Interactions of Low-Energy Negative Pions with Nuclei*†

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A boundary condition method is used to correlate the level shifts in π-mesonic atoms with the scattering of slow pions by light nuclei; carbon is chosen as an example. Up to 5-Mev kinetic energy, the s-phase shifts are dominant and a calculation of the scattering cross section is possible without much ambiguity. In the absence of sufficient experimental data on the p-level shifts, a best estimate is attempted in order to include p-wave scattering and to extend the predictions up to 10-Mev kinetic energy.

I. INTRODUCTION

THE short-range character of the interaction of pions with nuclear matter makes it possible to relate the energy level shifts of π-mesonic atoms to scattering phase shifts for the pion-nucleus system. We shall derive this relation by using a method similar to that of Wigner and Eisenbud.[1] Taking the nucleus to be a sphere of radius R, we assume that when the pion is at a distance greater than R from its center only Coulombic interaction is present. Then, in this region of configuration space, the pion can be described by a wave function whose form is known. Pion absorption and inelastic scattering are taken into account by allowing the phase shifts of the continuum wave functions and the energies of the bound-state wave functions to be complex numbers. We treat each angular momentum state l individually, and represent the nuclear interaction by a phenomenological parameter χ_l, which is the deviation of the logarithmic derivative of the pion wave function at the nuclear surface from the value it would have if only Coulombic interaction with a point source were present. This parameter is convenient for our purpose because it can be expected to be nearly constant, i.e., energy-independent, near zero kinetic energy. In this region, χ_l can be evaluated from mesonic x-ray data and used to obtain scattering phase shifts.

The energy dependence of χ in the low-energy region is discussed in some detail in Sec. II. Our analysis there is based upon the assumption that the interaction in each angular momentum state can be effectively described by a spherically symmetric potential. We find that for s states, if the potential is energy-independent, χ remains nearly constant up to energies of the order of 10 Mev. This is a consequence of the relatively weak s-state interaction. For strong interactions, as the p-state interaction may be expected to be, the energy dependence of χ can be taken into account. Therefore, we are able to predict low-energy pion-nucleus scattering cross sections.

A treatment of the type we employ always involves the nuclear radius as a parameter. We define the nuclear radius to be the minimum separation distance at which the nuclear interaction can be neglected. The relation between the level shift and the scattering phase shift which we obtain is then insensitive to the precise value of R. We assume that the nuclear radius as defined here is approximately the same as that which is obtained from experiments such as high-energy neutron-nucleus scattering, and take its value as $R=r_0A^{\frac{1}{3}}$, where r_0 is the pion Compton wavelength, $\hbar/mc$.

We have calculated the angular distributions for the elastic scattering of negative pions of energy 5 and 10 Mev from carbon. The results of these calculations are presented in Sec. III. The numerical predictions for the s-state scattering were obtained using π-mesonic x-ray data. The deviation of the angular distribution from pure Coulomb scattering at 5 Mev is primarily due to the s-phase shift for angles less than 120°. For the larger angles and at 10 Mev, the effect of the p-state interaction is important. We can only roughly estimate these effects in the absence of precise data on the $2P$ level shift. Such estimates are included in order to indicate what might be expected at 10 Mev. Our estimates for the nuclear p-phase shift correspond to a $2P$ level shift in carbon of -4.3 ev.

In Sec. IV, we discuss the origin of the $1S$ level shift according to the theories of Deser, Goldberger, Baumann, and Thirring,[2] Brueckner,[3] and Karplus and Halpern.[4] We compare the scattering predictions of our phenomenological treatment with results obtained from these theories.

II. RELATION OF THE LEVEL SHIFTS TO THE SCATTERING AMPLITUDES

For simplicity, we restrict our considerations to the interaction of pions with zero-spin nuclei, and allowing the interaction to be l-dependent, treat each angular momentum state individually. The relative motion of

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[1] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).
[2] S. Deser *et al.*, Phys. Rev. **96**, 774 (1954). This paper will be referred to as DGBT.
[3] K. A. Brueckner, Phys. Rev. **98**, 769 (1955).
[4] R. Karplus and F. Halpern, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. IX, p. 33.

the pion and the nucleus for separation distances, r, greater than the range of the nuclear interaction is described by wave functions

$$\psi_{n,l}=r^{-1}u_{n,l}(r)Y_{l,m}(\theta,\varphi), \quad \text{for the bound states } (E=E_n); \quad (1)$$

$$\psi_{k,l}=r^{-1}u_{k,l}(r)Y_{l,0}(\theta), \quad \text{for continuum states } (E=E_k).$$

If the nuclear interaction is switched off adiabatically, $u_{n,l}$ becomes $F_{n,l}$, a radial wave function (principal quantum number n) for the pion bound in the Coulomb field of a point source of strength Ze. We denote by $F_{k,l}$ the corresponding continuum Coulomb wave function which has the same energy as $u_{k,l}$. All deviations of the pion-nucleus interaction from that which would be present if the nucleus were a mere point charge are included in what we call the nuclear interaction. Therefore, if the nuclear interaction can be represented by a potential it would have the form $U+\delta\phi$, where U is the potential in the absence of charge and $\delta\phi$ is the change in the electrostatic potential due to the extended charge distribution and radiative corrections. For our purposes, the nuclear interaction is described by the quantity,[5]

$$\chi_l(E)=R\left(\frac{1}{F_l}\frac{dF_l}{dr}-\frac{1}{u_l}\frac{du_l}{dr}\right)_{r=R}. \quad (2)$$

