Beta Decay of C^{14} and Nuclear Forces*

WILLIAM M. VISSCHER† AND RICHARD A. FERRELL
University of Maryland, College Park, Maryland

(Received October 3, 1955; revised manuscript received April 12, 1957)

The anomalously long lifetime of C^{14} is interpreted as being due to an accidental cancellation in the β-decay matrix element which is complete to about one part in 160. In the shell model with oscillator wave functions, general two-nucleon interactions are introduced which are linear in the velocities and which satisfy various invariance requirements. As many of the force parameters are determined from low-energy experimental nuclear data as possible. The strength of the spin-orbit force is fixed by the $p_{\frac{1}{2}}-p_{\frac{3}{2}}$ splitting in N^{15}, while the tensor force is taken from meson theory. The conditions for cancellation are then shown to be satisfied, although uncertainties in the central force prevent an exact calculation of the wave functions. These uncertainties are therefore eliminated by requiring cancellation, in addition to fitting the three lowest energy levels of N^{14}. The resulting N^{14} ground state wave function is mainly D-state, containing only 13% P- and 3% S-component, and checks satisfactorily against various experimental properties of N^{14}. In addition, the β decay of O^{14} is investigated and it is shown that the $C^{14}-O^{14}$ difference in ft-values is not adequately accounted for by the Coulomb repulsion theory of Jancovici and Talmi. The discrepancy is removed by including the slight difference in spin-orbit coupling strength for protons and neutrons, due to electromagnetic interaction with their magnetic moments. It is also pointed out that a nonlinearity of about 10% should be expected in the O^{14} Fermi-Kurie plot. In the appendices are a treatment of nucleon hole conjugation, an extension of a proof by Inglis on the impossibility of cancellation without the tensor force, a calculation of the $n-\alpha$ spin-orbit splitting, and an approximate treatment of the binding energy of the α particle.

I. INTRODUCTION

EXPERIMENTAL evidence shows conclusively that the β-decay of C^{14} is allowed. The spins of both the parent and daughter nuclei have been measured to be zero and one, respectively. Thus the spin change is one, which corresponds to an allowed Gamow-Teller transition. The electron spectrum has also been investigated[1] and found to have an allowed shape. The Gamow-Teller coupling constant is known to be nonzero from various sources, e.g., the fast decay of He^6, also a $0+\rightarrow 1+$ transition. A theoretical calculation, using the presently establishing value of the coupling constant, would lead one to expect the C^{14} half-life to be a few weeks instead of the observed 5580 years. Thus the observed half-life is about 10^5 times longer than expected, which can result only from an anomalously small β-decay matrix element. Since the individual terms in the matrix element cannot be expected to be so small, such a small net value can only be accounted for by an accidental cancellation. The conditions for this cancellation, in which the two-nucleon tensor interaction plays an essential role, are studied in Sec. II. Out of the large variety of possibilities, the one actually fulfilled by the C^{14} and N^{14} ground-state wave functions can be determined only by calculating these wave functions from first principles. This requires numerical values of the Hamiltonian matrix elements for the s^4p^{10} configurations. These are computed in Secs. III and IV from a basic two-nucleon interaction containing central, spin-orbit, and tensor parts. The energy-level spectrum thus obtained for N^{14} is in satisfactory qualitative agreement with experiment, with the deviations being attributed to uncertainties in the central-force parameters. Since these uncertainties affect only the diagonal elements, we adopt in Sec. V a semiempirical approach which fixes the diagonal elements by fitting the three lowest N^{14} energy levels, as well as requiring cancellation in the β-decay matrix element. The resulting N^{14} ground state wave function is mainly D-state, and contains only 13% P- and 3% S-component. From this wave function other properties of N^{14} are calculated and shown to be consistent with experiment.

In Sec. VI the β decay of O^{14} is studied. It is shown that the Coulomb interaction of the proton holes is too weak a perturbation to explain the observed difference in the ft-values of C^{14} and O^{14}. There is an additional perturbation of the same order of magnitude due to the slightly different spin-orbit splittings of a proton hole and a neutron hole in the p-shell. The sum of the two perturbations is sufficient to account for the observed $C^{14}-O^{14}$ difference. Expected deviations from linearity in the Fermi-Kurie plots are also discussed in Sec. VI. A brief summary constitutes Sec. VII.

II. BETA DECAY IN THE SHELL MODEL

We shall use the shell model throughout, and assume that C^{14} and N^{14} are pure s^4p^{10} configurations. The wave functions for the 14-nucleon systems are simpler to write in the hole representation than in the particle representation, since there are only two holes compared to ten p-shell nucleons. The transformation to the hole representation (hole-conjugation) can be performed quite simply with the aid of second-quantization, and is discussed in Appendix I. The results are, as far as the forces between the two holes are concerned, that the holes may be treated as if they were particles. In their

* Investigation supported by the Office of Naval Research and the National Science Foundation.

