] By means of this assumption we make the special relativity theory solution of a problem the zero order approximation to the general theory one.

In dealing with the example to be described below it is convenient to use a noninertial coordinate system in the Minkowski space-time and therefore we do not suppose that the $g_{\mu\nu}^{(0)}$ are constants.

The example to which we apply the approximation procedure is that of solving the Eqs. (1.2) for a plane-symmetric space-time where the stress-energy tensor describes a perfect fluid with an arbitrary caloric equation of state in isentropic motion. It has been shown earlier[2] that for this case a co-moving coordinate system can be introduced in which the field equations become partial differential equations for the metric tensor alone.

Because the special theory of relativity is to provide a zero-order approximation to the solution obtained, we devote the first part of the paper to a discussion of the formulation and solution of the one dimensional motion of a perfect fluid in terms of co-moving coordinates.

In addition to leading to an example to which the approximation procedure can be applied in some detail, a plane-symmetric distribution of a perfect fluid in isentropic motion leads to another problem of interest. In classical theory and in the special theory of relativity shocks occur in certain cases. These cases are discussed in the general theory and it is shown that the same difficulties arise in this theory as do in the special theory. That this is so, is evident from the fact that the nonlinearity of the problem is confined to the zero order approximation and that approximation is given by the special theory of relativity.

2. LAGRANGIAN COORDINATES IN SPECIAL RELATIVITY

In an inertial coordinate system in special relativity, the line element is taken to be

$$ds^2=dt^2-\frac{1}{c^2}(dx^2+dy^2+dz^2)=g_{\mu\nu}dx^\mu dx^\nu, \tag{2.1}$$

with $x^1=x$, $x^2=y$, $x^3=z$, and $x^4=t$. The equations of hydrodynamics in this coordinate system are the five conservation laws: The four conservation laws involving energy and momentum,

$$T^{\mu\nu}{}_{,\nu}=0, \tag{2.2}$$

and the conservation of mass

$$(\rho u^\mu)_{,\mu}=0, \tag{2.3}$$

where the comma denotes the partial derivative, the

[1] A. H. Taub, Ann. Math. **53**, 472 (1951).
[2] A. H. Taub, Phys. Rev. **103**, 454 (1956).

summation convention is used, and

$$T^{\mu\nu}=\rho\left(1+\frac{\epsilon}{c^2}+\frac{p}{\rho c^2}\right)u^\mu u^\nu-g^{\mu\nu}\frac{p}{c^2}=\sigma u^\mu u^\nu-g^{\mu\nu}\frac{p}{c^2}, \quad (2.4)$$

u^μ is the four-velocity of the fluid, ρ the rest density, p the pressure, and

$$\epsilon=\epsilon(p,\rho) \quad (2.5)$$

is the caloric equation of state of the fluid. The vector field u^μ satisfies the equations

$$g_{\mu\nu}u^\mu u^\nu=1. \quad (2.6)$$

It may be shown that Eqs. (2.2) imply that

$$TS,_\mu u^\mu=0, \quad (2.7)$$

where T is the temperature and S is the specific entropy, these quantities being defined as functions of p and ρ by the equation

$$TdS=d\epsilon+pd(1/\rho). \quad (2.8)$$

For the one-dimensional motion of a gas, we have $u^2=u^3=0$ and u^1 and u^4 functions of x and t alone. We may write

$$u^4=1/(1-u^2)^{\frac{1}{2}}, \quad u^1=cu/(1-u^2)^{\frac{1}{2}}, \quad (2.9)$$

where u is a function of x and t and represents the three-dimensional particle velocity in units in which the velocity of light is one. Equation (2.6) is then satisfied.

When the motion is isentropic, that is when

$$S(p,\rho)=\text{constant}, \quad (2.10)$$

Eq. (2.7) is satisfied, and p is a function of ρ alone. The five equations (2.2) and (2.3) then reduce to the two equations

$$\frac{1}{c}\left(\frac{\rho}{(1-u^2)^{\frac{1}{2}}}\right)_t+\left(\frac{\rho u}{(1-u^2)^{\frac{1}{2}}}\right)_x=0, \quad (2.11)$$

and

$$\frac{1}{c}\left(\frac{\sigma c^2 u}{1-u^2}\right)_t+\left(\frac{\sigma c^2u^2}{1-u^2}+p\right)_x=0, \quad (2.12)$$

where the subscript denotes partial differentiation with respect to the variable indicated. Equations (2.11) and (2.12) are obtained from (2.3) and (2.2) respectively by using (2.9) and by setting $\mu=1$ in the latter equations.

The particle paths, the curves describing the world history of a point of the fluid, are defined as solutions of the ordinary differential equation

$$dx/dt=cu(x,t). \quad (2.13)$$

Let us write the solution of this equation as

$$x=x(X,t), \quad (2.14)$$

where X is the value of x on this curve at $t=0$. Equation (2.14) then is said to give the world line of the "particle X." The variable X is the Lagrange coordinate of classical hydrodynamics.

We write Eq. (2.14) as a pair of equations

$$x=x(X,t'), \quad t=t', \quad (2.15)$$

and consider these as a transformation from the coordinates x, t to X, t'. Then Eq. (2.13) becomes

$$x_{t'}=\partial x/\partial t'=cu, \quad (2.16)$$

and the conservation of mass equation is equivalent to the statement that

$$x_X=\partial x/\partial X=\rho_0(1-u^2)^{\frac{1}{2}}/\rho, \quad (2.17)$$

where ρ_0 is the density at $t=0$. It follows from the second of equations (2.15) that

$$t_X=\partial t/\partial X=0, \quad t_{t'}=\partial t/\partial t'=1. \quad (2.18)$$

Equations (2.16), (2.17), and (2.18) together with the rules of differentiation then imply that

$$\begin{aligned} X_x&=\frac{\partial X}{\partial x}=\frac{\rho}{\rho_0(1-u^2)^{\frac{1}{2}}},\\ X_t&=\frac{\partial X}{\partial t}=\frac{-cu\rho}{(1-u^2)^{\frac{1}{2}}\rho_0},\\ t_x'&=\partial t'/\partial x=0,\\ t_t'&=\partial t'/\partial t=1. \end{aligned} \quad (2.19)$$

Equation (2.1), the conservation-of-mass equation, is simply the statement that X is a function of x and t defined by the transformation (2.15). Differentiating Eq. (2.17) with respect to t' gives

$$cu_X=(\rho_0(1-u^2)^{\frac{1}{2}}/\rho)_{t'}. \quad (2.20)$$

The rules of partial differentiation and Eq. (2.20) enable us to write Eq. (2.12) as

$$\left(\frac{\rho_0\sigma c^2u}{(1-u^2)^{\frac{1}{2}}}\right)_{t'}+cp_X=0. \quad (2.21)$$

The coordinates X, t' may be called Lagrangian coordinates in special relativity. Since

$$\begin{aligned} dx&=x_XdX+x_{t'}dt'=(\rho_0/\rho)(1-u^2)^{\frac{1}{2}}dX+cudt',\\ dt&=t_XdX+t_{t'}dt'=dt', \end{aligned}$$

the line element (2.1) becomes

$$ds^2=\left((1-u^2)^{\frac{1}{2}}dt'-\frac{\rho_0}{\rho}\frac{u}{c}dX\right)^2-\frac{1}{c^2}\frac{\rho_0^2}{\rho^2}dX^2-\frac{1}{c^2}(dy^2+dz^2). \quad (2.22)$$

3. ORTHOGONAL CO-MOVING COORDINATES

The Lagrangian coordinates, X, t', are co-moving since in this coordinate system

$$u^{1*}=u^\nu\partial X/\partial x^\nu=u^4X_t+u^1X_x=0,$$

as follows from Eqs. (2.19) and (2.9). However, they are not orthogonal. An orthogonal co-moving coordinate system may be introduced as follows: For isentropic motions the integral

$$\phi=-\int_{p_0}^{p}dp/\sigma c^2 \tag{3.1}$$

defines a function $\phi(p)$ since σ is a function of p alone due to the isentropy condition. The quantity p_0 appearing in Eq. (3.1) is that value of the pressure corresponding to the given entropy and associated with the density ρ_0.

Now consider the equation

$$dT=e^{-\phi}\left((1-u^2)^{\frac{1}{2}}dt'-\frac{\rho_0}{\rho}\frac{u}{c}dX\right). \tag{3.2}$$

This defines a function $T(t',X)$ if the right-hand side is a perfect differential. The condition for this is of course

$$[e^{-\phi}(1-u^2)^{\frac{1}{2}}]_X+\left[e^{-\phi}\frac{\rho_0}{\rho}\frac{u}{c}\right]_{t'}=0,$$

or (3.3)

$$[(1-u^2)^{\frac{1}{2}}]_X-\phi_X(1-u^2)^{\frac{1}{2}}+\left(\frac{\rho_0}{\rho}\frac{u}{c}\right)_{t'}-\phi_{t'}\frac{\rho_0}{\rho}\frac{u}{c}=0.$$

Now from Eq. (3.1) we have

$$\phi_X=-(1/\sigma c^2)p_X,\quad \phi_{t'}=-(1/\sigma c^2)p_{t'}.$$

Moreover, since the motion is isentropic,

$$\epsilon_{t'}=-p(1/\rho)_{t'}=-(p/\rho)_{t'}+(1/\rho)p_{t'}.$$

That is,

$$(1/\rho)p_{t'}=(\epsilon+p/\rho)_{t'}=(c^2\sigma/\rho)_{t'}.$$

By using these results and Eq. (2.21), we may verify that Eq. (3.3) is satisfied. Hence the equations

$$T=T(X',t'),\quad X'=X, \tag{3.4}$$

where the function T is defined by Eq. (3.2), define a coordinate transformation to an orthogonal co-moving coordinate system in which the line element is given by

$$ds^2=e^{2\phi}dT^2-\frac{\rho_0^2}{(\rho c)^2}dX^2-\frac{1}{c^2}(dy^2+dz^2). \tag{3.5}$$

where we have dropped the primes since they are no longer needed for clarity. The quantity ϕ is a function of X and T, ρ is a function of ϕ, determined through the relations $\phi(p)$ and $p=p(\rho)$, and ρ_0 which represents the original density may be a function of X. However by a suitable replacement of X by a function of itself we may replace the line element by one of the form of (3.5) with ρ_0 a constant. In the coordinate system in which Eq. (3.5) holds, the velocity field is given by

$$u^\mu=e^{-\phi}\delta_4{}^\mu, \tag{3.6}$$

since the coordinate system is obtained by a time transformation from a co-moving one and since the velocity vector must be a unit time-like vector.

4. EQUATIONS OF MOTION IN CO-MOVING COORDINATES

We may of course formulate any problem in special relativistic hydrodynamics in terms of the X, T coordinate system since that theory is an invariant one. If this is done we must obtain a method for determining the function $\phi(X,T)$ which determines the velocity field and the pressure field (and hence the density field). Conversely, for any problem in hydrodynamics for which the velocity fields and the pressure field is known in an inertial frame, we may determine the function $\phi(X,T)$.

In the X, T coordinate system the conservation equations, Eqs. (2.2) and (2.3), which describe the motion of the fluid become

$$T^{\mu\nu}{}_{;\nu}=0, \tag{4.1}$$

and

$$(\rho u^\mu)_{;\mu}=0, \tag{4.2}$$

respectively, where the semicolon represents the covariant derivative with respect to the metric tensor given by Eq. (3.5). These equations are *identically* satisfied. We verify this statement for Eq. (4.2). A similar argument applies to Eq. (4.1).