We should like first to discuss the energy dependence of χ. In order to do this, we write

$$\chi(E)=\chi(m)-k^2R^2a(E), \quad E^2=k^2+m^2. \quad (3)$$

The quantity $\chi(m)$ depends only on the strength of the nuclear interaction at zero kinetic energy; we assume that this is appreciable. Then there will be an energy range about zero kinetic energy in which $k^2R^2a(E)$ is small compared to $\chi(m)$. If the effective potential inside the nucleus and the value of the Coulomb potential on its surface are both large compared to $k^2/2m$, both $(1/u)(du/dr)$ and $(1/F)(dF/dr)$ near $r=R$ will be insensitive to changes in kinetic energy. Therefore, the error made in assuming

$$\chi(E)=\chi(m) \quad (4)$$

will be small. The energy levels of π-mesonic atoms corresponding to the observed spectral lines lie in this energy range. If the nuclear interaction energy is very much larger than the Coulomb potential at the surface, there will be a range of energies for which $k^2/2m$ is small in comparison with the nuclear interaction energy but not small in comparison with the Coulomb potential on the nuclear surface. In this energy range, the second term in (2) will be much less energy-sensitive than the Coulombic term, and the variation with energy of the latter alone will provide a good estimate of the correction term $k^2R^2a(E)$ in (3). The variation with energy of the Coulombic term is, nonrelativistically,[6,7]

$$R\left(\frac{1}{F_0}\frac{dF_0}{dr}-\frac{1}{F_k}\frac{dF_k}{dr}\right)_{r=R}=k^2R^2\int_0^R\frac{F_0(r)F_k(r)}{F_0(R)F_k(R)}\frac{dr}{R}\equiv k^2R^2\alpha_F(E_k). \quad (5)$$

(If k is taken imaginary, this equation holds also for bound states.)

More generally, assume that the nuclear interaction can be replaced by an effective potential $U+\delta\phi$. Then the wave functions $u(r)$ can be continued[1] into the region $r<R$, and since they will be solutions of equations of the same form as are satisfied by the $F(r)$, the energy dependence of the second term in (2) will be given by the quantity $\alpha_u(E)$ which can be obtained from Eq. (5) by replacing F by u. Then the energy dependence of χ will be given by[8]

$$a(E)=\alpha_F(E)-\alpha_u(E). \quad (6)$$

Note that, in a power series expansion, both F and u have the leading term r^{l+1}. Therefore, it is evident from (5) that in (6) the terms α_F and α_u tend to cancel, unless the nuclear interaction energy U is extremely large. As a result, the energy dependence of χ will be weakened. As an example, consider the $1S$ state in carbon where the assumption of a repulsive potential (for $r<R$) of magnitude 3 Mev roughly corresponds to the observed line shift [Fig. 1(a); see also Sec. III]. For this case, α_F and α_u were calculated numerically as functions of k^2, and are plotted in Fig. 1(b). Generally, a repulsive nuclear potential implies that $\alpha_u<\alpha_F$, $a>0$, as can be seen from Eq. (5). In this example, where χ is negative, $|\chi|$ increases by about 17% as the kinetic energy increases from zero to 10 Mev.

We have thus far tacitly assumed that the nuclear interaction, U, is energy-independent. This restriction is not necessary. An energy dependence for U may be taken into account by modification of Eq. (5). For instance, if the nuclear potential is of the form $V+(bk^2/2m)$, $a(E)$ becomes

$$a(E)=\alpha_F(E)-(1-b)\alpha_u(E). \quad (7)$$

We shall now show how χ may be obtained from the level shifts and used to compute the scattering phase shifts. By relating the level shifts directly to the phase shifts, we shall display the effect of a nonvanishing $a(E)$. To obtain the $\chi(E_n)\approx\chi(m)$ from the energy level shifts, we need assume only that the wave functions

[5] Subscripts n or k will be appended only when it is necessary to distinguish between bound states and continuum states, and hereafter the subscript l is to be understood when it does not appear.

[6] H. A. Bethe, Phys. Rev. **76**, 38 (1949).

[7] Units are chosen such that $\hbar=c=1$.