† Present address: Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

[1] Pohm, Waddell, Powers, and Jensen, Phys. Rev. **97**, 432 (1955). We have not made a systematic survey of the literature on this point, but in addition see A. Moljk and S. C. Curran, Phys. Rev. **96**, 395 (1954), and J. P. Mize and D. J. Zaffarano, Phys. Rev. **91**, 210(A) (1953).

energy of interaction with the rest of the nucleus, however, holes and particles differ in sign, since a hole is nothing but the absence of a particle and its interactions.

The two holes in the p shell may combine in singlet or triplet spin states, and in S, P, or D space states to give wave functions with $J=0$, 1, 2, or 3, symmetric or antisymmetric under interchange of space and spin coordinates—corresponding to singlet and triplet isotopic spin states, respectively. Charge-independent forces will be used throughout (we neglect the relatively small Coulomb repulsions between protons), so that isotopic spin will be a good quantum number.

The ground state wave functions are [we shall use throughout the notation (J,T) for specifying the quantum numbers]

$$\begin{aligned} &N^{14}: \ \Psi(1,0)=C_S\psi(^3S_1)+C_P\psi(^1P_1)+C_D\psi(^3D_1), \\ &C^{14}: \ \Psi(0,1)=C_S'\psi(^1S_0)+C_P'\psi(^3P_0), \end{aligned} \tag{1}$$

where

$$\psi(^3D_1)=\frac{1}{\sqrt{10}}(D^0\chi_1^1-\sqrt{3}D^1\chi_1^0+(6)^{\frac{1}{2}}D^2\chi_1^{-1})T_0^0, \text{ etc.} \tag{2}$$

The space and spin functions and the other LS-basis wave functions are exhibited explicitly by Inglis.[2] The first, second, and third factors in each term of Eq. (2) are the space, spin, and isotopic spin parts, respectively, in the $L^2M_LS^2M_ST^2T_z$ representation.

In Appendix I it is proved that the β-decay operator which acts on the holes is just minus that which acts on the nucleons. The square of the Gamow-Teller operator, summed over final states, is

$$\left|\int\boldsymbol{\sigma}\right|^2=6\left(C_SC_S'-\frac{C_PC_P'}{\sqrt{3}}\right)^2. \tag{3}$$

In general, for any allowed β decay,

$$\left|\int 1\right|^2+R\left|\int\boldsymbol{\sigma}\right|^2=(ft)^{-1}\times(6.55\pm0.15)\times10^3 \text{ sec}. \tag{4}$$

The universal constant on the right hand side has been determined by Gerhart[3] from the O^{14} β decay. R is the ratio of the squares of the Gamow-Teller and Fermi coupling constants. From the neutron β decay,[4] $R=1.42\pm0.19$. Since the experimental ft-value[5] for C^{14} is 1.12×10^9 sec, and the Fermi term vanishes in this case, we find

$$|C_SC_S'-(C_PC_P'/\sqrt{3})|=(8.3\pm0.6)\times10^{-4}. \tag{5}$$

To account for this small experimental value it is necessary either that the terms on the left-hand side are individually very small, which cannot be the case because of the strong spin-orbit force present, or that they cancel one another. Inglis[2] considered this latter possibility in some detail for a central interaction between the holes, and single-hole spin-orbit force of the form

$$V_{s.o.}=a\sum_i\mathbf{l}_i\cdot\mathbf{s}_i, \tag{6}$$

and proved that the two terms cannot cancel, but always have the same sign.[6] From his work[7] the numerical value of the left-hand member of Eq. (5) can be estimated at 0.275.

There are other ways of accounting for the small matrix element, namely, accidental cancellation induced by different kinds of forces than those heretofore considered, and cancellation of the terms occurring in Eq. (4) by others arising from higher configurations. The latter possibility was considered by Inglis who showed that in order for configuration mixing to be solely responsible, the amplitude of the admixture would have to be quite large (about 0.25 in both the initial and final states). On experimental grounds such a large admixture seems to be excluded. Standing,[8] in analyzing the results of his measurement of the angular distribution of dueterons from the $N^{14}(p,d)N^{13}$ reaction, has set upper limits of 1% and 3% on the probabilities of the higher configurations $(1p)^{-4}(2s)^2$ and $(1p)^{-4}(2s)(1d)$, respectively, in the ground state of N^{14}.

The former possibility is therefore more promising, and it is clear that the tensor force should be considered, since it is known to be strong from the two-body data. Jancovici and Talmi[9] have, in fact, demonstrated that the tensor force is capable of causing cancellation in in Eq. (4), but, as they point out, they used a tensor force so large that the energy level spectrum of N^{14} was completely wrong.[10] They employed oscillator wave

[2] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).
[3] J. B. Gerhart, Phys. Rev. **95**, 288 (1954).
[4] J. M. Robson, Phys. Rev. **83**, 349 (1951), and private communication.
[5] A. M. Feingold, Revs. Modern Phys. **23**, 10 (1951).

