

Uniqueness of Solutions to Dispersion Relations for Potential Scattering*

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Some properties of the dispersion relations for potential scattering are examined. It is shown that even for a potential of finite extent, the dispersion and unitarity relations do not define a unique scattering amplitude, so that they do not contain all the information that is derivable from the Schrödinger equation.

INTRODUCTION

DISPERSION relations connecting the real and imaginary parts of scattering amplitudes have been deduced from the assumptions of Lorentz invariance, microscopic causality, and certain symmetry properties quite independently of any particular Hamiltonian.¹ The connection with a model is made in the assignment of the mass spectrum, the postulated threshold behavior of the amplitudes, and in the assumed behavior of the amplitudes for infinite momenta. It has been conjectured that a quantum field theory might be completely defined by such dispersion relations together with the unitarity condition.² In such an approach it is assumed that the dispersion relations, together with the unitarity condition, contain all the relevant information contained in the Hamiltonian, and that they yield a unique solution if the Hamiltonian does. One would then have a formulation of a field theory essentially in terms of observables, which would require none of the renormalization prescriptions of the canonical Hamiltonian theories.

However, this happy situation is not realized for certain simple models for which all solutions of the dispersion relations can be exhibited. Thus the dispersion-type equations that describe the "one-meson approximation" to a static-source meson theory have been solved by Castillejo, Dalitz, and Dyson,³ who find that these equations have an infinite number of solutions. It is not known whether this ambiguity remains if the static-source model is treated without the "one-meson approximation." This leads to an infinite number of coupled dispersion relations for which no solution has yet been exhibited. Similarly the two dispersion relations obtained for the Lee model⁴ do not possess a unique solution, although the Hamiltonian formulation of this model does. Again, however, as in the "one-meson" model, the theory is a mutilation of a canonical

field theory, so that the significance of the nonuniqueness is obscure.

It is therefore of interest to investigate the uniqueness of the solution to the exact dispersion relations which describe a reasonable, complete, and causal model, namely the scattering of a particle by a static scalar local potential of finite range. The Schrödinger (or Klein-Gordon) equations can be solved to give the scattering amplitude. Conversely the scattering amplitude determines the potential⁵ and so contains all the information that is in the Hamiltonian. This scattering amplitude is a function of two variables, the momentum and the momentum transfer, and therefore is described by a much more complicated dispersion relation than that of the Lee model or the "one-meson approximation." Indeed, if the scattering amplitude is expandable into a convergent series of partial waves, this dispersion relation (plus the unitarity condition) decomposes into an infinite number of coupled nonlinear integral equations. In addition to their multiplicity, these integral equations differ in an important way from those previously studied: the inhomogeneous term need no longer be a rational function of its argument, but may have an essential singularity at infinity. Those mathematical procedures which produce the spectrum of solutions for the Lee model and the "one-meson approximation" are not then applicable. Nevertheless it is possible to show by specific examples that even for a square-well potential the solutions of the dispersion relation are not unique. Although these extra solutions cannot correspond to the scattering from any "reasonable" potential, they do demonstrate that the dispersion relations and the unitarity condition do not exhaust the content of the Hamiltonian theory.

DISPERSION RELATIONS

The assumed dispersion and unitarity relations for the scattering of a particle by a spherically symmetric static potential (with no bound states) are

$$f(k, \Delta) = V(\Delta) + \frac{1}{\pi} \int_{-\infty}^{\infty} dk' \frac{\text{Im}f(k', \Delta)}{k' - k - i\epsilon}, \quad (1)$$

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¹ M. L. Goldberger, *Phys. Rev.* **99**, 979 (1955). For further references see, e.g., R. H. Capps and G. Takeda, *Phys. Rev.* **103**, 1877 (1956).

² M. Gell-Mann, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Nuclear Physics, Rochester, 1956* (Interscience Publishers, Inc., New York, 1956).

³ Castillejo, Dalitz, and Dyson, *Phys. Rev.* **101**, 453 (1956).