Equation (4.2) may be written as

$$\frac{1}{\surd(-g)}\frac{\partial}{\partial x^\mu}[\surd(-g)\rho u^\mu]=0.$$

On substituting from Eqs. (3.5) and (3.6), we see that since

$$\surd(-g)=e^\phi\rho_0/(c^3\rho),$$

the above equation becomes

$$(\rho_0)_T=0,$$

which is identically satisfied as a consequence of ρ_0 being a constant.

Since Eqs. (4.1) and (4.2) do not impose any restrictions on the function $\phi(X,T)$, it would appear that this function could be arbitrary. However this is not the case. Since we are dealing with special relativity, the space-time described by the line element (3.5) must be the Minkowski one. That is, the Riemann-Christoffel curvature tensor must vanish. To each $\phi(X,T)$ for which this condition is satisfied, there then corresponds a hydrodynamic motion determined by the world lines of the particles of the fluid which are the curves

$$x=x(t),$$

given parametrically by the equations

$$x=x(X,T), \quad t=t(X,T), \tag{4.3}$$

where X is kept fixed to obtain the world line of the particle X.

Equations (4.3) are the transformation equations from the orthogonal co-moving coordinates (X,T) to the inertial ones x, t. They may be determined from a knowledge of the function $\phi(X,T)$ by solving the metric tensor transformation equations. We shall discuss these equations later.

5. DETERMINATION OF ϕ

The situation encountered here, of having the conservation equations satisfied identically as a result of a condition imposed on the metric tensor and the choice of the coordinate system, is precisely that found in the general theory of relativity. However, in the latter theory the requirement of the vanishing of the Riemann-Christoffel tensor is replaced by the requirement that the Einstein gravitational field equations be satisfied. Since we shall discuss the field equations of general relativity by an approximation scheme in which the zeroth order approximation to the metric is given by Eq. (3.5), where ϕ is determined as in the special theory of relativity, we evaluate the Riemann-Christoffel tensor for the case where Eq. (3.5) holds.

It may be shown that in this coordinate system the only nonvanishing component of this tensor is[3]

$$R_{4114}=-\frac{\rho_0}{\rho}e^{\phi}\left\{\left[\frac{e^{-\phi}}{c^2}\left(\frac{\rho_0}{\rho}\right)_T\right]_T-\left[\frac{\rho}{\rho_0}(e^{\phi})_X\right]_X\right\}.$$

Hence the requirement that the space-time be flat is equivalent to the requirement that

$$\left[\frac{e^{-\phi}}{c^2}\left(\frac{\rho_0}{\rho}\right)_T\right]_T-\left[\frac{\rho}{\rho_0}(e^{\phi})_X\right]_X=0, \tag{5.1}$$

and hence there must exist a function W such that

$$W_X=\frac{e^{-\phi}}{c}\left(\frac{\rho_0}{\rho}\right)_T, \tag{5.2}$$

$$W_T=\frac{c\rho}{\rho_0}(e^{\phi})_X. \tag{5.3}$$

Equations (5.2) and (5.3) are equivalent to Eqs. (2.20) and (2.21) if the latter equations are expressed in terms of X, T as independent variables and if the function W is given by the equation

$$W=\tfrac{1}{2}\log\frac{1+u}{1-u}, \tag{5.4}$$

and u is the quantity entering those equations. In verifying this result one uses the rules of partial differentiation, and Eqs. (3.1) and (3.2) in the form

$$T_{t'}=e^{-\phi}(1-u^2)^{\frac{1}{2}} \tag{5.5}$$

and

$$T_X=-\frac{\rho_0}{\rho}e^{-\phi}\frac{u}{c}. \tag{5.6}$$

Thus, starting with the co-moving coordinate system in which Eq. (3.5) holds, we find that the requirement that space-time be flat is equivalent to Eq. (5.1), that is, to an equation for the determination of ϕ. This equation is in turn equivalent to Eqs. (5.2) and (5.3), the form that the equations of motion in the original coordinate system take when we introduce X and T as independent variables. In the next section we show how to define the function u (and hence W) by starting from the co-moving coordinate system.

6. FUNCTION $u(X,T)$

This function represents the tangent in the x, t plane of the curve given parametrically by the Eq. (4.3) where X is kept fixed. That is,

$$cu=x_T/t_T. \tag{6.1}$$

Hence if we can determine the transformation from the coordinate system X, T to the inertial one x, t, that is determine the functions $x(X,T)$ and $t(X,T)$ we would then determine the function u from Eq. (6.1).

Many methods exist for determining these functions. For example, we may say that this as the classical equivalence problem between two metrics. We consider another one. Let x be defined as a solution of the equation

$$\left(e^{-\phi}\frac{\rho_0}{\rho c^2}x_T\right)_T-\left(e^{\phi}\frac{\rho}{\rho_0}x_X\right)_X=0, \tag{6.2}$$

such that

$$e^{-2\phi}x_T^2-c^2(\rho^2/\rho_0^2)x_X^2=-c^2. \tag{6.3}$$

Define

$$t_X=e^{-\phi}(\rho_0/\rho c^2)x_T, \tag{6.4}$$

and

$$t_T=e^{\phi}(\rho/\rho_0)x_X. \tag{6.5}$$

Equation (6.2) is the integrability condition of these equations. Then it follows that

$$e^{-2\phi}t_T^2-c^2\frac{\rho^2}{\rho_0^2}t_X^2=-\frac{1}{c^2}\left(e^{-2\phi}x_T^2-c^2\frac{\rho^2}{\rho_0^2}x_X^2\right)=1. \tag{6.6}$$

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[3] L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, 1926), p. 44.

The functions x and t so defined are the transformation functions that were being sought, for Eqs. (6.3) and (6.6), respectively, give the g^{11} and g^{44} components of the metric tensor in the x, t coordinate system. We also have

$$g^{14}=e^{-2\phi}x_T t_T-(c^2\rho^2/\rho_0^2)x_X t_X=0. \quad (6.7)$$

It follows from Eqs. (6.1) and (6.5) that

$$cu=\frac{e^{-\phi}x_T}{(\rho/\rho_0)x_X}=\frac{(\rho c^2/\rho_0)t_X}{e^{-\phi}t_T}. \quad (6.8)$$

Equation (6.3) may be written as

$$x_X=\rho_0/[\rho(1-u^2)^{\frac{1}{2}}]. \quad (6.9)$$

Equation (6.2) may be written as

$$\left(\frac{u}{c}x_X\right)_T-\left(e^{\phi}\frac{\rho}{\rho_0}x_X\right)_X=0.$$

On substituting for x_X, we have

$$\left(\frac{u}{c}\frac{\rho_0}{\rho(1-u^2)^{\frac{1}{2}}}\right)_T-\left(\frac{e^{\phi}}{(1-u^2)^{\frac{1}{2}}}\right)_X=0. \quad (6.10)$$

It is a consequence of Eqs. (6.2) to (6.5) that

$$\left(e^{-\phi}\frac{\rho_0}{\rho c^2}t_T\right)_T-\left(e^{\phi}\frac{\rho}{\rho_0}t_X\right)_X=0.$$

Equation (6.5) may be written as

$$t_T=e^{\phi}/(1-u^2)^{\frac{1}{2}}. \quad (6.11)$$

Hence we have

$$\left(\frac{\rho}{\rho_0 c}\frac{1}{(1-u^2)^{\frac{1}{2}}}\right)_T-\left(\frac{e^{\phi}u}{(1-u^2)^{\frac{1}{2}}}\right)_X=0. \quad (6.12)$$

Equations (6.10) and (6.12) may be solved for u_T and u_X to give

$$u_T=(1-u^2)c(\rho/\rho_0)(e^{\phi})_X, \quad u_X=(1-u^2)(e^{-\phi}/c)(\rho_0/\rho)_T.$$

These equations are just Eqs. (5.2) and (5.3).

The formal results of Sec. 5 and this section may be summarized as follows. In order to solve the conservation equations determining the isentropic motion of a fluid in an inertial coordinate system, we determine a function $\phi(X,T)$ and a function $x(X,T)$ in which ϕ is determined in terms of x_X and x_T. Since the coefficients of Eq. (6.2) are known when ϕ is known, the equation for $x(X,T)$ is determined. The function ϕ determined by the method outlined will satisfy Eq. (5.2) since Eq. (5.3) are a consequence of (6.2) and (6.3) and the definition of u, as a function of x_T and x_X.

Another method consists in determining $\phi(X,T)$ as a solution of Eq. (5.1) and then determining $x(X,T)$ and $t(X,T)$ by solving the equation

$$\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\nu}=\left\{\begin{matrix}\beta\\ \mu\nu\end{matrix}\right\}'\frac{\partial x^\alpha}{\partial x'^\beta}, \quad (6.13)$$

subject to Eqs. (6.6), (6.7), and (6.3), where the primed variables refer to the co-moving coordinate system, the unprimed ones to the inertial one, and the quantity in parentheses is the Christoffel symbol computed from the metric tensor given by Eq. (3.5). In this notation, we have

$$\partial x^4/\partial x'^1=t_X, \quad \partial x^4/\partial x'^4=t_T,$$
$$\partial x^1/\partial x'^1=x_X, \quad \partial x^1/\partial x'^4=x_T,$$
$$\partial x^\alpha/\partial x'^\beta=\delta_\beta{}^\alpha, \quad (\alpha,\beta=2,3).$$

It may be verified that Eqs. (6.13) are equivalent to Eqs. (5.3) with u defined by Eq. (6.1).

Still another method is that used in classical hydrodynamics, namely, to determine functions W and ϕ satisfying Eqs. (5.2) and (5.3) subject to the given boundary conditions. The function $x(X,T)$ is then determined from the function $u(X,T)$ by an integration.

7. BOUNDARY CONDITIONS

Before reviewing the methods for integrating Eqs. (5.2) and (5.3), we state the various conditions that the functions u and ϕ may be required to satisfy. These will be called boundary conditions here and they are of two sorts: (1) requirements on these functions on the surface $t(X,T)=0$, and (2) requirements on these functions at certain fixed values of X.

For example, consider a tube of length L in an inertial coordinate system filled with gas, closed at one end by a rigid wall and at the other by a movable piston. The tube and gas are initially at rest in the inertial coordinate system and at constant pressure, density and entropy. Move the piston in accordance with the equation

$$x=x_0(t), \quad (7.1)$$

where x_0 is a given function of t. It is required to find the motion of the gas. The boundary conditions for this problem are

$$\begin{matrix} u(x,0)=0 & u(x_0(t),t)=dx_0/dt, \\ \phi(x,0)=\text{constant}, & u(L,t)=0. \end{matrix} \quad (7.2)$$

That is, on the initial surface $t(X,T)$, u and ϕ are prescribed and in addition u is prescribed along two "particle paths" given by the parametric equations

$$x=x(X,T), \quad t=t(X,T), \quad (7.3)$$

with x equal to L in one case and in the other case $X=X_0$, where X_0 is such that the curve described by Eq. (7.1) is given parametrically by Eqs. (7.3) with $X=X_0$.

The first two of Eqs. (7.2) are boundary conditions of type (1) and the second two are of type (2). In other problems Eqs. (7.2) are replaced by corresponding statements.

8. CHARACTERISTIC FORM OF THE EQUATION

We now turn to the integration of Eqs. (5.3) which may be written as

$$W_X=-\frac{e^{-\phi}}{c}\frac{\rho_0}{\rho^2}\rho'\phi_T,\quad W_T=c\frac{\rho}{\rho_0}e^{\phi}\phi_X, \tag{8.1}$$

where the prime denotes the derivative with respect to ϕ.