[6] R. Schulten and R. A. Ferrell [Phys. Rev. **94**, 739 (1954)], making use of the shell-model wave functions obtained by R. Schulten [Z. Naturforsch. **8a**, 759 (1953)] in a calculation of the energy-levels of p-shell nuclei, have calculated the β-decay lifetimes of the radioactive ones, including C^{14}. Using an interparticle spin-orbit force of the form

$$V_{s.o.}(1,2)=\text{const}\times(\mathbf{r}_{12}\times\mathbf{p}_1\cdot\boldsymbol{\sigma}_1+\mathbf{r}_{21}\times\mathbf{p}_2\cdot\boldsymbol{\sigma}_2)$$

they found that a partial cancellation in the β-decay matrix element did occur, raising the ft-value by about a factor of one hundred from that obtained with pure j-j coupling. Were it not for the fact that this interparticle spin-orbit force does not have exactly the same matrix elements as the one-nucleon force of Eq. (6), this result would be directly contradictory to Inglis' theorem. However, they must be still in error because Inglis' theorem can be extended to include spin-orbit forces of quite a general form. (See Appendix II.) Schulten worked in the j-j representation and a relatively small error in his expansion coefficients for the N^{14} ground state, when transformed to L-S representation, could have led to the wrong sign for the small $^3S_1$ or $^1P_1$ component.
[7] Reference 2, Fig. 20 ($a/K=-5.6$).
[8] K. G. Standing, Phys. Rev. **101**, 152 (1956).
[9] B. Jancovici and I. Talmi, Phys. Rev. **95**, 289 (1954).
[10] A. M. Lane and J. P. Elliott (unpublished manuscript) have had some success in reducing the strength of the tensor force necessary to produce cancellation. Their tensor force is still so large, however, as to affect adversely the energy level spectrum. This difficulty can be attributed to the Yukawa shape, which they also used. The off-diagonal element H_{SD} which produces the

functions and Yukawa interaction potentials with a Serber (half-ordinary, half-Majorana) exchange character, and a spin-orbit force of the form of Eq. (6). We have done the same calculation, in an attempt to find how much variation in the relative force strengths is permissible, using Gaussian rather than Yukawa potentials (still with Serber exchange), and have found that quite a variety of strengths is consistent with a zero β-decay matrix element. Our results, which are not inconsistent with those of Jancovici and Talmi (they are not really comparable, because we used a different well-shape, and did not consider tensor forces as strong as theirs) are given in Fig. 1.[11] It is evident from the figure that, for a small spin-orbit force a small tensor force suffices, while a large tensor force is necessary to give cancellation with a large spin-orbit force. That this relation between cancellation-giving forces should hold can be made reasonable by the following argument.

Let us assume that the N^{14} ground state is to a large extend $^3D_1$ (this is indicated by Standing's[8] experimental results and will be justified below), and use first-order perturbation theory to get expressions for the C's of the left-hand side of Eq. (5) from the secular equations

$$\begin{aligned}(H_{SS}-E)C_S+H_{SP}C_P+H_{SD}C_D=0, \quad \text{etc.},\\ (H'_{SS}-E')C_S'+H'_{SP}C_P'=0, \quad \text{etc.},\end{aligned} \tag{7}$$

where H'_{SS}, etc., are matrix elements of the Hamiltonian. Approximately, the left-hand member of Eq. (5) is then

$$C_SC_S'-\frac{C_PC_P'}{\sqrt{3}}=C_S'\left(\frac{H_{SD}}{E-H_{SS}}-\frac{1}{\sqrt{3}}\frac{H_{PD}}{E-H_{PP}}\frac{H'_{SP}}{E'-H'_{PP}}\right).$$

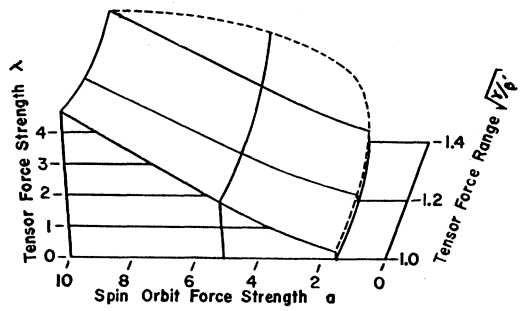

FIG. 1. Values of the tensor-force strength parameters which give zero β-decay matrix element as a function of spin-orbit strength and the tensor-force range. (S.o. strength is in Mev.) Cancellation does not occur for values of the parameters beyond the dashed line. Gaussian shape and Serber exchange were used for both tensor and central forces, and the central-force strength was such that $L=6K=-4.5$ Mev. λ is the ratio of the tensor to the central force strength. The naturally occurring case is given by $a\approx 4$, $\lambda\approx 1$.