⁴ T. D. Lee, *Phys. Rev.* **95**, 1329 (1954). See also K. W. Ford, *Phys. Rev.* **105**, 320 (1957).

⁵ R. Jost and W. Kohn, *Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd.* **27**, No. 9 (1953).

where

$$\operatorname{Im}f(k,\Delta) = \frac{1}{4\pi m} \int d^3\Delta' \delta(\Delta'^2 + 2\mathbf{k} \cdot \Delta') \times f^*(k, |\Delta - \Delta'|) f(k, \Delta'), \quad (k \text{ real}) \quad (2)$$

and

$$f^*(k, \Delta) = f(-k, \Delta), \quad (k \text{ real}). \quad (3)$$

The scattering amplitude $f(k, \Delta)$ is a function of the momentum k of the incident particle, and of its momentum transfer Δ . The inhomogeneous term of Eq. (1), $V(\Delta)$, is the Born approximation for the scattering with momentum transfer Δ , and of course depends only upon Δ for a static local potential. The unitarity relation (2) and the symmetry condition (3) follow directly from the reality of the potential, but the dispersion relation (1) holds only for certain classes of potentials. It is a consequence of Cauchy's theorem provided that

- (a) $f(k, \Delta)$ is a regular function of k for fixed Δ , Δ real and $\operatorname{Im}k > 0$,
- (b) $f(k, \Delta) - V(\Delta) \rightarrow 0$ as $|k| \rightarrow \infty$, $\operatorname{Im}k \geq \delta > 0$,
- (c) $k^{-1}f(k, \Delta)$ is bounded on the real axis.

Wong⁶ has shown that these conditions hold for each term in a perturbation expansion of the scattering amplitude. More generally, Khuri⁷ has derived Eq. (1) under the conditions

$$\int_0^\infty r |V(r)| dr \text{ is finite,} \quad (A)$$

and

$$\int_0^\infty e^{\alpha r^2} |V(r)| dr \text{ is finite,} \quad (B)$$

provided

$$\frac{1}{2}\Delta < \alpha.$$

Thus Eq. (1) has been demonstrated to follow from the Schrödinger theory only for values of momentum transfer $\Delta < 2\alpha$. For a well of finite range Eq. (1) is valid for all Δ , but, if α is bounded from above, the Schrödinger amplitude for large momentum transfers need not be a solution of Eq. (1).

The conditions (A) and (B) are also sufficient to insure that the scattering amplitude for an s -wave does not possess a "redundant pole" in the strip $0 < \operatorname{Im}k < M$, where M is the upper bound for those values of α which satisfy (B).⁸ However, even if the s -wave scattering amplitude, $f_s(k)$, does have "redundant poles" in the upper half plane for $\operatorname{Im}k \geq M$, $f(k, \Delta)$ does not for $\Delta < 2M$ even if $\operatorname{Im}k \geq M$. Thus if the s -wave scattering is separated from the total amplitude in the manner

$$f(k, \Delta) = f_s(k) + \varphi(k, \Delta) \quad (4)$$

we see that the remainder of the scattering amplitude $\varphi(k, \Delta)$ must also possess "redundant poles" when $\Delta < 2M$, such as to cancel those from $f_s(k)$. This does not necessarily imply that the scattering amplitude for higher partial waves must also possess redundant poles whenever and wherever the s -wave amplitude does. Indeed the partial wave expansion of $f(k, \Delta)$,

$$f(k, \Delta) = \sum_{l=0}^{\infty} f_l(k) Y_l(\cos\theta), \quad \theta \text{ real} \quad (5)$$

is necessarily convergent only if $|k| < M$.

UNIQUENESS

Clearly one solution of the dispersion and unitarity relations is that solution which results from solving the Schrödinger equation with the potential that appears in Eq. (1). If there is another solution of Eqs. (1)–(3), then it cannot correspond to the scattering by another potential which satisfies conditions (A) and (B), for if it did, then it would satisfy Eq. (1) with a different inhomogeneous term, which is impossible. Nevertheless, alternate solutions can be constructed. Let us consider that solution of Eqs. (1)–(3) which is also the scattering amplitude calculated from the Schrödinger equation: $f_0(k, \Delta)$. We shall show that even for a square-well potential there exist an infinite number of other solutions identical to $f_0(k, \Delta)$ in all phase shifts except one, which we shall take to be the s wave.