[8] In order to obtain the relativistic correction to (5), we used the Klein-Gordon equation instead of the Schrödinger equation and followed the same steps as lead to (5). The result then is, for an arbitrary potential U,

$$\alpha_F(E_k)=\int_0^R\left(1-\frac{2U(r)}{E_k+m}\right)\left(\frac{F_0(r)F_k(r)}{F_0(R)F_k(R)}\right)\frac{dr}{R}.$$

$u(r)$ are valid descriptions of our system for $r>R$. In this region they are solutions of the Klein-Gordon equation with a Coulomb potential, Ze^2/r. (Requiring that the bound-state solutions vanish as r goes to infinity and satisfy (2) at $r=R$ defines an eigenvalue problem.) The energy eigenvalue E_n (obtained from line-shift data) determines the function $u_n(r)$, and χ may be evaluated directly from $[(1/u)(du/dr)]_{r=R}$. The functions which satisfy the radial Klein-Gordon equations with the Coulomb potential and an arbitrary energy parameter E, and vanish as r goes to infinity, are known as Whittaker functions.[9] The following formal expression of this relation between χ and the level shift, δE_n, which is useful for small level shifts, may be obtained in a manner similar to the derivation of Eq. (5):

$$\chi(E_n)=-2mR\delta E_n\int_R^\infty \frac{u_n(r)F_n(r)}{u_n(R)F_n(R)}dr. \tag{8}$$

In the above, the level shift refers to the shift in the total energy, i.e., $\delta E_n=E_n-E_n{}^0$, where $E_n{}^0=m\times(1-Z^2e^4/n^2)$. Equation (8) is nonrelativistic so terms of order Ze^2/mR and $(1-E/m)$ are neglected. A relativistic generalization is easily obtained with the use of the Klein-Gordon equation. For the work presented in this paper, relativistic corrections are not significant. It will be convenient to define the quantity $N_n(R)$ by rewriting Eq. (8) in the form

$$\delta E_n=-\left(\frac{\chi(E_n)}{2mR}\right)N_n(R). \tag{9}$$

The scattering amplitude in a given angular-momentum state is related to $\chi(E)$ in a particularly simple way. The deviation of this amplitude from the well-known Coulomb scattering amplitude is of interest here. It is given by quantities called "nuclear phase shifts," τ_l, which are related to the total phase shift, δ_l, for the lth partial wave by the relation $\tau_l+\sigma_l=\delta_l$; σ_l is the phase shift for pure Coulomb scattering. The radial wave function for the pion with $E>m$ has the form

$$u_{k,l}(r)=F_l(kr)+\tan\tau_l G_l(kr), \tag{10}$$

where $F_l(kr)$ and $G_l(kr)$ are the Coulomb functions that, for large r, have the asymptotic behavior $\sin(kr-\frac{1}{2}l\pi+\eta\ln 2kr+\sigma_l)$ and $\cos(kr-\frac{1}{2}l\pi+\eta\ln 2kr+\sigma_l)$, respectively. $[F_l(kr)\equiv F_{k,l}(r).]$ Inserting (10) into (2), and using the Wronskian relation

$$G_l(kr)\partial F_l(kr)/\partial r-F_l(kr)\partial G_l(kr)/\partial r=k,$$

one finds that

$$\frac{\sin\tau_l}{k}=\frac{\chi(E_k)}{R}\frac{F_k{}^2(R)\sin[\tau+\varphi(kR)]}{\sin\varphi(kR)}\equiv\frac{\chi(E_k)}{R}N_k(R), \tag{11}$$

[9] See, for instance, H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics* (Cambridge University Press, Cambridge, 1950), second edition, pp. 607–620.

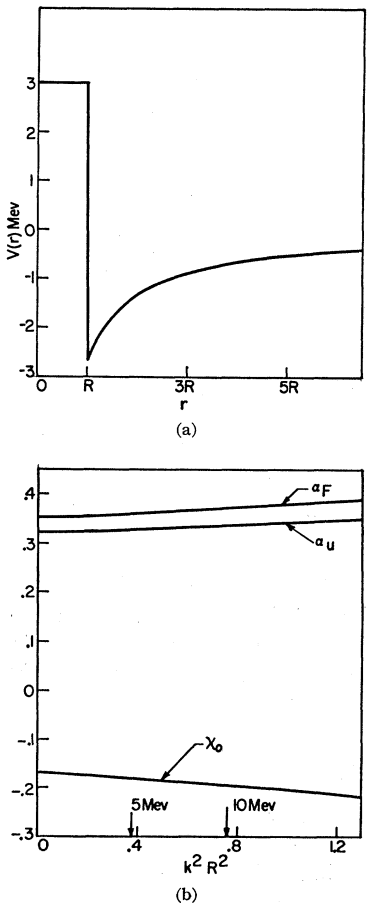

FIG. 1. (a) A potential which yields a 1S level shift in carbon approximately equal to that observed. (b) Illustration of cancellation in the energy-dependent contributions to χ_0. $[\chi_0(E)=\chi_0(m)-k^2R^2(\alpha_F-\alpha_u).]$ These curves are based on the potential of Fig. 1(a).

where we have used the usual notation

$$\varphi_l(kR)\equiv\tan^{-1}[F_l(kR)/G_l(kR)]$$

and suppressed the subscript l again. The quantity $N_k(R)$ is defined above in analogy with the $N_n(R)$ of Eq. (9).