It may be shown that[4] α, the ratio of the velocity of sound to the velocity of light, is given by the equation

$$\alpha^2=-\rho/\rho'. \tag{8.2}$$

We define the quantity

$$\psi=\int_{\rho_0}^{\rho}\alpha\frac{d\rho}{\rho}=-\int_0^{\phi}\left(-\frac{\rho'}{\rho}\right)^{\frac{1}{2}}d\phi. \tag{8.3}$$

Hence

$$\psi_X=-(-\rho'/\rho)^{\frac{1}{2}}\phi_X,\quad \psi_T=-(-\rho'/\rho)^{\frac{1}{2}}\phi_T. \tag{8.4}$$

Multiplying the first of Eqs. (8.1) by α, one obtains

$$\frac{e^{-\phi}}{c}\psi_T+\frac{\alpha\rho}{\rho_0}W_X=0. \tag{8.5}$$

The second of Eqs. (8.1) may be written as

$$\frac{e^{-\phi}}{c}W_T+\alpha\frac{\rho}{\rho_0}\psi_X=0. \tag{8.6}$$

Adding and subtracting these equations gives

$$\frac{e^{-\phi}}{c}r_T+\alpha\frac{\rho}{\rho_0}r_X=0,\quad \frac{e^{-\phi}}{c}s_T-\alpha\frac{\rho}{\rho_0}s_X=0, \tag{8.7}$$

where

$$r=\psi+W,\quad s=\psi-W. \tag{8.8}$$

Equations (8.7) are the characteristic form of the equations of hydrodynamics in terms of the Riemann functions r and s. They state that r is a constant along the curve given by

$$dX/dT=ce^{\phi}(\rho/\rho_0)\alpha, \tag{8.9}$$

and that s is constant along the curve

$$dX/dT=-ce^{\phi}(\rho/\rho_0)\alpha. \tag{8.10}$$

9. PROGRESSIVE WAVES

Solutions of Eqs. (8.7) which are such that either r or s is constant are called progressive waves. They propagate in one direction alone. We shall discuss the case

$$s=\psi_0=\text{constant}.$$

Then

$$W=\psi-\psi_0,$$

and hence

$$u=\tanh(\psi-\psi_0). \tag{9.1}$$

The first Eq. (8.7) may be written as

$$(1/c)\phi_T+\alpha e^{\phi}(\rho/\rho_0)\phi_X=0. \tag{9.2}$$

This equation has as its general solution

$$f(\phi)=X-c\alpha e^{\phi}(\rho/\rho_0)T, \tag{9.3}$$

where $f(\phi)$ is an arbitrary function. It may be determined from a boundary condition such as the third of Eqs. (7.2) by using Eq. (9.1).

Equation (9.2) states that ϕ is constant along straight lines in the X, T plane with slope

$$\Gamma=c\alpha e^{\phi}(\rho/\rho_0), \tag{9.4}$$

and hence this quantity is the velocity of propagation of ϕ. Depending on the nature of the function $f(\phi)$ this set of straight lines may or may not intersect in the region of interest in the X, T plane. When they do, as they will for a compressive motion of a piston into a compressible gas, a shock forms and the differential equations no longer can apply. These equations must be replaced by the Rankine-Hugoniot equations.[4]

10. COMPOUND WAVES

Solutions of Eqs. (8.7) for which neither r nor s are constant are called compound waves. Such solutions are most readily obtained by interchanging the role of independent and dependent variables in these equations.[5] This may be done since

$$J=r_Xs_T-r_Ts_X=2c\alpha e^{\phi}(\rho/\rho_0)s_Xr_X\neq 0, \tag{10.1}$$

by the definition of compound waves.

It is a consequence of the rules of partial differentiation, that

$$JX_r=s_T,\quad -JX_s=r_T,\quad -JT_r=s_X,\quad JT_s=r_X. \tag{10.2}$$

Hence Eqs. (8.7) are equivalent to

$$X_s-\Gamma T_s=0,\quad X_r+\Gamma T_r=0. \tag{10.3}$$

On adding and subtracting these we obtain, after using Eq. (8.8), the equations,

$$X_\psi+\Gamma T_W=0,\quad X_W+\Gamma T_\psi=0. \tag{10.4}$$

Since Γ defined by Eq. (9.4) is a function of ϕ alone and hence of ψ alone, the second of these equations states that there is a function of $Z(\psi,W)$ such that

$$T=Z_W=Z_r-Z_s,\quad X=-\Gamma Z_\psi=-\Gamma(Z_r+Z_s). \tag{10.5}$$

The first equation is then the linear equation

$$Z_{WW}-(1/\Gamma)(\Gamma Z_\psi)_\psi=0, \tag{10.6}$$

[4] A. H. Taub, Phys. Rev. **74**, 328 (1948), p. 332.
[5] A. H. Taub, Ann. Math. **47**, 811 (1946).

which may be written as

$$Z_{WW}-Z_{\psi\psi}-(\log\Gamma)_{\psi}Z_{\psi}=0,$$

or as

$$Z_{rs}+[(\log\Gamma)_r+(\log\Gamma)_s](Z_r+Z_s)=0. \quad (10.7)$$

Solutions of this linear equation define X and T as functions of r and s by Eqs. (10.5). On inverting these, that is solving for r and s as functions of X and T we obtain solutions of Eqs. (8.7), that is solutions of our original problem. We may obtain x and t as functions of ψ and W (hence of r and s) by noting that the line element of the Minkowski space may be written as

$$ds^2=e^{2\phi}(T_\psi d\psi+T_W dW)^2-\frac{1}{c^2}\left(\frac{\rho}{\rho_0}\right)^2(X_\psi d\psi+X_W dW)^2-\frac{1}{c^2}(dy^2+dz^2).$$

That is,

$$ds^2=g_1 d\psi^2+g_2 d\psi dW+g_3 dW^2-\frac{1}{c^2}(dy^2+dz^2),$$

where

$$g_1=e^{2\phi}T_\psi{}^2-\frac{1}{c^2}\left(\frac{\rho}{\rho_0}\right)^2X_\psi{}^2=e^{2\phi}(Z_{\psi W})^2-\frac{1}{c^2}\left(\frac{\rho}{\rho_0}\right)^2[(\Gamma Z_\psi)_\psi]^2,$$

$$g_2=2Z_W\left[e^{2\phi}Z_{WW}-\frac{1}{c^2}\left(\frac{\rho}{\rho_0}\right)^2\Gamma(\Gamma Z_\psi)_\psi\right],$$

$$g_3=e^{2\phi}(Z_{WW})^2-\frac{1}{c^2}\left(\frac{\rho}{\rho_0}\right)^2(\Gamma Z_{\psi W})^2.$$

The functions $x(\psi,W)$ and $t(\psi,W)$ define a transformation of coordinates in the Minkowski space which may be evaluated by solving equations of the form of Eqs. (6.13). This is equivalent to solving the equations

$$x_X=\rho_0/(1-u^2)^{\frac{1}{2}}, \quad x_T=cue^\phi/(1-u^2)^{\frac{1}{2}},$$

which follow from the results of Sec. 6.

11. FIELD EQUATIONS IN CO-MOVING COORDINATES

It has been shown[2] that in a space-time with plane-symmetry in which the gravitational field is created by a perfect fluid at constant entropy, we may introduce a co-moving coordinate system in which the line element has the form

$$ds^2=e^{2\phi}dT^2-\frac{1}{c^2}\left(\frac{\rho_0}{\rho}\right)^2e^{-4H}dX^2-\frac{1}{c^2}e^{2H}(dy^2+dz^2), \quad (11.1)$$

where ρ_0 may be taken to be a constant, ϕ is related to the pressure by Eq. (3.1) and hence ρ is a function of ϕ, and H is a function of X and T. The velocity vector of the fluid is

$$u^\mu=e^{-\phi}\delta_4{}^\mu. \quad (11.2)$$

The Einstein field equations, in this coordinate system, may be taken to be

$$2e^{-\phi}(H_Te^{-\phi})_T+3(e^{-\phi}H_T)^2-c^2\left(e^{2H}\frac{\rho}{\rho_0}H_X\right)\left(\frac{\rho}{\rho_0}e^{2H}H_X+2\frac{\rho}{\rho_0}e^{2H}\phi_X\right)=-kp, \quad (11.3)$$

$$e^{-\phi}H_T\left[3e^{-\phi}H_T+2e^{-\phi}\frac{\rho_T}{\rho}\right]+2c^2e^{2H}\frac{\rho}{\rho_0}\left(e^{2H}\frac{\rho}{\rho_0}H_X\right)_X+3c^2\left(e^{2H}\frac{\rho}{\rho_0}H_X\right)^2=-k(p+p')=-k(p-\sigma c^2), \quad (11.4)$$

and

$$\frac{e^\phi\rho_0}{\rho}\left((e^{3H})_T\frac{\rho}{\rho_0}e^{-\phi}\right)_X=(e^{3H})_T\frac{\rho_X}{\rho}-(e^{3H})_X\frac{\rho_T}{\rho}, \quad (11.5)$$

where k is given by Eq. (1.1) and the prime denotes the derivation of p with respect to ϕ.

Equations (11.3) to (11.5) are derived from the equations

$$R^{\mu\nu}-\tfrac{1}{2}g^{\mu\nu}R=-kc^2T^{\mu\nu} \quad (11.6)$$

by using the line element (11.1) and Eq. (11.2).

The problem of determining the gravitational field and one-dimensional isentropic motion of a perfect gas subject to certain boundary conditions, when formulated in terms of the co-moving coordinate system, is analogous to the problems formulated and discussed for such motions in special relativity. Thus the line element (11.1) and that given by (3.5) differ only in the presence of the function $H(X,T)$. This function is related to ϕ by the field equations. The conservation equations, Eqs. (4.1) and (4.2) are again automatically satisfied. We shall show that an equation which reduces to (5.1) in the limit $k=0$, $H=0$ is a consequence of the field equations, Eqs. (11.3) to (11.5).

The particle paths, Eqs. (4.3) with X fixed, provide a mapping from the line element given by equation (11.1) to another one for a *curved* space time. For problems in which the boundary conditions may be expressed in terms of co-moving coordinates, the determination of these curves is unnecessary. In some problems, stated in a coordinate system which differs from the co-moving one, it may be necessary to determine these curves.

12. INTEGRABILITY CONDITIONS FOR THE FIELD EQUATIONS

The system of field equations, Eqs. (11.3) through Eqs. (11.5) contain only the first derivations of the function ϕ and if ϕ is known, they presumably determine the function H. However, the function ϕ may not be chosen arbitrarily. To see how this comes about, we recast these equations into another form. Let us define

$$U(X,T)=ce^{2H}(\rho/\rho_0)H_X \tag{12.1}$$

and

$$V(X,T)=e^{-\phi}H_T. \tag{12.2}$$

Then since we must have

$$H_{XT}=H_{TX},$$

it follows that the functions U and V must satisfy the equation

$$e^{-\phi}U_T-ce^{2H}\frac{\rho}{\rho_0}V_X-e^{-\phi}\frac{\rho'}{\rho}U\phi_T-ce^{2H}\frac{\rho}{\rho_0}\phi_X V=2UV. \tag{12.3}$$

In terms of the variables U and V the system of equations (11.3) through (11.5) may be written as

$$2e^{-\phi}V_T=2cUe^{2H}\frac{\rho}{\rho_0}\phi_X-kp+U^2-3V^2, \tag{12.4}$$

$$2ce^{2H}\frac{\rho}{\rho_0}U_X=-2\frac{\rho'}{\rho}Ve^{-\phi}\phi_T+k(p+p')-3U^2-3V^2, \tag{12.5}$$

and

$$2e^{-\phi}U_T=2V\left(ce^{2H}\frac{\rho}{\rho_0}\phi_X-U\right), \tag{12.6}$$

respectively. Subtracting Eq. (12.6) from (12.3), one obtains

$$2ce^{2H}\frac{\rho}{\rho_0}V_X=-2U\left(\frac{\rho'}{\rho}\phi_Te^{-\phi}+3V\right). \tag{12.7}$$

Thus the system of field equations may be considered as a system of first order equations consisting of Eqs. (12.1) and (12.2) and (12.4) through (12.7) for the quantities H, U, and V. In order that they admit a solution it must be so that $V_{XT}=V_{TX}$ and $U_{XT}=U_{TX}$. These integrability conditions imply conditions on the function ϕ, which we shall now obtain.