If the s -wave part of $f_0(k, \Delta)$ has a phase shift $\delta_0(k)$, then

$$f_0(k, \Delta) - (2ik)^{-1} \{ \exp[2i\delta_0(k)] - 1 \} + (2ik)^{-1} \{ \exp[2i\delta(k)] - 1 \} \quad (6)$$

is unitary and a solution of Eq. (1) if

$$k^{-1} \{ \exp[2i\delta_0(k)] - \exp[2i\delta(k)] \} \text{ is a regular function of } k, \text{ which approaches zero as } |k| \rightarrow \infty \text{ for } \operatorname{Im}k > 0. \quad (C)$$

We note first that if $V \equiv 0$ then $\exp[2i\delta_0(k)] \equiv S_0(k) = 1$, and

$$\exp[2i\delta(k)] = \prod_{i=1}^N \frac{(k - a_i)(k + a_i^*)}{(k + a_i)(k - a_i^*)} \equiv s(k); \quad \operatorname{Im}a_i > 0 \quad (7)$$

satisfies condition (C), so that there exist an infinite number of solutions in addition to $f_0(k, \Delta) \equiv 0$.

There is a class of potentials discussed by Bargmann⁹ for which $S_0(k)$ is a rational function of k , which approaches unity as $k \rightarrow \infty$ [for example, $V(r) = -V_0 e^{-\lambda r} \times (1 + \beta e^{-\lambda r})^{-2}$]. Here any expression of the type

$$\exp[2i\delta(k)] = s(k) \prod_n \frac{k + \alpha_n}{k - \alpha_n}, \quad \operatorname{Re}\alpha_n = 0, \quad \operatorname{Im}\alpha_n > 0 \quad (8)$$

will be a satisfactory solution provided the a_i are chosen so as to give the correct residues at the poles.

⁹ V. Bargmann, *Revs. Modern Phys.* **21**, 488 (1949).

⁶ D. Y. Wong, Physics Department Technical Report No. 62, University of Maryland, January 1957 (unpublished).

⁷ N. N. Khuri (to be published).

⁸ V. Bargmann, *Revs. Modern Phys.* **21**, 488 (1949), Appendix. See also R. Jost, *Helv. Phys. Acta* **20**, 256 (1947).

This is the kind of situation which occurs in the "one-meson approximation" and in the Lee model, at least in the case of no cutoff, for which an infinite number of solutions of form of Eq. (8) could be exhibited.

In general, however, the situation is a great deal more complicated by the fact that $S_0(k)$ may have an essential singularity at infinity.¹⁰ For example, for a potential of finite extent a , we have, quite generally,

$$S_0(k) = \exp(-2ika) \frac{1 + ikR_0(k^2)}{1 - ikR_0(k^2)}, \quad (9)$$

where $R_0(k^2)$ is a Wigner R function,¹¹ and we cannot generate another solution of the type $s(k)S_0(k)$ since $k^{-1}[s(k)-1]S_0(k)$ is still singular at infinity. In the case of the square-well potential, for which

$$R_0(k^2) = K^{-1} \tan(aK), \quad (10)$$

where $K = k(1 + \lambda/k^2)^{1/2}$ and $\lambda/2m$ is the depth of the potential, we can exhibit several representative examples of alternate solutions, which satisfy condition (C) and thus establish nonuniqueness, even though they have some obvious nonphysical properties.