In accordance with our previous discussion, we now take

$$\chi(E_n)=\chi(m),\quad \chi(E_k)=\chi(m)-k^2R^2a(E_k).$$

The elimination of $\chi(m)$ between Eqs. (11) and (9) yields

$$\delta E_n=-\frac{1}{2m}\left(\frac{\sin\tau}{k}\right)\left(\frac{N_n(R)}{N_k(R)}\right)-\frac{1}{2m}k^2R^2a(E_k)N_n(R). \tag{12}$$

As $k\to 0$, this expression becomes particularly simple for interactions which are not strong. If we take $u=F$ in $N_n(R)$ and ignore terms of order R/a_Z $[a_Z=\text{Bohr radius}=1/(Ze^2m)]$, $N_n(R)$ is just the square of the normalized radial wave function for the Bohr

orbit n, l evaluated at R;

$$N_n(R)\approx\left(\frac{2R}{na_Z}\right)^{2l+2}\left(\frac{2}{na_Z(2l+2)!}\right).$$

In this same approximation,

$$F_l(R)\approx C_l(kR)^{l+1}/(2l+1)!!$$

and

$$N_k(R)=k^{-2}F_k^2(R)\approx C_l^2(kR)^{2l+2}/k^2[(2l+1)!!]^2,$$

where

$$C_l^2=\frac{2\pi\eta}{1-e^{-2\pi\eta}}\prod_{p=1}^{l}\left(1+\frac{\eta^2}{p^2}\right),\quad \eta=\frac{Ze^2}{v},\ (v=\text{pion velocity}),$$

and $(2l+1)!!\equiv 1\times3\times5\times\cdots\times(2l+1)$. Therefore,

$$\frac{N_n(R)}{N_k(R)}\approx\frac{4(2l+1)!!}{(na_Z)^{2l+3}l!C_l^2k^{2l}}. \tag{12'}$$

(Actually, the first-order corrections to (12') due to the difference of u and F vanish, and the cancellation of the first-order R/a_Z correction is nearly complete.) In the approximation which leads to (12'), $C_l^2k^{-(2l+1)}\sin\tau_l$ may be replaced by $k^{-(2l+1)}\sin\delta_l$, where δ_l is the phase shift that would apply in the absence of Coulomb effects. With the use of this replacement and (12'), Eq. (12) yields the result given by Deser, Goldberger, Baumann, and Thirring[2] for s states.

It might be remarked here that nuclear absorption effects are included in the foregoing treatment. The fact that pions are absorbed by the nucleus can be formally expressed by writing the level shift as a complex number, $\delta E=\mathcal{E}-i\frac{1}{2}\gamma$, where γ is the reciprocal mean life of the state, the level width. In our treatment, complex values for the level shifts imply complex-valued "nuclear phase shifts." If $\tau_l=\alpha_l+i\beta_l$, the absorption cross section is given by[10]

$$\sigma_a=\pi k^{-2}\textstyle\sum_l(2l+1)(1-e^{-4\beta_l}). \tag{13}$$

In principle, inelastic scattering processes which involve nuclear excitations are included in what we have called nuclear absorption effects. However, these may be ignored safely at our energies because they are much less probable than processes in which the pion actually disappears. It will be shown in the next section that, in the low-energy region, the imaginary part of χ can be taken to be energy-independent.

III. PREDICTIONS FOR THE SCATTERING OF π^- MESONS FROM CARBON

A prediction of the scattering cross section on the basis of the level shifts is possible at present only for nuclei for which $R/a_Z<1$. Shifts of the $K_\alpha(2P\rightarrow1S)$ line have been observed in mesonic spectra from Li to F.[11] To the accuracy of these measurements, it is reasonable to attribute the line shifts entirely to a shift in the energy of the $1S$ state. This assumption is usually made because the centrifugal barrier effectively prevents the pion in a p state from interacting with the nucleus. Such an interpretation is supported by measurements of the $L_\alpha(3D\rightarrow2P)$ energies.[12] Taking the observed line shifts to be the $1S$ level shifts, we found that for s-states the effective nuclear potential is not very strong. Stearns' value of 6.7 kev for the level shift in carbon corresponds to $\chi_0=-0.20$. Assuming a potential of the form

$$\begin{aligned}V(r)&=U-(Ze^2/2R)(3-r^2/R^2),\quad r\leq R,\\&=-Ze^2/r,\quad r>R,\end{aligned} \tag{14}$$

we solved the Schrödinger equation in the region $r<R$ and found that the value of U which yields $\chi_0=-0.20$ is 7.2 Mev. Therefore, the total effective potential at the origin is 3.2 Mev. (The harmonic part of the potential was treated as a perturbation and found to produce small corrections.)

On the basis of this result and the considerations of Sec. II [see Fig. 1(b)], we have assumed that the real part of χ_0 is energy-independent in the range from the bound states to kinetic energies of the order of 10 Mev.

It is interesting to note that if the actual level shifts depend on Z like Z^4, U has approximately the same value (7.2 Mev) for the various atoms. This is suggested by the Born approximation and confirmed by a more exact calculation.