Here E is the lowest eigenvalue of the 3×3 $(J,T)=(1,0)$ Hamiltonian matrix and E' the lowest eigenvalue of the 2×2 $(J,T)=(0,1)$ matrix. Since the lowest eigenvalue of a matrix is less than any diagonal element, all the energy denominators are negative. H_{PD} and H'_{SP} arise from the spin-orbit force only, and are both negative. Hence the second term within the parentheses is negative. For cancellation H_{SD} must therefore be negative, which is true for an attractive force.[12] The behavior of the graph (Fig. 1) may be qualitatively understood from Eq. (7), especially for high spin-orbit strengths, when $E'-H'_{PP}$ is roughly proportional to H'_{SP}. In that case the tensor force required for cancellation increases in proportion to the spin-orbit strength.

To enable more quantitative statements to be made, we shall now turn to the calculation of the Hamiltonian matrix elements for the s^4p^{10} configurations in the harmonic-oscillator shell model.

III. HAMILTONIAN MATRIX ELEMENTS

Harmonic-oscillator wave functions are generally employed in shell-model calculations with two-body forces because they make the integrals encountered easy to evaluate and because they can be considered as approximations to the true Hartree-Fock wave functions for the system. Their use allows the integration over relative coordinates to be simply separated from that over the center-of-mass coordinates. This fact has been pointed out and exploited by Talmi[13] in his paper on harmonic-oscillator nuclear spectroscopy. The one-nucleon oscillator wave functions contain the Gaussian factor $\exp(-\frac{1}{2}\gamma r^2)$, where $\gamma^{-\frac{1}{2}}$ determines the size (i.e., root-mean-square radius) of the nucleus. From the $N^{15}-O^{15}$ Coulomb energy difference, Carlson and Talmi[14] find $\gamma^{-\frac{1}{2}}=1.68\times 10^{-13}$ cm. The tacit assumption

cancellation, is sensitive to the shape and range of the tensor force. The long-tailed Yukawa potential is less effective than the Gaussian shape and there is no Yukawa tensor force of reasonable strength which gives cancellation, [D. T. Goldman and R. A. Ferrell, Bull. Am. Phys. Soc. Ser. II, **2**, 27 (1957)]. There is no *a priori* preference for the Yukawa over the Gaussian shape. The meson theory tensor force, which is used in Secs. V and VI below, has a more singular radial dependence, and approaches the Yukawa shape only at large internucleon separations. While this manuscript was in revision we received a preprint of a paper by Elliott [J. P. Elliott, Phil. Mag. **1**, 503 (1956)], in which, by avoiding the assumption of any specific tensor shape, he reaches the same general conclusion as the one we had arrived at in reference 11, *viz.*, that a moderate tensor force can give cancellation. The present work is an effort to go beyond this stage and to give a more detailed and specific picture of the cancellation, as well as to relate it to other sources of information on the nuclear forces. *Note added in proof.*—R. Hueper [Z. Naturforsch. **12a**, 295 (1957)], has recently given a discussion of the shell model as applied to some of the p-shell nuclei, including N^{14}. He remarks that the effective central force necessary to account for the N^{14} ground state is stronger than that which agrees with the scattering data, and considers this a difficulty for the theory. But as discussed in Sec. III, C below, this situation is just what one would expect in the Brueckner theory, and we are inclined not to agree with Hueper's statement that the beta-decay of C^{14} remains an "unsolved problem."

[11] These results were reported at the 1955 Washington Meeting of the American Physical Society [W. M. Visscher and R. A. Ferrell, Phys. Rev. **99**, 649(A) (1955)].

[12] This is the case only for a tensor force of not too long range.

[13] I. Talmi, Helv. Phys. Acta **25**, 185 (1952).

[14] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954).

TABLE I. Hamiltonian matrix elements for the s^4p^{10} configurations. L and K are the central-force direct and exchange integrals in the p-shell, while I_l' and I_l'' are the tensor and two-nucleon spin-orbit interaction Talmi integrals, respectively; a is the strength of the one-nucleon spin-orbit force. Exchange coefficients are denoted by w, w', etc.