Example 1.—Consider

$$\exp[2i\delta(k)] = \exp(-2ika) \frac{1 + ikR(k^2)}{1 - ikR(k^2)}, \quad (11)$$

and choose

$$R(k^2) = K^{-1} \tan(bK). \quad (12)$$

where $b < \pi/2\sqrt{\lambda}$ if there are no bound states. This solution satisfies condition (C), provided $b > a$, and it is thus an acceptable alternative solution. It has the physically undesirable feature that the phase shift for (real) infinite momenta oscillates instead of vanishing, but this "unreasonable" behavior cannot be ruled out without imposing an additional condition to supplement Eqs. (1)–(3). Equation (12) can be generalized immediately to

$$R(k^2) = K^{-1} \sum_i \rho_i \tan(b_i K) / \sum_i \rho_i; \quad b_i \geq a. \quad (12')$$

Example 2.—Consider the form (11) with

$$R(k^2) = K^{-1} \tan[(a + b/k^2)K]. \quad (13)$$

¹⁰ If $V(r)$ satisfies $\int_0^\infty r^n |V(r)| dr < \infty$ ($n=1, 2$), then if $S_0(k)$ has no redundant poles (poles which do not correspond to bound states), it must have an essential singularity at infinity. The conditions on $V(r)$ imply a theorem of N. Levinson [Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 25, No. 9 (1949)] that $\int_{-\infty}^\infty dk S_0'(k)/S_0(k) = 0$ if there are no bound states. If $S_0(k) = 1 + O(k^{-1})$ at infinity in the upper half-plane, then the absence of poles for $S_0(k)$ in the upper half-plane implies an absence of zeros. But as we have $S_0(k)S_0(-k) = 1$, this implies $S_0(k) \equiv 1$. If, however, $S_0(k)$ is not identically equal to unity, and there are no redundant poles, then $S_0(k)$ does not approach unity at infinity. Moreover, since we have $S_0(k)S_0(-k) = 1$, its behavior at infinity depends on the direction of approach.

¹¹ E. P. Wigner, Ann. Math. 53, 36 (1950).

This satisfies condition (C), provided $b < 0$. As we approach the real axis at $k=0$ from above, $R(k^2)$ does not become real, but this does not violate unitarity because it is $kR(k^2)$, which vanishes as k approaches the origin, that enters into $\exp[2i\delta(k)]$ and so this is an acceptable solution.¹² It is clearly unphysical because the s -wave scattering cross section oscillates violently near $k=0$; the "pathological" region may be made arbitrarily small by choosing b small enough. A variety of other types of solutions have been found but they also have the property that $\delta(\infty) - \delta(0) \neq m\pi$ (m an integer).

The difference between the "Bargmann potentials" and the square well may be restated by writing the dispersion relation for the s -wave scattering amplitude $F(k)$ [$=F^*(-k)$ for real k],

$$F(k) = G(k^2) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dk'}{k' - k - i\epsilon} k' |F(k')|^2, \quad (14)$$

which may be derived from Eqs. (1)–(3) by separating the scattering amplitude

$$f(k, \Delta) = F(k) + \phi(k, \Delta), \quad (15)$$

$$\frac{1}{2k^2} \int_0^{2k} \Delta \phi(k, \Delta) d\Delta = 0,$$

and noting that

$$F(k) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dk'}{k' - k - i\epsilon} \text{Im} F(k')$$

$$= V(\Delta) - \phi(k, \Delta) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dk'}{k' - k - i\epsilon} \text{Im} \phi(k', \Delta)$$

$$\equiv G(k^2),$$

because the left-hand side is independent of Δ and even in k . [The unitarity condition yields the relation $\text{Im} F(k) = k |F(k)|^2$.] For the cases of no potential and the "Bargmann potential," $G(k^2)$ is zero and a rational function of k^2 , respectively, and the equation may be solved by the method used by Castillejo, Dalitz, and Dyson. On the other hand, for the square well $G(k^2)$ has an essential singularity at infinity, and nonuniqueness of the solutions of Eq. (15) cannot be established by this means.

If bound states are present, the inhomogeneous term of Eq. (1) must be appropriately altered to exhibit the poles of the scattering amplitude, but the conclusions of this paper are unchanged.

¹² It is the unitarity condition which rules out the otherwise acceptable solution $R(k^2) = K^{-1} \tan\{[a + \sum_n \alpha_n^2 / (\beta_n^2 - K^2)]K\}$. Such a solution is not unitary along the discrete set of points $K = \pm \beta_n$.