In lieu of data on the $2P$ level shift, we attempt an estimate of $\chi_1(m)$ using the known pion-nucleon interactions in p states. Following the method used in DGBT[2] for s states (see Sec. IV), we assume that near zero kinetic energy the effects of the individual nucleons are simply additive in the scattering amplitude and that, neglecting Coulomb effects, the $l=1$ phase shift is given by

$$\begin{aligned}k^{-3}\tan\delta_1=&\tfrac{1}{9}Z(2\lambda_{33}+4\lambda_{13}+\lambda_{31}+2\lambda_{11})\\&+\tfrac{1}{3}(A-Z)(2\lambda_{33}+\lambda_{31}),\end{aligned} \tag{15}$$

where

$$\lambda_{ij}=\lim_{k\rightarrow0}k^{-3}\tan\alpha_{ij},$$

and α_{ij} is a pion-nucleon phase shift with the conventional notation to denote angular momentum and isotopic spin states (α_{13}=phase shift for isospin state $\frac{1}{2}$ and angular momentum $\frac{3}{2}$). The expression (15) is at best a rough approximation to the scattering amplitude,

[10] See, for instance, J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 321.

[11] M. B. Stearns and Martin Stearns, Phys. Rev. **103**, 1534 (1956). See also *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. IX, pp. 37–41.

[12] D. West and E. F. Bradley report on upper bound of 0.1 kev for the L_α shift in fluorine [*Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. IX, pp. 40–43]. This is to be contrasted with the K_α shift for F of 25.3 kev reported by Stearns, reference 11.

and we shall only assume that it holds in the limit of zero kinetic energy. Corrections to (15) should arise from binding effects and a form factor for the spatial distribution of the nucleons. Taking (15) in the limit $k\to 0$, we can obtain a corresponding logarithmic derivative $[(1/u_1)(d/dr)u_1]_{r=R}$. To do this we use Eq. (10); neglecting all Coulomb effects, direct evaluation yields

$$\mathfrak{L}\equiv\lim_{k\to 0}\left(\frac{r}{u_1}\frac{du_1}{dr}\right)_{r=R}=\lim_{k\to 0}\left(\frac{2-3(kR)^{-3}\tan\delta_1}{1+3(kR)^{-3}\tan\delta_1}\right). \quad (16)$$

A value $\mathfrak{L}=2$ corresponds to the familiar behavior $u(r)\approx r^2$ (for p states) near the origin. The actual value of $\mathfrak{L}$ as calculated from the observed pion-nucleon p-phase shifts (see below) turns out to be appreciably smaller than 2 (1.26 for carbon). One sees easily that a very large attractive potential ($U\approx -50$ Mev) would be needed to produce this effect, and that the addition of the electric field cannot possibly alter the effect, so that Eq. (16) may be used even in the presence of the electric potential. Also, raising the kinetic energy, $E-m$, up to 10 Mev will presumably not affect the value of the wave function's logarithmic derivative near the nuclear surface. Therefore, we shall, in Eqs. (3) and (6), neglect the term α_u for p states.

On account of the resonance in the $(\frac{3}{2},\frac{3}{2})$ state of the pion-nucleon system, all λ_{ij} are small compared to λ_{33}. We have taken $\lambda_{33}=0.248$,[13] and the other $\lambda_{ij}=0$. The value for $\chi_1(m)$ thus obtained gives for carbon a $2P$ level shift of -4.3 ev $[\chi_1(m)=0.69]$. If West and Bradley's upper limit[12] is extrapolated to carbon assuming $\delta E_{2P}\sim Z^6$, a value of 9 ev is obtained for $|\delta E_{2P}|$. This guarantees that we have not underestimated the p-phase shift by more than a factor of about 2. The Z^6 dependence for δE_{2P} is analogous to the Z^4 dependence of the $1S$ level shifts, one factor Z arises from the number of nucleons in the nucleus and Z^5 from the fact that the p-state interaction energy will be proportional to the square of the gradient of the normalized Bohr orbit wave function evaluated at the origin.

It remains now to take into account the absorptive effects. As can be seen from Eq. (11), if the imaginary part of χ_l is large it will make a significant contribution to the real part, α_l, of the "nuclear phase shift." However, the widths of the K_α lines are certainly not larger than the shifts[11]; therefore, the imaginary part of χ_0 will contribute appreciably to α_0 only if it has a strong energy dependence. To investigate this, we compare the level widths with the absorption cross sections of nuclei for free pions. Writing Eq. (13) as

$$\sigma_a=4\pi k^{-2}(\beta_0+3\beta_1+\cdots), \quad (17)$$

we compare it with the relation proposed by Brueckner Serber, and Watson[14]:

$$\sigma_a=\Gamma Z\sigma(\pi^-+d\to 2n)=\Gamma Z k^{-2}(ak+bk^3), \quad (18)$$

where Γ is a factor depending mainly on the nuclear wave functions. The idea underlying this equation is that the absorption of pions is effected primarily by deuteron-like structures in nuclear matter. Analysis of stars initiated by pions has led many authors[15] to conclude that such a mechanism can account for a majority of the events they have observed. Therefore, for the sake of comparison, we shall identify (18) with (17). As Brueckner and co-authors showed,[14] $\sigma(\pi^-+d\to 2n)$ can be evaluated from the inverse (and charge-conjugate) reaction. Using the cross section for this process given by Rosenfeld,[16] for $k^2\ll 1$, we write