(J,T)	$^{(2S+1)}L_J$			
(1,0)	$^3S_1$	$L+2K$	$(\frac{2}{3})^{\frac{1}{2}}a$	$(3I_1'-5I_2')/\sqrt{5}$
	$^1P_1$	...	$(w-m-b+h)(L-3K)$	$-(\frac{5}{6})^{\frac{1}{2}}a$
	$^3D_1$	...	...	$L-K-I_2'-\frac{3}{2}a-3I_2''$
(2,0)	$^3D_2$	$L-K+I_2'-\frac{1}{2}a-I_2''$		
(3,0)	$^3D_3$	$L-K-(2/7)I_2'+a+2I_2''$		
(0,1)	$^1S_0$	$(w+m-b-h)(L+2K)$		$-\sqrt{2}a$
	$^3P_0$	...		$(w-m+b-h)(L-3K)+2(w'-m')I_1'$ $-a-2(w''-m'')I_1''$
(1,1)	$^1P_1$	$(w-m+b-h)(L-3K)-(w'-m')I_1'$ $-\frac{1}{2}a-(w''-m'')I_1''$		
(2,1)	$^3P_2$	$(w-m+b-h)(L-3K)+\frac{1}{5}(w'-m')I_1'$ $+\frac{1}{2}a+(w''-m'')I_1''$		$a/\sqrt{2}$
	$^1D_2$	...		$(w+m-b-h)(L-K)$

that $\gamma^{-\frac{1}{2}}$ differs only slightly for neighboring mass numbers is borne out by the high-energy electron scattering experiments on C^{12}, which yield a value of 1.58×10^{-13} cm.[15] That it is probably justified to use the same Gaussian factor in both the s and p shells follows from an investigation by Jancovici.[16]

Subject to the restrictions of charge and velocity independence and the simplifying assumption of exchange-independent shape, the most general form for the two-body central and tensor potential is

$$V_C+V_T=-V_0(w+mP_{12}+bQ_{12}+hP_{12}Q_{12})U_C(r_{12}) -V_0'(w'+m'P_{12})S_{12}U_T(r_{12}), \quad (8)$$

where P_{12} and Q_{12} are the Majorana and Bartlett exchange operators, S_{12} is the usual tensor operator

$$S_{12}=3\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{r}_{12})(\boldsymbol{\sigma}_2\cdot\mathbf{r}_{12})}{r_{12}^2}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2, \quad (9)$$

and $w+m+b+h=w'+m'=1$.

In the choice of the interparticle spin-orbit force we follow Eisenbud and Wigner,[17] and write the following most general charge-independent expression linear in the relative velocity which is invariant under rotation, Galilean acceleration, space inversion, time reversal, and exchange of the two nucleons:

$$V_{s.o.}=-V_0''\hbar^{-1}(w''+m''P_{12})(\mathbf{s}_1+\mathbf{s}_2) \cdot(\mathbf{r}_{12}\times\mathbf{p}_{12})U_{s.o}(r_{12}), \quad (10)$$

where

$$\mathbf{p}_{12}=\mathbf{p}_1-\mathbf{p}_2.$$

P_{12} is actually a velocity-dependent operator, but in a system obeying the generalized Pauli principle, it is equivalent to minus the charge-exchange operator because $(\mathbf{s}_1+\mathbf{s}_2)$ is nonzero only between even spin states, for which the Bartlett operator is equivalent to unity. Summing this spin-orbit force over the closed shell in the hole-representation is illustrated in Appendix I, and leads to an effective one-body force of the form used by Inglis and Jancovici and Talmi, and which is expressible in the form of Eq. (6). We note that an expression of this form describes the entire spin-orbit energy of a single hole [Eq. (12) below gives the value of a for N^{15}], thus affording a way of determining one of the spin-orbit parameters from experimental data. In the 14-nucleon system, however, part of the contribution of the spin-orbit energy to the diagonal matrix elements stems directly from the interaction between two holes. This interaction, of course, cannot be expressed as a one-body force, but will be shown in Sec. IV to be small.

The matrix elements of $(V_C+V_T+V_{s.o.})$ in the s^4p^{10} configuration are given in Table I.[18] L and K are the central-force direct and exchange integrals and can be expressed as

$$L=\tfrac{3}{4}I_0-\tfrac{1}{2}I_1+\tfrac{3}{4}I_2, \qquad K=\tfrac{1}{4}I_0-\tfrac{1}{2}I_1+\tfrac{1}{4}I_2, \quad (11)$$

[15] J. H. Fregeau, Phys. Rev. **104**, 225 (1956); R. A. Ferrell and W. M. Visscher, Bull. Am. Phys. Soc. Ser. II, **1**, 17 (1956) and Phys. Rev. **104**, 475 (1956). Three corrections have been applied to the published value of $\gamma^{-\frac{1}{2}}=1.63\times10^{-13}$ cm. First of all, correction for the Born approximation brings in a reduction of 1.4% according to Ravenhall (as reported by Fregeau). Allowing for the proton root-mean-square radius of 0.72×10^{-13} cm further reduces $\gamma^{-\frac{1}{2}}$ by 4.5%, while correcting the theoretical C^{12} rms radius expression for the lack of center-of-mass motion (see Appendix III) increases $\gamma^{-\frac{1}{2}}$ by 2.8%. Thus there is a net reduction of 3.1% in the previous value of $\gamma^{-\frac{1}{2}}$.

[16] B. Jancovici, Compt. rend. **240**, 1608 (1955). (Jancovici's s-p interaction energy is in error. The correction is given in footnote 5 of a paper by the present authors [Phys. Rev. **102**, 450 (1956)].)