We may solve Eq. (12.4) for V_T and differentiate the resulting expression with respect to X. After some manipulation involving the use of Eqs. (12.4) to (12.7), we obtain

$$V_{TX}=U\left[\left(ce^{2H}\frac{\rho}{\rho_0}e^{\phi}\phi_X\right)_X-Ue^{\phi}\phi_X+e^{\phi}U_X\right]-V\left(\frac{\rho'}{\rho}\phi_X\phi_T+3Ve^{\phi}\phi_X+3e^{\phi}V_X\right). \tag{12.8}$$

Similarly we may solve Eq. (12.7) for V_X and differentiate it with respect to T to obtain

$$V_{XT}=U\left\{\left[\left(e^{-2H}\frac{\rho_0}{\rho c}\right)_Te^{-\phi}\right]_T-\frac{e^{-2H}}{c}\frac{\rho_0}{\rho}(-e^{\phi}V^2+V_T)-2V\left(e^{-2H}\frac{\rho_0}{\rho c}\right)_T\right\}-V\left(\frac{\rho'}{\rho}\phi_X\phi_T+3e^{\phi}\phi_XV\right). \tag{12.9}$$

Subtracting Eq. (12.9) from (12.8), we obtain

$$U\left\{\left(ce^{2H}\frac{\rho}{\rho_0}e^{\phi}\phi_X\right)_X-\frac{1}{c}\left[\left(e^{-2H}\frac{\rho_0}{\rho}\right)_Te^{-\phi}\right]_T+e^{\phi-2H}\frac{\rho_0}{\rho c}(\tfrac{1}{2}kp'+V^2-U^2)\right\}=0. \tag{12.10}$$

Similarly the condition

$$U_{XT}=U_{TX},$$

may be shown to be equivalent to

$$V\left\{\left(ce^{2H}\frac{\rho}{\rho_0}e^{\phi}\phi_X\right)_X-\frac{1}{c}\left[\left(e^{-2H}\frac{\rho_0}{\rho}\right)_Te^{-\phi}\right]_T+e^{\phi-2H}\frac{\rho_0}{\rho c}(\tfrac{1}{2}kp'+V^2-U^2)\right\}=0. \tag{12.11}$$

Since U and V cannot both be zero when p and $p'\neq 0$ we must have as a condition on ϕ the equation

$$\Omega=\left(ce^{2H}\frac{\rho}{\rho_0}e^{\phi}\phi_X\right)_X-\frac{1}{c}\left[\left(e^{-2H}\frac{\rho_0}{\rho}\right)_Te^{-\phi}\right]_T+e^{\phi-2H}\frac{\rho_0}{\rho c}(\tfrac{1}{2}kp'+V^2-U^2)=0. \tag{12.12}$$

Note that when $k=0$ and $H=0$ the line element given by Eq. (11.1) becomes the same as that given by Eq. (3.5) and Eq. (12.12) is just Eq. (5.1), the condition that the space-time be flat.

13. EQUIVALENT FORM OF THE FIELD EQUATIONS

We have seen that the problem of solving the field equations is equivalent to solving the system of Eqs. (12.1), (12.2), (12.4) through (12.7) and for the functions H, ϕ, U, and V. In this section we show that Eqs. (12.4) through (12.7) may be written in another equivalent form.

If we multiply Eq. (12.4) by Ve^{ϕ} and Eq. (12.6) by $-Ue^{\phi}$ and add, we obtain

$$2(VV_T-UU_T)=-kpVe^{\phi}+3Ve^{\phi}(U^2-V^2).$$

On multiplying this equation by e^{3H}, we may write it as

$$[e^{3H}(V^2-U^2)]_T=-\tfrac{1}{3}kp(e^{3H})_T. \tag{13.1}$$

Similarly, if we multiply Eq. (12.5) by $-(1/c)e^{-2H}\times(\rho_0/\rho)U$ and multiply Eq. (12.7) by $(1/c)e^{-2H}(\rho_0/\rho)V$ and add, we obtain

$$[e^{3H}(V^2-U^2)]_X=-\tfrac{1}{3}k(p+p')(e^{3H})_X. \quad (13.2)$$

Equations (13.1) and (13.2) together with (12.1) and (12.2), the definitions of U and V, imply Eqs. (12.4) through (12.7) when $U\neq 0$ and $V\neq 0$. To see this, we note that it is a consequence of Eqs. (13.1) and (13.2) that

$$[p(e^{3H})_T]_X=[(p+p')(e^{3H})_X]_T.$$

That is,

$$p'(e^{3H})_{XT}-p'\phi_X(e^{3H})_T+(p+p')'\phi_T(e^{3H})_X=0.$$

However, it is a consequence of the definition of $p(\phi)$ [see Eqs. (2.7) and (3.1)] that

$$(p+p')'=p'\rho'/\rho. \quad (13.3)$$

Hence since $p'\neq 0$, we may write the above expression as

$$H_{XT}+3H_XH_T-\phi_XH_T+(\rho'/\rho)\phi_TH_X=0, \quad (13.4)$$

which is another form for Eq. (11.5); thus Eqs. (13.1) and (13.2) together with Eqs. (12.1) and (12.2) imply Eqs. (12.6) and (12.7).

If Eq. (12.6) is substituted into the equation preceding Eq. (13.1), we obtain

$$Ve^{\phi}[2e^{-\phi}V_T-2ce^{2H}(\rho/\rho_0)\phi_XU+kp-U^2+3V^2]=0.$$

When $V\neq 0$, this is Eq. (12.4). Similarly we may show that when $U\neq 0$ Eq. (13.2) implies Eq. (12.5).

14. APPROXIMATION PROCEDURE

We shall use an approximating procedure for solving the system of field equations, Eqs. (12.4) to (12.7), together with the defining Eqs. (12.1) and (12.2) and the integrability conditions, Eq. (12.12). The scheme we shall use is similar to that of Einstein, Hoffmann, and Infeld[6] and that of Einstein and Infeld[7] in discussing the motion of singularities in the gravitational field. However, it differs from the latter methods in that we use expansions of the gravitational field quantities in terms of the constant k instead of some velocity. This will free us of the necessity of assuming that time variations of the field quantities are small compared to spatial ones.

Expansions of the type used below have been used by McVittie[8] for determining integrals of the classical equations of hydrodynamics from solutions of the Einstein field equations. Thus McVittie has used the general formulation of relativistic hydrodynamics to obtain solutions of classical problems. The approach given below uses special relativistic solutions of hydrodynamical problems as the first terms in series expansions of general relativistic problems.

More precisely, our program is the following: we write

$$\phi=\phi^{(0)}+k\phi^{(1)}+\frac{k^2}{2!}\phi^{(2)}+\frac{k^3}{3!}\phi^{(3)}+\cdots=\sum_{i=0}^{\infty}\frac{k^i}{i!}\phi^{(i)}, \quad (14.1)$$

$$H=H^{(0)}+kH^{(1)}+\frac{k^2}{2!}H^{(2)}+\cdots=\sum_{i=0}^{\infty}\frac{k^i}{i!}H^{(i)}. \quad (14.2)$$

These expressions are substituted for ϕ and H in the equations listed in the above paragraph, and coefficients of like powers of k on both sides of each equation are equated. We then discuss solutions of the resulting system of equations.

It follows from Eqs. (14.1) and (14.2) that

$$\phi^{(i)}=(d^i\phi/dk^i)_{k=0} \quad (14.3)$$

and

$$H^{(i)}=(d^iH/dk^i)_{k=0}, \quad (14.4)$$

respectively. That is, the superscript (i) on each of these quantities represents the ith derivative of the quantity with respect to k evaluated at $k=0$.

If F is an analytic function of ϕ, H, and their first derivatives, we may write also:

$$F=\sum_{i=0}^{\infty}\frac{k^i}{i!}F^{(i)}, \quad (14.5)$$

and again

$$F^{(i)}=(d^iF/dk^i)_{k=0}. \quad (14.6)$$

The right-hand side of the last equation may be evaluated by using the rules for differentiating a function of a function. It then follows that the various sets of equations we must consider may be obtained by differentiating with respect to k both sides of Eqs. (12.4) to (12.7) an appropriate number of times and setting $k=0$. The equations resulting from differentiating n times with respect to k will be called the nth order equations.

15. ZERO-ORDER EQUATIONS

These equations are obtained by setting $k=0$ in the field equations and the defining equations. We then obtain

$$cH_X^{(0)}=(e^{-2H}\rho_0/\rho)^{(0)}U^{(0)}, \quad (15.1)$$

$$H_T^{(0)}=(e^{\phi})^{(0)}V^{(0)} \quad (15.2)$$

for Eqs. (12.1) and (12.2), respectively, where we use the notational device of denoting the rth derivative of a quantity with respect to k evaluated at $k=0$ by putting parentheses around the quantity and using a superscript $^{(r)}$. The set of Eqs. (12.4) to (12.7) inclusive becomes

$$2V_T^{(0)}=2cU^{(0)}(e^{2H}\rho/\rho_0)^{(0)}(e^{\phi})_X^{(0)}+(e^{\phi})^{(0)}(U^{(0)})^2-3(e^{\phi})^{(0)}(V^{(0)})^2, \quad (15.3)$$

$$2cV_X^{(0)}=-2(e^{-2H}\rho_0/\rho)^{(0)}U^{(0)}\times[(\rho'/\rho)\phi_Te^{-\phi}+3V]^{(0)}, \quad (15.4)$$

$$2U_T^{(0)}=2V^{(0)}[c(e^{2H}\rho/\rho_0)\phi_X-U]^{(0)}(e^{\phi})^{(0)}, \quad (15.5)$$

$$2cU_X^{(0)}=(e^{-2H}\rho_0/\rho)^{(0)}[2(\rho'/\rho)e^{-\phi}\phi_TV-3U^2-3V^2]^{(0)}. \quad (15.6)$$

[6] Einstein, Hoffmann, and Infeld, Ann. Math. 39, 65 (1938).
[7] A. Einstein and L. Infeld, Can. J. Math. 1, 209 (1949).
[8] G. C. McVittie, Quart. Appl. Math. 11, 327 (1953).

The integrability conditions of Eqs. (15.2) and (15.1) are satisfied as a consequence of (15.4) and (15.5).

Equations (15.3) to (15.6) may be derived from the equations

$$R^{\mu\nu}=0 \tag{15.7}$$

by using the coordinate system in which the line element is given by Eq. (11.1) with $\phi=\phi^{(0)}$ and $H=H^{(0)}$. This statement follows from the fact that the equations we are concerned with are a consequence of Eqs. (11.6) and the latter equations reduce to (15.7) when we set $k=0$. In general the zero-order equations are contained in the field equations for an empty space-time.

It is known[1] that plane-symmetric space-times satisfying (15.7) are one of two sorts: (1) those in which there exists a coordinate system in which the line element may be reduced to

$$ds^2=(a+bx)^{-\frac{1}{2}}\left(dt^2-\frac{1}{c^2}dx^2\right)-(a+bx)\frac{1}{c^2}(dy^2+dz^2), \tag{15.8}$$

and (2) those which are flat—that is, those for which the Riemann-Christoffel curvature tensor vanishes.