$$\begin{aligned}\beta_0&=(\Gamma Z/12\pi)M(0.014)kC_0^2,\\ \beta_1&=(\Gamma Z/36\pi)M(0.10)k^3C_1^2,\end{aligned} \quad (19)$$

where $m_{\text{pion}}=1$, $M=$nucleon mass. [We have included Coulomb attraction factors C_0^2 and C_1^2 in (19). These factors are appreciably different from one only near zero kinetic energy.] The above energy dependence of β_0 and β_1 at low energies is the same as is given by our formalism if the imaginary part of χ is taken to be independent of energy. Equation (12) in the approximation (12′) yields a $1S$ level width given by

$$\gamma_0=4a_Z^{-3}(\beta_0/kC_0^2)_{k=0}. \quad (20)$$

Measured pion absorption cross sections[17] indicate that $\Gamma=6$ is a reasonable value to use near zero energy. With this value, (19) and (20) predict a $1S$ level width for beryllium of 0.8 kev. West and Bradley[12] have measured the line width in Be^9 as $(1.15_{-0.4}{}^{+0.2})$ kev. In view of this agreement, we have assumed that β_0 and β_1 are given by (19) with $\Gamma=6$. The imaginary parts of χ_0 and χ_1 are then both roughly energy-independent, and their numerical values are sufficiently small so that their contribution to the real part of the "nuclear phase shifts," τ_0 and τ_1, may be neglected.

We have, therefore, calculated the real and imaginary parts of τ_0 and τ_1 separately. Their values at 5 and 10 Mev are given in Table I. The angular distributions to be expected on the basis of these phase shifts are plotted in Figs. 2(a) and 2(b). We have included for comparison the pure Coulomb cross sections (curves marked II) and have also plotted the angular distributions to be expected at these energies if only τ_0 were different from zero (curves III). The additional effect of the p-phase shift, which is somewhat uncertain on account of the

[13] H. L. Anderson and N. Metropolis, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. I, pp. 20–23. In this and the following section, units are chosen such that $\hbar=c=m_{\text{pion}}=1$.

[14] Brueckner, Serber, and Watson, Phys. Rev. **81**, 575 (1951) and **84**, 258 (1951).

[15] G. Bernardini and F. Levy, Phys. Rev. **84**, 610 (1951); Byfield, Kessler, and Lederman, Phys. Rev. **86**, 17 (1952); P. Ammiraju, thesis, Columbia University, 1956 (unpublished); A. Tomasini, Nuovo cimento **3**, 160 (1956).

[16] A. H. Rosenfeld, Phys. Rev. **96**, 139 (1954).

[17] F. H. Tenney and J. Tinlot [Phys. Rev. **92**, 974 (1953)] find that $\Gamma=(3.2_{-0.8}{}^{+1.1})$ and $(5.6_{-1.1}{}^{+2.1})$ for positive pions of 39 and 20 Mev, respectively, absorbed by beryllium.

assumptions we had to make, tends to obscure the depression due to the repulsive s-state interaction at 10 Mev. Regarding the s scattering, it should be recalled that in the calculations χ_0 was taken strictly energy-independent. If, instead, we had assumed an inside potential like (14) or Fig. 1(a), χ_0 would have a slight energy dependence as can be seen in Fig. 1(b). This would raise the values of $|\alpha_0|$ by about 5% at 5 Mev and about 15% at 10 Mev, and correspondingly causes the curves III to deviate more strongly from the Coulomb curves II.[18]