[17] L. Eisenbud and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **27**, 281 (1941); R. G. Sachs, *Nuclear Theory* (Addison-Wesley Press, Cambridge, 1953), p. 215.

[18] J. P. Elliott, Proc. Roy. Soc. (London) **A218**, 345 (1953), and T. Regge, Nuovo cimento **11**, 285 (1954), have tabulated most of these matrix elements.

where

$$I_l=-V_0\int_0^\infty \exp(-\tfrac{1}{2}\gamma r^2)U_C(r)r^{2l+2}dr \Big/ \int_0^\infty \exp(-\tfrac{1}{2}\gamma r^2)r^{2l+2}dr \quad (12)$$

are the central-force Talmi integrals. The I_l' and I_l'' which appear in Table I are the corresponding Talmi integrals for the tensor and spin-orbit force, respectively. They are obtained from Eq. (12) by replacing V_0 by V_0' or V_0'' and by replacing U_C by U_T or $U_{s.o.}$. The parameter a which also appears in Table I is just the single-hole spin-orbit strength parameter of Eq. (6). In terms of the basic two-nucleon spin-orbit force, it is (see Appendix I)

$$a=-9(w''-m'')I_1''-5I_2''. \quad (13)$$

We would be in a position to determine from Table I the ground-state wave functions as well as the properties of the other stationary states of C^{14} and N^{14} which arise by excitation within the p shell, if we knew the values of the exchange parameters w, m, etc., and of the force strengths V_0, V_0', and V_0''. Approximate numbers will now be attached to these.

IV. NUCLEAR FORCE DETERMINATION

A. Spin-Orbit Force

Hochberg *et al.*[19] have analyzed the scattering of neutrons by alpha-particles and have been able to fit the experimental phase shifts with a two-nucleon spin-orbit force of the form of Eq. (10), provided

$$(w''-\tfrac{1}{2}m'')V_0''=V_0''-1.5m''V_0''=4.5 \text{ Mev}. \quad (14)$$

They chose the Gaussian shape $U_{s.o.}=\exp(-\beta''r_{12}^2)$, with $\beta''=0.266\times10^{26}$ cm^{-2}, (corresponding to a range of $\beta''^{-\frac{1}{2}}=1.94\times10^{-13}$ cm). An alternative approximate derivation of Eq. (14) is given in Appendix III, where, instead of treating (as one should) the mass-five system as in a scattering state, we calculate the $p_{\frac{1}{2}}-p_{\frac{3}{2}}$ spin-orbit splitting using bound-state oscillator wave functions. The spin-orbit splitting in the mass-fifteen nuclei can also be expressed in terms of the basic interaction parameters. The Gaussian shape makes the Talmi integrals especially easy to evaluate (they become simply gamma functions of half-odd-integer argument), and one obtains

$$I_l''=-V_0''(1+2\beta''/\gamma)^{-l-\frac{3}{2}}. \quad (15)$$

Substituting into Eq. (13) and setting $a=4.22$ Mev (the splitting in N^{15} is $3a/2=6.33$ Mev) gives

$$1.11V_0''-1.82m''V_0''=4.22 \text{ Mev}. \quad (16)$$

One can solve the simultaneous equations (14) and (16) and obtain reasonable values for the quantities V_0'' and $m''V_0''$. It should be noted, however, that the corresponding coefficients in the two equations have roughly the same values. The relatively small differences cannot be regarded as significant. We therefore prefer to regard the two equations as redundant. We retain Eq. (16) and drop Eq. (14), since it provides no additional information.

It might be expected that the spin-orbit splitting in the mass-seventeen nuclei would yield additional information on the spin-orbit interaction, but, designating the one-nucleon coupling parameter in the d-shell by a_d, we find[20,21]

$$(15/2)I_1''+(21/2)I_3''-m''(15I_1''-3I_2''+21I_3'')=-0.93V_0''+1.74m''V_0''=a_d. \quad (17)$$

Thus once again we are disappointed by the practical identity of the coefficients in the left hand member with those of Eq. (16). More serious than a mere redundancy, however, is a discrepancy in the values of the right hand members. The $d_{\frac{3}{2}}-d_{\frac{5}{2}}$ splitting in O^{17} is $-(5/2)a_d=5.08$ Mev. Thus $a_d=-2.03$ Mev, putting Eq. (17) into stark contradiction with Eq. (16). According to Eq. (16) the coupling in the d-shell should be of about the same magnitude as in the p-shell. If this were the case, the $d_{\frac{3}{2}}$ state in O^{17} would lie at 10.6 Mev, or at about twice the observed excitation energy.[22,23] This discrepancy can be removed by noting that in addition to the spin-orbit splitting calculated here there are two other effects which tend to compensate and shift the $d_{\frac{3}{2}}$ level back down. First, there is the Ehrman-Thomas shift.[24] The last neutron in O^{17} is very weakly bound (with a binding energy of only 4.14 Mev in the ground state), and an excitation energy of 10.6 Mev would take it far above the threshold for free-neutron emission. As a consequence the wave function will accommodate to more relaxed boundary conditions, thus lowering the energy of the (virtual) $d_{\frac{3}{2}}$-level. The second effect is the "pushing-down" of the single-nucleon level by mixing with other $J=\frac{3}{2}$ levels which arise from 2+ core excitation, coupled to the $d_{\frac{5}{2}}$ neutron orbital. Since neither of these shifts has been calculated, we can only