We shall assume as the zero-order solution of Eqs. (15.3) to (15.6), the solution of Eqs. (15.7) corresponding to the flat-space time. There seems to be no general justification for this assumption. In its favor, it may be said that the zero-order solution equivalent to the metric given by Eq. (15.8) has a singularity in it that makes it plausible that it corresponds to a gravitational field outside of matter. In the present discussion we are concerned with the gravitational field inside of matter and therefore discard the possibility represented by the metric given by Eq. (15.8).

It is evident from Eqs. (15.1) to (15.6) that

$$H^{(0)}=0 \tag{15.9}$$

is a solution of these equations. The argument given above shows that this is not the only solution. However, from the discussion given of special relativistic hydrodynamics it is clear that Eq. (15.8) is consistent with the assumption we are making, namely that the space-time determined by $H^{(0)}$ and $\phi^{(0)}$ is flat. The equation satisfied by $\phi^{(0)}$ will be determined from the first order equations.

It is a consequence of Eqs. (15.1), (15.2), and (15.9) that

$$U^{(0)}=V^{(0)}=0. \tag{15.10}$$

16. HIGHER ORDER EQUATIONS

By solving Eqs. (12.1) and (12.2) for H_X and H_T, differentiating the resulting equation n times with respect to k (where $n\geqslant 1$), setting $k=0$, and making use of Eqs. (15.8) and (15.9), we obtain

$$cH_X{}^{(n)}=\frac{\rho_0}{\rho^{(0)}}U^{(n)}+\sum_{r=1}^{n-1}C_{n,r}\left(\frac{\rho_0 e^{-2H}}{\rho}\right)^{(r)}U^{(n-r)}, \tag{16.1}$$

$$H_T{}^{(n)}=(e^{\phi})^{(0)}V^{(n)}+\sum_{r=1}^{n-1}C_{n,r}(e^{\phi})^{(r)}V^{(n-r)}, \tag{16.2}$$

where $C_{n,r}$ denote the binomial coefficients.

Similarly the system of Eqs. (12.4) to (12.7) lead to the equations

$$V_T{}^{(n)}=c[(e^{\phi})^{(0)}]_X\frac{\rho^{(0)}}{\rho_0}U^n+F_{1n}, \tag{16.3}$$

$$cV_X{}^{(n)}=-U^{(n)}\frac{\rho_0}{\rho^{(0)}}\left(\frac{\rho'}{\rho}\phi_T e^{-\phi}\right)^{(0)}+F_{2n}, \tag{16.4}$$

$$U_T{}^{(n)}=V^{(n)}\left(ce^{\phi}\frac{\rho}{\rho_0}\phi_X\right)^{(0)}+F_{3n}, \tag{16.5}$$

$$cU_X{}^{(n)}=-V^{(n)}\left(\frac{\rho_0\rho'}{\rho^2}e^{-\phi}\phi_T\right)^{(0)}+F_{4n}. \tag{16.6}$$

The functions $F_{\alpha n}$ ($\alpha=1, 2, 3, 4$) depend on $\phi^{(r)}$ and $H^{(r)}$ with $r=1, 2, \cdots, n-1$, and the derivatives of these functions with respect to X and T.

The explicit expressions for the $F_{\alpha n}$ are:

$$F_{1n}=c\sum_{r=1}^{n-1}C_{n,r}\left(\frac{\rho}{\rho_0}e^{2H+\phi}\phi_X\right)^{(r)}U^{(n-r)}-\frac{n}{2}(e^{\phi}p)^{(n-1)}+\sum_{r=1}^{n-1}C_{n,r}(e^{\phi})^{(r)}\left(\frac{U^2}{2}\right)^{(n-r)}-3\sum_{r=1}^{n-1}C_{n,r}(e^{\phi})^{(r)}\left(\frac{V^2}{2}\right)^{(n-r)}, \tag{16.7}$$

$$F_{2n}=-\sum_{r=1}^{n-1}C_{n,r}U^{(r)}\left(e^{-\phi}\frac{\rho'}{\rho}-e^{-2H}\frac{\rho_0}{\rho}\phi_T+3V\right)^{(n-r)}, \tag{16.8}$$

$$F_{3n}=\sum_{r=1}^{n-1}C_{n,r}V^{(r)}\left(ce^{2H}\frac{\rho}{\rho_0}e^{\phi}\phi_X-U\right)^{(n-r)}, \tag{16.9}$$

$$F_{4n}=\sum_{r=1}^{n-1}C_{n,r}\left(e^{-2H}\frac{\rho_0\rho'}{\rho^2}e^{-\phi}\phi_T\right)^{(r)}V^{(n-r)}+\frac{n}{2}\left((p+p')e^{-2H}\frac{\rho_0}{\rho}\right)^{(n-1)}-\tfrac{3}{2}\sum_{r=1}^{n-1}C_{n,r}\left(e^{-2H}\frac{\rho_0}{\rho}\right)^{(r)}(U^2+V^2)^{(n-r)}. \tag{16.10}$$

If f is any analytic function of H, ϕ and their derivatives with respect to X and T, we assume that we may represent it by the convergent power series

$$f=\sum_{i=0}^{\infty}\frac{k^i}{i!}f^{(i)}.$$

Hence

$$f_X=\sum_{i=0}^{\infty}\frac{k^i}{i!}f_X^{(i)}$$

and

$$(f_X)^{(n)}=\left(\frac{d^n}{dk^n}f_X\right)_{k=0}=(f^{(n)})_X.$$

Similarly

$$(f_T)^{(n)}=(f^{(n)})_T.$$

These observations may be used to study the integrability conditions of the pairs of equations (16.1) and (16.2), (16.3) and (16.4), and (16.5) and (16.6). Since $H_{XT}{}^{(n)}=(H_{XT})^{(n)}$ and $H_{TX}{}^{(n)}=(H_{TX})^n$, we have

$$H_{XT}{}^{(n)}-H_{TX}{}^{(n)}=(H_{XT}-H_{TX})^{(n)}.$$

Hence the integrability condition of Eqs. (16.1) and (16.2) is satisfied if the nth derivative with respect to k evaluated at $k=0$ of the integrability condition of Eqs. (12.1) and (12.2) is satisfied, that is, if such an nth derivative of Eq. (12.3) is satisfied. However, this condition is satisfied for it is merely the difference of Eqs. (16.4) and (16.5).

Similarly the integrability condition of Eqs. (16.3) and (16.4) is given by

$$\sum_{r=0}^{n}C_{n,r}U^{(r)}\Omega^{(n-r)}=0, \tag{16.11}$$

and that of Eqs. (16.5) and (16.6) is given by

$$\sum_{r=0}^{n}C_{n,r}V^{(r)}\Omega^{(n-r)}=0, \tag{16.12}$$

where Ω is defined by Eq. (12.12).

If $n=1$, Eqs. (16.11) and (16.12) together with (15.10) imply that

$$\Omega^{(0)}=0. \tag{16.13}$$

If Eqs. (16.11) and (16.12) are satisfied from $n=1, 2, \cdots, m$, then we must have

$$\Omega^{(n)}=0,\quad n=0, 1, \cdots m-1. \tag{16.14}$$

When the functions $H^{(0)}=0$, $H^{(1)}$, $\cdots H^{(m-1)}$, and the functions $\phi^{(0)}$, $\phi^{(1)}$, $\cdots\phi^{(m-2)}$ are known, we may determine $\phi^{(m-1)}$ by solving Eq. (16.14) with $n=m-1$. Then the quantities $F_{\alpha n}$ in Eqs. (16.3) to (16.6) are known as well as the coefficients of $U^{(n)}$ and $V^{(n)}$ appearing in these equations. These are nonhomogeneous linear first order partial differential equations. The only nonlinear equation we have to deal with is Eq. (16.13) which, as was pointed out earlier, is Eq. (5.1), the equation which holds in special relativity for the determination of the function ϕ and whose solution was discussed in the earlier part of this paper. Thus the nonlinearity of the field equations has been concentrated in the zero-order equation for ϕ, that is, in the special relativistic approximation to the problem.

17. FIRST-ORDER EQUATIONS

On setting $n=1$ in the above equations, we obtain for Eqs. (16.1) and (16.2) the equations

$$cH_X{}^{(1)}=U^{(1)}\rho_0/\rho^{(0)}, \tag{17.1}$$

and

$$H_T{}^{(1)}=(e^{\phi})^{(0)}V^{(1)}, \tag{17.2}$$

respectively. Equations (16.3) to (16.6) become

$$V_T{}^{(1)}=c[(e^{\phi})^{(0)}]_X(\rho^{(0)}/\rho_0)U^{(1)}-\tfrac{1}{2}(e^{\phi})^{(0)}p^{(0)}, \tag{17.3}$$

$$cV_X{}^{(1)}=U^{(1)}(e^{-\phi})^{(0)}(\rho_0/\rho^{(0)})_T, \tag{17.4}$$

$$U_T{}^{(1)}=V^{(1)}c(e^{\phi})^{(0)}(\rho^{(0)}/\rho_0)\phi_X{}^{(0)}, \tag{17.5}$$

$$cU_X{}^{(1)}=V^{(1)}(\rho_0/\rho^{(0)})_T(e^{-\phi})^{(0)}+\tfrac{1}{2}(p^{(0)}+p'^{(0)})\rho_0/\rho^{(0)}, \tag{17.6}$$

respectively.

It may be verified directly that the integrability conditions of Eqs. (17.1) and (17.2) are satisfied as a consequence of Eqs. (17.4) and (17.5). The integrability conditions of the pair of equations (17.3) and (17.4) and the pair of equations (17.5) and (17.6) are satisfied as a consequence of Eq. (16.13) which is the same as Eq. (5.1), namely

$$c[(e^{\phi})_X\rho/\rho_0]_X{}^{(0)}-[e^{-\phi}(\rho_0/\rho)_T]_T{}^{(0)}=0. \tag{17.7}$$

The function $\phi^{(1)}$ may be determined by Eq. (16.14) with $n=1$, that is, by the equation

$$c^2\left[\left(\frac{\rho}{\rho_0}e^{\phi}\right)^{(0)}\phi^{(1)}\right]_{XX}+\left[\left(e^{-\phi}\frac{\rho_0\rho'}{\rho^2}\right)^{(0)}\phi^{(1)}\right]_{TT}$$
$$+2c^2\left[\left(\frac{\rho}{\rho_0}e^{\phi}\phi_X\right)^{(0)}H^{(1)}\right]_X+2\left[\left[(H^{(1)}\frac{\rho_0}{\rho^{(0)}}\right]_T(e^{-\phi})^{(0)}\right]_T$$
$$+\left(e^{\phi}\frac{\rho_0}{2\rho}p'\right)^{(0)}=0. \tag{17.8}$$

If we write

$$\Phi^{(1)}=\left(\frac{\rho}{\rho_0}\right)^{(0)}\frac{(e^{\phi})^{(0)}}{\Gamma}\phi^{(1)},$$

we may write Eq. (17.8) as

$$[\Gamma\Phi^{(1)}]_{XX}-[\Phi^{(1)}/\Gamma]_{TT}+D^{(1)}=0, \tag{17.9}$$

where $D^{(1)}$ is given in terms of $H^{(1)}$ and $\phi^{(0)}$ by the sum of the last three terms of Eq. (17.8) divided by c^2 and

$$\Gamma=\left[ce^{\phi}\frac{\rho}{\rho_0}\left(\frac{-\rho}{\rho'}\right)^{\frac{1}{2}}\right]^{(0)}, \tag{17.10}$$

and is the value obtained from Eq. (9.4) by setting $\phi=\phi^{(0)}$ in that equation. The quantity Γ is therefore the special theory of relativity velocity of sound in orthogonal Lagrangian coordinates.