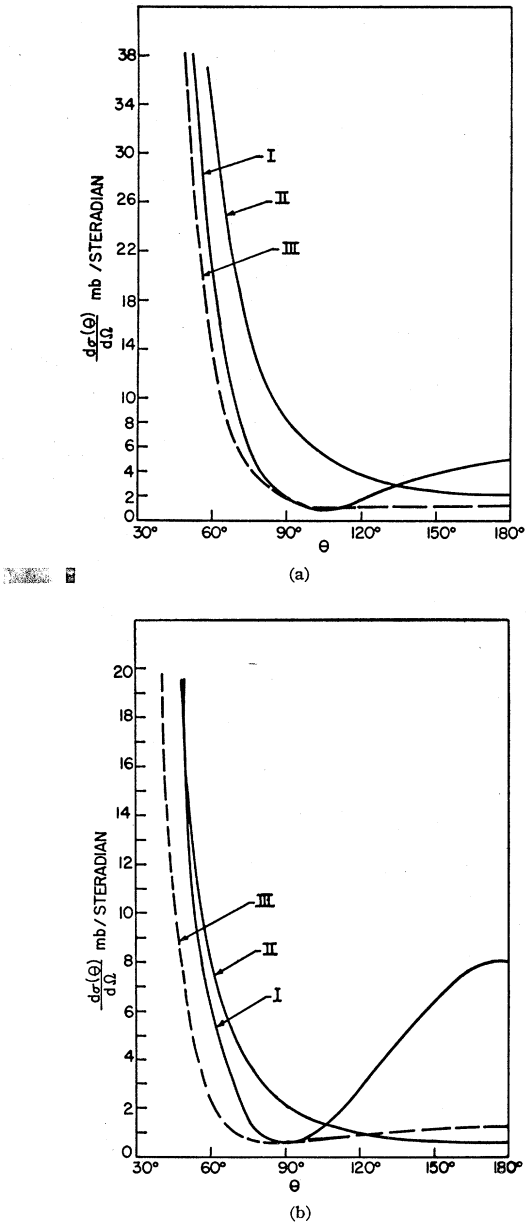

FIG. 2. Differential cross sections for the elastic scattering of negative pions from carbon. (I) Predicted cross section (nuclear interaction in both s and p states taken into account, see Sec. III); (II) pure Coulomb scattering; (III) the effect on the Coulomb scattering due to τ_0 alone. (a) $E-m=5$ Mev, (b) $E-m=10$ Mev.

IV. COMPARISON WITH MODELS BASED ON MESON FIELD THEORY

DGBT[2] have suggested that the $1S$ level shift may be understood in terms of the pion-nucleon s-wave scattering phase shifts α_1 and α_3 (isotopic spin $\frac{1}{2}$ and $\frac{3}{2}$). Assuming that the contributions to the level shift from the various nucleons are additive, they predicted that δE_{1S} should be given by $\delta E_{1S}=-\lambda/2ma_Z^3$, where $\lambda=\frac{1}{3}Z(2\lambda_1+3\lambda_3)+(A-Z)\lambda_3$ and $\lambda_i=\lim_{k\to 0}(k^{-1}\tan\alpha_i)$. For $A=2Z$, λ reduces to $\frac{2}{3}Z(\lambda_1+2\lambda_3)$.[19] Taking $\alpha_1=\lambda_1 k$ and $\alpha_3=\lambda_3 k$, Orear[20] obtained the values $\lambda_1=0.167\pm0.012$, $\lambda_3=-0.105\pm0.010$ from an analysis of all the data on low-energy pion-nucleon interactions available at that time. These values give, according to the theory of DGBT, a $1S$ level shift in carbon of 4.0 kev. Hence, with allowance for the rather large error, it seems probable that the major contribution to this level shift stems from the same interaction as gives rise to pion-nucleon s-state scattering. Additional contributions may be due to effects of the type discussed by Brueckner, and Karplus and Halpern (see below).

TABLE I. "Nuclear phase shifts" and the scattering cross section for angles greater than 30° corresponding to curves I, II, and III of Figs. 2(a) and 2(b). The last column contains the absorption cross sections (σ_a) corresponding to curves I.

$E-m$ (Mev)	τ_0 (radians)	τ_1 (radians)	σ_{I} (mb)	σ_{II} (mb)	σ_{III} (mb)	σ_a (mb)
5	$-0.122+0.039i$	$0.0285+0.0067i$	297	356	244	202
10	$-0.139+0.049i$	$0.0564+0.017i$	106	91.6	55	165

DBGT discussed the bound states only, but it may be of interest to generalize their theory to apply to energies $E>m$ for the sake of comparison with our more phenomenological approach. The crucial point will be the energy dependence of the generalized scattering length, $\lambda(k)=k^{-1}\tan(\delta_0)_{\mathrm{eff}}$. Unfortunately, there is no unique way to generalize (for $k\neq 0$) the DGBT additivity hypothesis, but it appears most natural to assume that λ, or the tangent of the phase shift, is additive. This automatically preserves the unitarity of

[18] If $\tau_1=0$, then, nonrelativistically,

$$\frac{d\sigma}{d\Omega}=\frac{Z^2e^4m^2}{4(k\sin\frac{1}{2}\theta)^4}[1-A(\sin\tfrac{1}{2}\theta)^2+B(\sin\tfrac{1}{2}\theta)^4].$$

(The angular dependence of the phase of the Coulomb scattering amplitude gives A a weak angular dependence.) At 5 Mev, the term containing B contributes substantially less than the interference term and the cross section is roughly a linear function of α_0 and β_0. At 10 Mev, however, the term containing B becomes appreciable for angles greater than 90°. Therefore, if $|\alpha_0|$ is underestimated at 10 Mev, one can expect the actual cross section to be further depressed for angles less than 90°, but increased somewhat for angles greater than 90°.

[19] Note that the two terms will tend to cancel, resulting in the rather "weak" s-state interaction we have emphasized.