[19] Hochberg, Massey, Robertson, and Underhill, Proc. Phys. Soc. (London) **A68**, 746 (1955).

[20] According to the $O^{17}-F^{17}$ Coulomb energy difference calculation of Carlson and Talmi (reference 14) the d-oscillator wave function has exactly the same Gaussian factor as the p-orbitals.

[21] Both Eq. (13) and Eq. (17) for the special case of $m''=0$ can be obtained from Talmi's paper (reference 13, Sec. 10) after correcting his results by a factor of two. His expressions for the spin-orbit splitting are too small by this amount, an error which seems to be due to his identifying the operator $\mathbf{r}_{12}\times(\mathbf{p}_1-\mathbf{p}_2)$ with the relative angular momentum. [$\mathbf{r}_{12}\times(\mathbf{p}_1-\mathbf{p}_2)$ is actually *twice* the relative angular momentum operator.]

[22] In first presenting this result [Bull. Am. Phys. Soc. Ser. II, **1**, 16 (1956)] we used a specific choice of strength and exchange coefficients. It is clear, however, from the above discussion, that the result is independent of such details, provided only that the p-shell splitting be correctly accounted for.

[23] This same result has also been obtained by C. A. Pearse, [Phys. Rev. **106**, 544 (1957)]. See also J. P. Elliott and A. M. Lane, Phys. Rev. **96**, 1160 (1954) and references to earlier work given there.

[24] J. B. Ehrman, Phys. Rev. **81**, 412 (1951); R. G. Thomas, Phys. Rev. **88**, 1109 (1952).

assume here that when the observed $d_{\frac{3}{2}}$ excitation energy is corrected for them, the intrinsic d-shell spin-orbit splitting will be found to be of the order of 10 Mev, and thus consistent with the p-shell splitting.

Since the d-shell splitting provides no new information, we have only the single Eq. (16) to determine the two unknowns V_0'' and $m''V_0''$. We are forced to make an arbitrary assumption, and we choose as the simplest, $m''=0$. Thus we have only a Wigner-type spin-orbit interaction with strength $V_0''=3.80$ Mev. This is a relatively weak interaction and the two-hole terms in the matrix elements of Table I, which depend on the Talmi integrals $I_1''=-0.38$ Mev and $I_2''=-0.15$ Mev, can be neglected compared to the cumulative contributions of the interaction, which depend on the much larger quantity, $a=4.22$ Mev. The error introduced by this approximation is of the order of a few tenths of an Mev, and is well within the accuracy which can be expected from the present rather rough shell-model calculations.

B. Tensor Force

According to the meson theory of nuclear forces[25] the tensor interaction between nucleons is given, to lowest order in the coupling constant f, by

$$V_T=-f^2\mu c^2(\tfrac{1}{3}+\tfrac{2}{3}P_{12})S_{12}U_T(r_{12}), \tag{18a}$$

where

$$U_T(r)=\left(1+3\frac{\lambda\!\!\!^{-}}{r}+3\frac{\lambda\!\!\!^{-2}}{r^2}\right)\frac{\exp(-r/\lambda\!\!\!^{-})}{r/\lambda\!\!\!^{-}}. \tag{18b}$$

Here $\lambda\!\!\!^{-}=1.41\times10^{-13}$ cm and $\mu c^2=140$ Mev are the pion Compton wavelength and rest energy, respectively. Thus the tensor force is two-thirds Majorana and only one-third Wigner. Its strength is fixed by the recent determination of the coupling constant from dispersion theory and experimental scattering data. Haber-Schaim[26] finds $f^2=0.082$, which gives $V_0'=f^2\mu c^2=11.4$ Mev.