Methods for dealing with Eq. (17.7) were described earlier. In discussing the solution of Eqs. (17.3) to (17.7) it is convenient to introduce the characteristic curves in the X, T plane (see Sec. 8). We define the curves of parameter α as the solutions of the ordinary differential equation, Eq. (8.9). Thus

$$X_\alpha-\Gamma T_\alpha=0. \tag{17.11}$$

Similarly we define the curves of parameter β as the solutions of

$$X_\beta+\Gamma T_\beta=0. \tag{17.12}$$

Then, for any function $f(X,T)$, we have

$$f_\alpha=f_T T_\alpha+f_X X_\alpha=T_\alpha(f_T+\Gamma f_X),$$

and

$$f_\beta=f_T T_\beta+f_X X_\beta=T_\beta(f_T-\Gamma f_X).$$

Equations (17.1) and (17.2) are then equivalent to the equations

$$H_\alpha^{(1)}=(e^\phi)^{(0)}T_\alpha\{V^{(1)}+[(-\rho/\rho')^{\frac{1}{2}}]^{(0)}U^{(1)}\}. \tag{17.13}$$

$$H_\beta^{(1)}=(e^\phi)^{(0)}T_\beta[V^{(1)}-[(-\rho/\rho')^{\frac{1}{2}}]^{(0)}U^{(1)}]. \tag{17.14}$$

Similarly Eqs. (17.3) to (17.6) become

$$V_\alpha^{(1)}=-U^{(1)}\psi_\alpha^{(0)}-\tfrac{1}{2}(e^\phi p)^{(0)}T_\alpha, \tag{17.15}$$

$$V_\beta^{(1)}=U^{(1)}\psi_\beta^{(0)}-\tfrac{1}{2}(e^\phi p)^{(0)}T_\beta, \tag{17.16}$$

$$U_\alpha^{(1)}=-V^{(1)}\psi_\alpha^{(0)}+\tfrac{1}{2}[(-\rho/\rho')^{\frac{1}{2}}e^\phi(p+p')]^{(0)}T_\alpha, \tag{17.17}$$

and

$$U_\beta^{(1)}=V^{(1)}\psi_\beta^{(0)}-\tfrac{1}{2}[(-\rho/\rho')^{\frac{1}{2}}e^\phi(p+p')]^{(0)}T_\beta, \tag{17.18}$$

respectively, where ψ is defined as a function of ϕ by Eq. (8.3). By adding and subtracting Eqs. (17.15) and (17.17), we obtain

$$[(e^\psi)^{(0)}(V^{(1)}+U^{(1)})]_\alpha =\tfrac{1}{2}T_\alpha\{[(-\rho/\rho')^{\frac{1}{2}}(p+p')-p]e^{\psi+\phi}\}^{(0)}, \tag{17.19}$$

and

$$[(e^{-\psi})^{(0)}(V^{(1)}-U^{(1)})]_\alpha =-\tfrac{1}{2}T_\alpha\{[(-\rho/\rho')^{\frac{1}{2}}(p+p')+p]e^{-\psi+\phi}\}^{(0)}, \tag{17.20}$$

respectively.

Similarly, by adding and subtracting Eqs. (17.16) and (17.18),

$$[(e^{-\psi})^{(0)}(V^{(1)}+U^{(1)})]_\beta =-\tfrac{1}{2}T_\beta\{[(-\rho/\rho')^{\frac{1}{2}}(p+p')+p]e^{-\psi+\phi}\}^{(0)}, \tag{17.21}$$

and

$$[(e^\psi)^{(0)}(V^{(1)}-U^{(1)})]_\beta =\tfrac{1}{2}T_\beta\{[(-\rho/\rho')^{\frac{1}{2}}(p+p')-p]e^{\psi+\phi}\}^{(0)}, \tag{17.22}$$

respectively. Equations (17.19) to (17.22) are of course equivalent to Eqs. (17.3) to (17.6). Hence the solutions of the latter may be determined by quadratures when the function $\phi^{(0)}(X,T)$ or $\phi^{(0)}(\alpha,\beta)$ is determined by the methods discussed in the earlier part of the paper.

Thus, for example, if $\phi^{(0)}$ is such that it describes a progressive wave (see Sec. 9), then it follows from Eq. (9.2) that

$$\phi_\alpha^{(0)}=0.$$

Thus $\phi^{(0)}$ and hence $\psi^{(0)}$, $\rho^{(0)}$, and $p^{(0)}$ are functions of β alone. Equations (17.19) and (17.20) may then be integrated immediately to give

$$V^{(1)}+U^{(1)} =\tfrac{1}{2}T\{[(-\rho/\rho')^{\frac{1}{2}}(p+p')-p]e^\phi\}^{(0)}+A(\beta), \tag{17.23}$$

$$V^{(1)}-U^{(1)} =-\tfrac{1}{2}T\{[(-\rho/\rho')^{\frac{1}{2}}(p+p')+p]e^\phi\}^{(0)}+B(\beta), \tag{17.24}$$

where $A(\beta)$ and $B(\beta)$ are two functions of the variable β which must be chosen so that Eqs. (17.21) and (17.22) are satisfied. Substituting from Eqs. (17.23) and (17.24) into the latter equations we obtain first order ordinary differential equations for $A(\beta)$ and $B(\beta)$. These are

$$(TAe^{-\psi})_\beta=-T_\beta\{[(-\rho/\rho')^{\frac{1}{2}}(p+p')+p]e^{-\psi+\phi}\}^{(0)} -\{T[(-\rho/\rho')^{\frac{1}{2}}(p+p')-p]e^{-\psi+\phi}\}_\beta^{(0)}, \tag{17.25}$$

and

$$(TBe^{\psi})_\beta=T_\beta\{[(-\rho/\rho')^{\frac{1}{2}}(p+p')-p]e^{\psi+\phi}\}^{(0)} +\{T[(-\rho/\rho')^{\frac{1}{2}}(p+p')+p]e^{\psi+\phi}\}_\beta^{(0)}. \tag{17.26}$$

The solutions of these equations depend on a single arbitrary constant.

It follows from Eqs. (17.23) and (17.24) that

$$V^{(1)}=-\tfrac{1}{2}Tpe^\phi+\tfrac{1}{2}[A(\beta)+B(\beta)], \tag{17.27}$$

$$U^{(1)}=\tfrac{1}{2}T(-\rho/\rho')^{\frac{1}{2}}(p+p')e^\phi+\tfrac{1}{2}[A(\beta)-B(\beta)]. \tag{17.28}$$

When these quantities are substituted with Eqs. (17.13) and (17.14), we obtain equations for the determination of $H_\alpha^{(1)}$ and $H_\beta^{(1)}$, namely

$$H_\alpha^{(1)}=\tfrac{1}{2}e^{2\phi}TT_\alpha[p-(\rho/\rho')(p+p')] +\tfrac{1}{2}A(\beta)[1+(-\rho/\rho')^{\frac{1}{2}}e^\phi T_\alpha +\tfrac{1}{2}B(\beta)[1-(-\rho/\rho')^{\frac{1}{2}}]e^\phi T_\alpha, \tag{17.29}$$

and

$$H_\beta^{(1)}=\tfrac{1}{2}e^{2\phi}TT_\beta[p+(\rho/\rho')(p+p')] +\tfrac{1}{2}A(\beta)[1-(-\rho/\rho')^{\frac{1}{2}}e^\phi T_\beta +\tfrac{1}{2}B(\beta)[1+(-\rho/\rho')^{\frac{1}{2}}]e^\phi T_\beta. \tag{17.30}$$

It is evident from the last two equations that even if ϕ is characterized as a progressive wave, that is, even if $\phi=\phi(\beta)$, $H^{(1)}$ is a compound wave, that is, a function of α and β. Hence the determination of $V^{(n)}$, $U^{(n)}$, and $H^{(n)}$ ($n\geqslant 1$) involves the solution of equations of the form of (17.19) to (17.22) where the right hand sides are known functions of both α and β.

18. DETERMINATION OF $\phi^{(1)}$

Equation (17.9) is equivalent to the statement that there exists a function $\Psi^{(1)}$ such that

$$\Psi_T^{(1)}=(\Gamma\Phi^{(1)})_X+\mu(X,T), \tag{18.1}$$

$$\Psi_X^{(1)}=(\Phi^{(1)}/\Gamma)_T+\nu(X,T), \tag{18.2}$$

where the functions μ and ν are given by

$$\mu=2H^{(1)}\left[\frac{\rho}{\rho_0}(e^{\phi})_X\right]^{(0)}+\frac{K}{4c^2}X, \tag{18.3}$$

$$\nu=-\frac{2}{c^2}H^{(1)}\left[\left(\frac{\rho_0}{\rho}\right)_T e^{-\phi}\right]^{(0)}-\frac{K}{4c^2}T, \tag{18.4}$$

where

$$K=[e^{\phi}(\rho_0/\rho)p']^{(0)}, \tag{18.5}$$

and it follows from Eqs. (2.7) and (3.1) that K is a constant. For, the integrability condition of Eqs. (18.1) and (18.2) is just Eq. (17.9).

If we multiply Eq. (18.2) by $+\Gamma$ and $-\Gamma$, and add the ensuing equations to (18.1), we obtain

$$(\Psi^{(1)}-\Phi^{(1)})_T+\Gamma(\Psi^{(1)}-\Phi^{(1)})_X=-(\Gamma_T-\Gamma\Gamma_X)\frac{\Phi^{(1)}}{\Gamma}+\mu+\Gamma\nu,$$

$$(\Psi^{(1)}+\Phi^{(1)})_T-\Gamma(\Psi^{(1)}+\Phi^{(1)})_X=(\Gamma_T+\Gamma\Gamma_X)\frac{\Phi^{(1)}}{\Gamma}+\mu-\Gamma\nu,$$

or

$$(\Psi^{(1)}-\Phi^{(1)})_\alpha=-\frac{\Gamma_\beta}{\Gamma}\frac{T_\alpha}{T_\beta}\Phi^{(1)}+(\mu+\Gamma\nu)T_\alpha$$
$$=-\frac{\Gamma_\beta}{\Gamma}\frac{T_\alpha}{T_\beta}\Phi^{(1)}-\frac{2H^{(1)}}{c}\psi_\beta{}^{(0)}\frac{T_\alpha}{T_\beta}+\frac{K}{4c^2}(X-\Gamma T)T_\alpha, \tag{18.6}$$

$$(\Psi^{(1)}+\Phi^{(1)})_\beta=\frac{\Gamma_\alpha}{\Gamma}\frac{T_\beta}{T_\alpha}\Phi^{(1)}+(\mu-\Gamma\nu)T_\beta$$
$$=\frac{\Gamma_\alpha}{\Gamma}\frac{T_\beta}{T_\alpha}\Phi^{(1)}-\frac{2H^{(1)}}{c}\psi_\alpha{}^{(0)}\frac{T_\beta}{T_\alpha}+\frac{K}{4c^2}(X+\Gamma T)T_\beta. \tag{18.7}$$

These equations determine the change in $\Psi^{(1)}-\Phi^{(1)}$ along the characteristic curve (17.11) and the change in $\Psi^{(1)}+\Phi^{(1)}$ along the other characteristic. Hence the value of $\Psi^{(1)}-\Phi^{(1)}$ at a point α,β in the characteristic plane may be determined from its value at α_0,β. Similarly the value of $\Psi^{(1)}+\Phi^{(1)}$ at (α,β) may be determined from its value at α,β_0. We may then determine $\Psi^{(1)}$ and $\Phi^{(1)}$ at α,β in terms of the values of $\Psi^{(1)}-\Phi^{(1)}$ at α_0,β and $\Psi^{(1)}+\Phi^{(1)}$ at α,β_0.