[20] J. Orear, Nuovo cimento 4, 856 (1956).

the S matrix, whose diagonal form is

$$e^{2i\delta}=\frac{1+i\tan\delta}{1-i\tan\delta}=\frac{1+ik\lambda(k)}{1-ik\lambda(k)}.$$

Therefore, we choose (for $A=2Z$)

$$\lambda(k)=\tfrac{2}{3}Z[\lambda_1(k)+2\lambda_3(k)]\equiv k^{-1}\tan(\delta_0)_{\text{eff}}. \quad (21)$$

The k dependence of λ_1 and λ_3 was derived by Goldberger[21] from dispersion relations for pion-nucleon reactions. His estimates give, for $k^2\ll 1$,

$$\lambda_1(k)=\lambda_1-\tfrac{1}{3}k^2(\lambda_3+0.8\lambda_1),$$
$$\lambda_3(k)=\lambda_3+\tfrac{1}{3}k^2(0.17\lambda_3-0.50\lambda_1). \quad (22)$$

(Incidentally, it can be surmised from the weak energy dependence of these expressions that corrections to the DGBT theory[2] due to the momentum distribution of the nucleons in the nucleus will be small. Form factors, however, may not be quite negligible.) Using the values $\lambda_1=0.16$ and $\lambda_3=-0.11$ in (22) and substituting the resulting expressions into (21), we calculated the variation with k of $\tan(\delta_0)_{\text{eff}}$. It is shown in Fig. 3. For comparison, we have included in this figure the energy variation of the $l=0$ phase shifts obtained under the assumption that an energy-independent χ_0 describes the nuclear interaction, and that an energy-independent potential U describes the interaction. All curves correspond to the same zero-energy scattering length and do not include Coulomb effects.

As can be seen from the figure, in all three cases the deviation from linearity becomes appreciable only for kinetic energies greater than 5 Mev. The curvature in the cases of a potential or constant χ_0 is proportional to $A^{\frac{1}{3}}$. Therefore, for nuclei lighter than carbon, the linearity will persist to higher energies. The curvature for the effective phase shift derived from the DGBT theory, on the other hand, is independent of A and corresponds to an energy-dependent potential. Referring to Eq. (7) and Fig. 1(b), it can be seen that if b is sufficiently positive, a curvature of this type will result. An effective nuclear potential of the form $V+\frac{1}{2}bk^2$ with $V\approx 5$ Mev and $b\approx 0.1$ will give rise to the $(\delta_0)_{\text{eff}}$ of Fig. 3.

Brueckner[3] has shown that there should be a contribution to the $1S$ level shift resulting from the reaction $\pi^-+A\rightarrow$star, the nuclear absorption of pions. Karplus and Halpern[4] extended the work of Brueckner by observing that the nonrelativistic coupling of pseudoscalar pions to nucleons should be taken of the form

$$\boldsymbol{\sigma}\cdot(\mathbf{v}_\pi-\mathbf{v}_N) \quad (23)$$

[21] M. L. Goldberger, *Proceedings of the Sixth Annual Rochester Conference on High-Energy Physics, 1956* (Interscience Publishers, Inc., New York, 1956), Sec. I, p. 10.

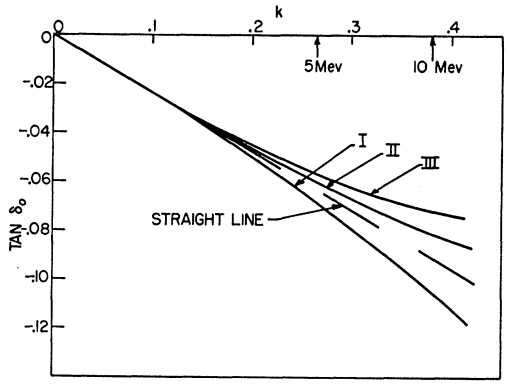

FIG. 3. The s-phase shift, neglecting Coulombic effects, for the scattering of pions from carbon as a function of pion momentum (in units $m_{\text{pion}}=c=1$). (I) Derived from DGBT theory (see Sec. IV); (II) derived from an energy-independent potential; (III) derived from an energy-independent χ_0.

in order to be invariant under Galilean transformations. The first term stems from the usual form of the interaction, namely $(f/m)\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\varphi$. To this must be added a term $\sim\partial\varphi/\partial t$, which gives, in nonrelativistic approximation, the correction term $\mathbf{v}_N$ in (23), where $\mathbf{v}_N$ stands for the average velocity of the nucleon before and after the Yukawa absorption (emission) of the pion. Karplus and Halpern used the interaction (23) to study the $1S$ level shift. Preliminary results[22] indicate that this "virtual absorption" interaction does not contribute much to the observed line shifts. An accurate estimate of this contribution, however, must await better knowledge of the momentum distribution of the nucleons in the nucleus. We have calculated the $l=0$ pion-deuteron scattering amplitude from (23), using second-order perturbation theory, and found that it is very sensitive to the wave function chosen for the nucleus. Because of the at present unavoidable uncertainties in such calculations, we should like to leave open the question of whether our phenomenological approach is still valid if the interaction represented by (23) contributes substantially to the $1S$ level shift. In such a case, a k dependence quite different from those described in Fig. 3 might arise.

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[22] R. Karplus (private communication). See also, R. Karplus and F. Halpern, Bull. Am. Phys. Soc., Ser. II, 2, 5 (1957). We would like to express our thanks to Dr. Karplus for informing us of these results.