In the X, T plane we may determine $\Phi^{(1)}$ and $\Psi^{(1)}$ at a point X, T as follows. Through this point draw the two characteristics, the characteristic of parameter α and the characteristic of parameter β. On the first of these characteristics find a point where $\Psi^{(1)}-\Phi^{(1)}$ is known. From this value and Eq. (18.6) compute the value of $\Psi^{(1)}-\Phi^{(1)}$ at X, T. Similarly on the second characteristic find a point where $\Psi^{(1)}+\Phi^{(1)}$ is known, and from this value and Eq. (18.7) compute $\Psi^{(1)}+\Phi^{(1)}$ at X, T.

The methods employed for discussing the first order equations hold for the nth-order ones equally well. The equations for $U^{(n)}$ and $V^{(n)}$ in terms of the characteristic parameters are of the form of (17.19) to (17.22) with the right-hand sides known as functions of α and β in virtue of the determination of $U^{(m)}$, $V^{(m)}$, $H^{(m)}$, and $\phi^{(m)}$ for $m=0, 1, \cdots n-1$ by methods similar to those discussed above.

19. PARTICLE PATHS

The preceding discussion has treated the determination of the gravitational field, the pressure distribution, and the density distribution, for the isentropic motion of a perfect fluid in its own gravitational field in a particular coordinate system—the co-moving one. In such a coordinate system the world-line of any particle of the fluid has as its tangent vector the four-velocity vector

$$u^\mu=e^{-\phi}\delta_4{}^\mu.$$

We may obtain the solution of the same problem in any other coordinate system by using the tensor transformation laws and the equations of transformation

$$x=x(X,T),\quad t=t(X,T), \tag{19.1}$$

once these functions are known. It is the purpose of this section to discuss the determination of these functions for the case where the x, t coordinates are such that the line element is given by

$$ds^2=e^{2F}\left(dt^2-\frac{1}{c^2}dx^2\right)-\frac{e^{2H}}{c^2}(dy^2+dz^2). \tag{19.2}$$

Every line element in a space-time with plane symmetry may be reduced to this form.

Consider the function $x(X,T)$ satisfying the equation

$$\left(e^{-2H-\phi}\frac{\rho_0}{\rho}x_T\right)_T-c^2\left(e^{\phi+2H}\frac{\rho}{\rho_0}x_X\right)_X=0. \tag{19.3}$$

Then there exists a function $t(X,T)$ such that

$$t_T=e^{\phi+2H}\frac{\rho}{\rho_0}x_X,\quad t_X=\frac{1}{c^2}e^{-2H-\phi}\frac{\rho_0}{\rho}x_T. \tag{19.4}$$

It follows that $t(X,T)$ also satisfies the differential equation (19.3).

If we consider the functions x and t satisfying (19.3) and (19.4) as defining the transformation (19.1), the metric tensor in the new coordinate system is given by

$$g^{44*}=g^{44}t_T^2+g^{11}t_X^2=e^{4H}\frac{\rho^2}{\rho_0^2}x_X^2-\frac{e^{-2\phi}}{c^2}x_T^2=e^{-2F},$$

$$g^{11*}=g^{44}x_T^2+g^{11}x_X^2=-c^2e^{4H}\frac{\rho^2}{\rho_0^2}x_X^2+e^{-2\phi}x_T^2=-c^2e^{-2F},$$

$$g^{14*}=g^{44}x_Tt_T+g^{11}x_Xt_X=\left(ce^{-2\phi}e^{\phi+2H}\frac{\rho}{\rho_0}-c^2e^{+4H}\frac{\rho^2}{\rho_0^2}\frac{1}{c}e^{-2H-\phi}\frac{\rho_0}{\rho}\right)x_Tx_X=0. \tag{19.5}$$

Hence in the x, t coordinate system the line element of space-time is given by Eq. (19.2), where the function F is defined by the first of Eqs. (19.5).

By adding and subtracting Eq. (19.4), we obtain

$$\begin{aligned}(t_T+x_T)-ce^{\phi+2H}(\rho/\rho_0)(t_X+x_X)=0,\\(t_T-x_T)+ce^{\phi+2H}(\rho/\rho_0)(t_X-x_X)=0.\end{aligned} \tag{19.6}$$

The first of these equations states that $t+x$ is constant along the curve in the X, T plane given by

$$dX/dT=-ce^{\phi+2H}\rho/\rho_0, \tag{19.7}$$

and the second states that $t-x$ is constant along the curve in the X, T plane given by

$$dX/dT=ce^{\phi+2H}\rho/\rho_0. \tag{19.8}$$

Equations (19.7) and (19.8) describe the curves in which the X, T plane intersect the light-cone of the space-time.

If $x(0,T)$ and $t(0,T)$ are known as well as $x(X,0)$, $t(X,0)$, then we may determine $x(X,T)$ and $t(X,T)$ at any point of the X, T plane as follows. Through this point of the plane there exists an integral curve of the family defined by Eq. (19.7) which intersects the curve $T=0$, say at the point X_0. Then

$$t(X,T)+x(X,T)=t(X_0,0)+x(X_0,0). \tag{19.9}$$

Similarly through X, T there is a curve of the family given by Eq. (19.9) which intersects the curve $X=0$ in the point $(0,T_0)$. Then

$$t(X,T)-x(X,T)=t(0,T_0)+x(0,T_0). \tag{19.10}$$

Hence $x(X,T)$ and $t(X,T)$ are determined by the values of these functions along the curves $T=0$ and $X=0$, provided Eqs. (19.7) and (19.8) can be integrated and the quantities X_0 and T_0 detemined as indicated above.

20. FUNCTION $u(X,T)$ IN GENERAL RELATIVITY

Just as in special relativity (see Sec. 6), we may interpret $u(X,T)$ defined by the equation

$$cu=x_T/t_T \tag{20.1}$$

as the three-dimensional velocity of the particle X at time T. In terms of u, we define the function

$$W=\tfrac{1}{2}\ln\left(\frac{1+u}{1-u}\right)=\tfrac{1}{2}\ln\left(\frac{ct_T+x_T}{ct_T-x_T}\right). \tag{20.2}$$

Hence

$$W_X=(-cx_Tt_{XT}+ct_Tx_{XT})/(c^2t_T^2-x_T^2).$$

However, it follows from Eqs. (19.4) and (19.5) that

$$c^2t_T^2-x_T^2=c^2e^{2\phi+4H}(\rho^2/\rho_0^2)x_X^2-x_T^2=c^2e^{2\phi-2F},$$

and that

$$\begin{aligned}-cx_Tt_{XT}+ct_Tx_{XT}&=-(1/c)x_T[e^{-2H-\phi}(\rho_0/\rho)x_T]_T+ce^{\phi+2H}(\rho/\rho_0)x_Xx_{XT}\\&=ce^{\phi-F}(e^{-2H-F}\rho_0/\rho)_T.\end{aligned}$$

Therefore

$$W_X=\frac{e^{F-\phi}}{c}\left(e^{-2H-F}\frac{\rho_0}{\rho}\right)_T=\frac{e^{-\phi}}{c}\left[\left(e^{-2H}\frac{\rho_0}{\rho}\right)_T-e^{-2H}\frac{\rho_0}{\rho}F_T\right]. \tag{20.3}$$

Similarly we may derive the equation,

$$W_T=c(e^{\phi-F})_Xe^{F+2H}\rho/\rho_0=ce^{\phi+2H}(\rho/\rho_0)(\phi_X-F_X). \tag{20.4}$$

Equations (20.3) and (20.4) are the general relativity theory analogs of Eq. (5.3) of the special theory of relativity. The former equations reduce to the latter when $H=F=0$. The former equations may be interpreted as the equations of motion of the fluid in terms of the orthogonal Lagrangian coordinates X, T. In this interpretation, the terms involving H and F are attributed to the effects of the gravitational field. It follows from Eqs. (19.5) and (19.4) that

$$e^{-2F}=c^2(1-u^2)e^{4H}(\rho^2/\rho_0^2)x_X^2. \tag{20.5}$$

This equation reduces to Eq. (6.9) the special theory of relativity form of the conservation of mass when $H=F=0$.

When the approximation method discussed in earlier sections is used to obtain approximate functions to represent ϕ, H, and ρ, these approximations may be used in Eqs. (19.3) and (19.4) to define approximate values of the functions $x(X,T)$ and $T(X,T)$. In terms of the latter (approximate) functions, we may define an (approximate) three-dimensional velocity field $u(X,T)$ and even $u(x,t)$, where in the latter expression we use the inverse transformation to that defined by Eqs. (19.1) to determine $X(x,t)$ and $T(x,t)$.

21. BOUNDARY CONDITIONS

As pointed out in Sec. 11, in general relativity, just as in special relativity, the solution of a particular physical problem is determined by supplementing the field equations with specific boundary conditions. That is, by specifying the values of various functions on certain surfaces in space-time. It is the purpose of this section to illustrate by means of an example the determination of the boundary data and in the next section to describe their use in determining solutions of the field equations for a specific problem.

The example we shall consider is the analog of the isentropic motion of a gas in a tube set into motion by moving a piston (see Sec. 7). The tube filled with gas will be replaced by a region of space-time occupied by a free surface on which the pressure vanishes; that is,

$$p(\phi)=0. \tag{21.1}$$

Outside of this surface the space-time is empty.

The condition requiring the tube and gas to be initially at rest and at constant pressure, density, and entropy is replaced by the condition that the gas be at constant entropy and in equilibrium under the hydrodynamic and gravitational forces acting.

It has been shown in an earlier paper[2] that under these conditions there exists a co-moving coordinate system in which the line element is of the form given by Eqs. (11.1) and in which the functions H and ϕ are independent of T. Methods for determining the functions H and ϕ as functions of X for different gases, each of which is described by a specific caloric equation of state, $\epsilon=\epsilon(p,\rho)$ were described in that paper.

Thus, the assumption of equilibrium is equivalent to assuming a knowledge of $H(X,T)=H_e(X)$ and $\phi(X,T)=\phi_e(X)$ for $T\leq T_0$, where T_0 is a time at which we shall assume that the equilibrium is disturbed in the following way: We shall assume that $\phi(X,T)$ departs from its equilibrium value for $X=X_0$ and all $T>T_0$. Thus, we specify the function

$$\phi(X_0,T).$$

In the special theory of relativity, this is equivalent to prescribing a piston motion. For moving a piston into a gas at rest sends a progressive wave into the gas and hence is determined at the piston via Eq. (9.1). The specification of the function (X_0,T) is also equivalent to giving the pressure at the "particle X_0" as a function of time since, for isentropic motion,

$$p(X_0,T)=p(\phi(X_0,T)).$$

Thus, in the example we are considering, we have the following boundary data: On the surface $T=T_0$,

$$\phi(X,T)=\phi_e(X),\quad H(X,T)=H_e(X), \tag{21.2}$$

where $\phi_e(X)$ and $H_e(X)$ are the equilibrium solutions obtained in the earlier paper.[1] On the surfaces $X=X_0$,

$$\phi=\phi(X_0,T); \tag{21.3}$$

and on the surface $X=X_b>X_0$, which is the boundary between the fluid and empty space-time,

$$\phi=\phi_0,$$

where ϕ_0 is a constant such that

$$p(\phi_0)=0. \tag{21.4}$$

We shall further assume, as is generally done in the general theory of relativity, that across any surface the metric tensor is continuous as are its derivatives. This assumption together with Eqs. (21.2) implies that on $T=T_0$

$$\phi_T=0 \quad\text{and}\quad H_T=0. \tag{21.5}$$

Equation (21.5) will be consistent with Eq. (21.3) if and only if

$$\partial\phi(X_0,T)/\partial T=0. \tag{21.6}$$

We shall assume that Eq. (21.6) holds.

With the above sequence of assumptions, we have specified the values of ϕ, H, U, and V on the surface $T=T_0$. In particular, we have specified the values of $\phi^{(i)}$, $H^{(i)}$, $U^{(i)}$, and $V^{(i)}$ on this surface for X in the range

$$X_0\leq X\leq X_b, \tag{21.7}$$

In addition, we have specified $\phi^{(i)}(X_0,T)$ and $\phi^{(i)}(X_b,T)$.

22. METHOD OF SOLUTION

The function $\phi^{(0)}(X,T)$ is determined as is the function $\phi(X,T)$ in the special theory of relativity. That is, in the region of the X, T plane between the line $T=0$ and the line $X=\Gamma T$, where Γ is defined by Eq. (17.10), we have

$$\phi^{(0)}(X,T)=\phi_e^{(0)}(X). \tag{22.1}$$

This region will be called region I.

We define region II as follows: It is bounded by the lines

$$X=\Gamma T, \tag{22.2}$$

$$X=X_0, \tag{22.3}$$

and the characteristic defined by the equation

$$dX/dT=-\Gamma \tag{22.4}$$

which passes through the point

$$(X_b,X_b/\Gamma).$$

In the region I the functions $\phi^{(0)}$, $H^{(1)}$, and $\phi^{(1)}$ are given by

$$\phi^{(0)}=\phi_e^{(0)}(X),\quad H^{(1)}=H_e^{(1)}(X),\quad \phi^{(1)}=\phi_e^{(1)}(X),$$

since these functions satisfy the field equations and the boundary conditions.

In region II, $\phi^{(0)}$ is determined by Eq. (9.3) in which $\phi=\phi^{(0)}$ and in which $f(\phi)$ is chosen so that $\phi^{(0)}(X_0,T)$ is the prescribed function. Thus, in region II, $\phi^{(0)}$ is constant along one family of characteristics which is the family of straight lines (22.2), which have varying slopes. The second family of characteristics, the curves defined by Eqs. (22.4), can now be determined.

In region III, the remainder of the X, T plane between the lines $X=X_0$ and $X=X_b$, $\phi^{(0)}$ is given by a compound wave which may be determined by the methods described in Sec. 10.

When $\phi^{(0)}(X,T)$ is determined, the characteristic curves described by Eqs. (17.11) and (17.12) are determined. As is evident from the discussion in Secs. 17 and 18, these curves play an important role in the determination of $U^{(1)}$, $V^{(1)}$, $H^{(1)}$, and $\phi^{(1)}$. We shall illustrate how the characteristics may be used to determine these quantities at a point X, T in region II. Through this point, the characteristic of parameter α is a straight line

$$X=\Gamma T,$$

and the curve of parameter β is the solution of equation (22.4). We shall assume that the data given on the curve $X=X_0$ are such that there is one characteristic of each family through the point X, T. This assumption will be examined in some detail in the next section.

A curve of parameter β through X, T will intersect the curve $T=0$ at a point which we shall call $X^{(\beta)}(X,T)$. Along this curve the variation of $U^{(1)}$ and $V^{(1)}$ may be determined from Eqs. (17.21) and (17.22). Hence, we may determine $V^{(1)}$ and $V^{(1)}$ by integrating these equations along the (known) curve of parameter β and by using as initial conditions the known values of $U^{(1)}$ and $V^{(1)}$ at $X=X^{(\beta)}$ and $T=0$.

Similarly, by integrating Eq. (17.14) along the same curve of parameter β and using the (known) value of H at $X=X^{(\beta)}$ and $T=0$, we may determine $H^{(1)}(X,T)$.

The value of $\phi^{(1)}$ at X, T may be determined from Eqs. (18.6) and (18.7) in a similar fashion. However, in this case we must use both the curve of parameter β and the curve of parameter α. The latter curve is a straight line which intersects the curve $X=X_0$ in a point X^0, $T^{(\alpha)}(X,T)$. By integrating the two equations (18.6) and (18.7) along the curves of parameter α and β, respectively, we may determine $\Phi^{(1)}$ (and hence $\phi^{(1)}$) and $\Psi^{(1)}$ at X, T provided we know the values of these quantities at $(X^{(\beta)},0)$ and $(X_0,T^{(\alpha)})$.

However, $\Psi^{(1)}(X,0)$ may be determined by integrating equation (18.2) with respect to X along the curve $T=0$. In this equation the right-hand side is a known function of X (actually zero) specified by the boundary conditions.

The values of $\Psi^{(1)}(X^{(\beta)},0)$ may be determined by making use of Eq. (18.7). Since, in region II, $\phi^{(0)}$ is given by a progressive wave for which $\phi_\alpha{}^{(0)}=0$, Eq. (18.7) may be written as

$$(\Psi^{(1)}+\Phi^{(1)})_\beta=(K/4c^2)(X+\Gamma T)T_\beta.$$

However, throughout region II we have, from equation (9.3),

$$X=\Gamma(\beta)T+f(\phi(\beta)).$$

Hence by integrating this equation we have

$$\Psi^{(1)}=-\Phi^{(1)}+\frac{K}{4c^2}\int[2\Gamma(\beta)T+f(\phi(\beta))]T_\beta d\beta.$$

The constant of integration in this equation is chosen so that the value of $\Psi^{(1)}$ at $X=X_0$, $T=0$ obtained from this formula agrees with that obtained by use of Eq. (18.2).

Similar methods enable one to determine $\phi^{(i)}$ and $H^{(i)}$ for $i>1$ in region II of the X, T plane.

23. SHOCKS

It is well known in classical theory and in the special theory of relativity that when $\Gamma(\phi)$ is an increasing function of ϕ, as it is for many gases, then the family of straight-line characteristics in the X, T plane along which ϕ is a constant and which are defined by Eq. (9.3) is such that at least two of them intersect in a point in region II when the specified functions $\phi^{(0)}(X_0,T)$ is an increasing function of T.

Thus, in classical theory when the motion of a piston is such as to compress the gas contained in a tube, the theory analogous to that described above breaks down because two characteristics intersect. This failure in mathematical description is known to be associated with the physical phenomenon of the formation of shock waves. Since the theory of a perfect fluid represents that idealization of a compressible fluid which ignores viscosity and heat conductivity, the shock waves are represented as mathematical discontinuities instead of transition zones with large gradients in velocity, pressure and density. However, it is known that the results obtained in classical theory by treating shocks as mathematical discontinuities give the same relations between the hydrodynamical variables on both sides of the continuous transition zone as the theory which takes viscosity and heat conduction into account. It is to be expected that a corresponding relation between the general relativistic theory of perfect fluids and that of physically realizable compressible fluids will obtain.

When the prescribed boundary data, $\phi(X_0,T)$ is such that the family of straight lines in region II of the X, T plane, on which $\phi^{(0)}(X,T)=\text{constant}$ and which are described by Eq. (9.3), has a real envelope in region II, then the functions $\phi^{(i)}$ and $H^{(i)}$ $(i=0, 1, \cdots)$ determined in the manner described above will not be single-valued throughout region II and hence will not be physically acceptable. In this case, we must change the theory described in the preceding sections.

The change that seems most plausible is to generalize the theory of general relativity so as to allow for the existence of surfaces across which the hydrodynamical (and hence the gravitational) fields change discontinuously and in such a manner that an isentropic flow becomes merely adiabatic. That is, the Einstein field equations will be said to hold in regions of space-time but the possibility of having surfaces (shocks) across which these equations are not defined will be allowed. The relations that will hold across these surfaces will be a generalization of the Rankine-Hugoniot equations. The derivation of these relations will be discussed elsewhere.

This proposed change is suggested by classical hydrodynamics and special relativistic hydrodynamics. In fact, if the general theory of relativity is going to admit an approximation procedure in which either of these theories are going to be obtained as low-order approximations to the general theory, we must modify the general theory in the manner proposed, since this is the manner in which those theories discuss the physically unacceptable solutions of the equations of motion. Note that these unacceptable solutions appear in the zero-order (special relativistic) approximation, and it can be shown that if we have means of determining physically acceptable zero-order solutions of the field equations, then higher order approximations are determined by linear equations alone. Therefore, the resolution of the difficulty for special relativity together with an acceptance of the approximation procedure discussed earlier gives a resolution of this difficulty for general relativity.

If shocks are allowed in general relativity, then we must expect the flow behind the shock to be non-isentropic. However, if the Einstein field equations (11.6), with $T^{\mu\nu}$ given by Eqs. (2.4), are to hold in such regions and if the conservation of mass is also required to hold, then the flow will be such that entropy is conserved along a world line of a particle. For the plane-symmetric case, these assumptions imply the existence of a co-moving coordinate system in terms of which the field equations may be shown to imply a set of equations which involve only one set of dependent variables. The approximation procedure described above may be applied to this set of variables. Again, the nonlinearity of the problem is concentrated in the special relativistic approximation and this approximation determines linear equations for higher order corrections.

Thus, the physical and geometrical interpretation of shocks in general relativity may be based on their interpretation in special relativity. We note that in that theory a singularity in the function $\phi(X,T)$ which implies that the metric tensor in the co-moving Lagrangian coordinate system in which the line element is given by Eq. (3.5) is not interpreted as a singularity in space-time, which is always the Minkowski space-time. Instead, we interpret the singularity as a singularity in the transformation between the inertial coordinates x, t and the coordinates X, T. In the inertial coordinates the equations describing the world line of a particular element of the fluid are given parametrically as

$$x=x(X,T), \quad t=t(X,T),$$

with X fixed. These curves are allowed to have points at which the tangent is discontinuous. The totality of such points make up the shock.

Similarly in general relativity we may expect that the shocks which represent surfaces on which the line element is singular are not essential singularities in the following sense: By making an appropriate (singular) transformation we may introduce a coordinate system in which the line element does not have singularities. The coordinate transformation will determine the world lines of elements of the fluid, and these world lines may have points at which the tangents are discontinuous.

24. GRAVITATIONAL WAVES

In the plane-symmetric case, equations determining these transformations in regions where the flow is isentropic are Eqs. (19.3) and (19.4). In regions where the flow is adiabatic, similar equations obtain. When the metric tensor is determined in the co-moving coordinate system, say by the approximation procedure, these equations may be solved subject to appropriate boundary conditions by the method described in Sec. 19.

That method involves the use of the characteristic curves associated with Eq. (19.3), that is, the curves given by Eqs. (19.7) and (19.8) which differ from the characteristic curves which enter into the solution of the equations determining the coefficients of the metric tensor. That is, changes in the variables ϕ and H and therefore in p and ρ are propagated along the "sound-cone," the surface which intersects the X, T plane in the characteristics associated with the field equations, whereas changes in the functions $X(X,T)$ and $T(X,T)$ are propagated along the light-cone.

In a general coordinate system, say one in which the line element is given by Eq. (19.2), changes in the metric tensor, the function F in Eq. (19.2), will be propagated along the light cone since F involves the functions $x(X,T)$ and $t(X,T)$. The line element given by Eq. (19.2) may be said to represent gravitational waves. However, it must be pointed out that the light cone enters the description of the variation of F in space-time only through the choice of coordinate system, and in the co-moving one the gravitational waves (variations in the metric tensor) are propagated along the hydrodynamical characteristics.

If shock waves occur, we may expect that, for very strong shocks, the surface across which discontinuous changes in various quantities may take place will practically coincide with the light cone. In that case, the hydrodynamic and gravitational phenomena even in the co-moving coordinate system involve the light-cone, and gravitational waves with a velocity of propagation related to the velocity of light may be said to exist.