It has recently been noted that this tensor force gives a good account of the quadrupole moment of the deuteron.[27] This may seem surprising in view of the fact that Eq. (18b) cannot be expected to be valid for internucleon separations less than $\lambda\!\!\!^{-}$. At these smaller distances there will also be appreciable contributions from terms in the interaction which are of higher order in $f^2\exp(-r/\lambda\!\!\!^{-})$, corresponding to the exchange of more than one meson. The operator S_{12}, however, is only effective for nucleon-nucleon encounters involving some relative angular momentum. In such cases the relative wave function vanishes at small distances, so that $U_T(r)$ is actually not "sampled" in the region where it is not correctly represented by Eq. (18b). Also for this reason the singular behavior of $U_T(r)$ causes no difficulty. Hence we feel that it is justified to use Eqs. (18a,b) to calculate the Hamiltonian matrix elements for the mass fourteen system. The Talmi integrals can be expressed in terms of error functions. Their numerical values are[28] $I_1'=-3.86$ Mev, $I_2'=-1.16$ Mev. The critical S-D off-diagonal element in the $(J,T)=(1,0)$ matrix has consequently the value $H_{SD}=-2.58$ Mev. As will be seen below, this is quite sufficient to change the sign of the S-component in the N^{14} ground state and bring about cancellation in the β-decay matrix element.

C. Central Force

The central potential which is deduced from meson theory in lowest order is of Yukawa shape but with an additional repulsive hard core. The latter is difficult to take into account properly in the shell model, which is essentially the Hartree-Fock method of handling the many-nucleon problem by approximating the wave function by a single Slater determinant or at most only a few Slater determinants. This is because the hard core establishes nucleon-nucleon correlations in the many-nucleon wave function which the Hartree-Fock method can describe only by configuration mixing, involving in the wave function a sum of a large number of Slater determinants. An alternative method of dealing with the correlations is to apply the "model operator" of Brueckner.[29] In this approach the correlations in the wave function are transformed away, restoring the simple shell-model picture, and at the same time the interaction is replaced by an effective interaction. The latter is defined such that its matrix elements for nucleon-nucleon scattering in Born approximation are equal to the corresponding matrix elements of the so-called reaction matrix. Thus the effective nucleon-nucleon interaction in the shell model bears no *a priori* simple relationship to the basic meson-theoretic interaction potential.

Because of the difficulty of systematically following the Brueckner approach at the present time, we shall make the following arbitrary but nevertheless reasonable assumptions concerning the effective interaction:

1. The spin-orbit and tensor forces are unaltered by the model operator except for additional contributions which they make to the effective central force. These latter contributions occur only for triplet nucleon-nucleon interactions and vanish for singlet encounters.

2. The hard core of the central interaction is removed by the model operator so that the resulting effective interaction is satisfactorily described by the Gaussian $U_C(r)=\exp(-\beta r^2)$.

[25] H. A. Bethe and P. Morrison, *Elementary Nuclear Theory* (John Wiley and Sons, Inc., New York, 1956), second edition, p. 154.

[26] U. Haber-Schaim, Phys. Rev. **104**, 1113 (1956). The roughly 20% uncertainty in the value of f^2 does not greatly affect the present calculations.

[27] Iwadare, Otsuki, Tamagaki, and Watari, Progr. Theoret. Phys. Japan **15**, 86 (1956).

[28] We wish to thank Mr. D. T. Goldman for carrying out this computation for us.

[29] For a recent discussion of this theory see H. A. Bethe, Phys. Rev. **103**, 1353 (1956).

3. The Serber condition applies to the effective potential. These assumptions ride rough-shod over many important points, such as the fact that some, but probably not all, of the effective spin-orbit interaction arises from the tensor force.[30] In addition we now do further violence to the Brueckner method in that we determine the effective central strength for singlet interactions by simply using the results of the low-energy scattering calculations of Blatt and Jackson,[31] instead of working with the Born approximation. From their graphs we find that a scattering length and an effective range for singlet proton-proton scattering of -16×10^{-13} cm and 2.66×10^{-13} cm, respectively, are accounted for by a Gaussian potential of strength

$$(w+m-b-h)V_0=32.9 \text{ Mev}, \tag{19}$$

and range $\beta^{-\frac{1}{2}}=1.73\times10^{-13}$.

The effective strength for triplet interactions could be determined from the binding energy of the deuteron. We prefer, however, to get this information from a case in which the nuclear matter is more nearly saturated and thus more closely approximates the situation in the mass-fourteen nuclei. He^4 is the simplest such nucleus and is studied by an approximate method in Appendix IV. There it is shown that the binding energy is accounted for provided

$$(w+m)V_0=42.4 \text{ Mev}. \tag{20}$$

From Eqs. (19) and (20) we find $V_0=51.9$ Mev, which is actually about the correct strength to reproduce the deuteron binding energy. Further we have $w+m=0.817$ and $b+h=0.183$, so with assumption 3 above we obtain $w=m=0.408$ and $b=h=0.092$. The Talmi integrals can be calculated from Eq. (15) above, simply by dropping the primes. We find $I_0=-10.58$ Mev, $I_1=-3.67$ Mev, and $I_2=1.27$ Mev. Substituted into Eq. (11) these yield $L=-7.05$ Mev and $K=-1.12$ Mev, with a ratio of direct integral to exchange integral of $L/K=6.29$.

D. Energy Levels

With the above nuclear force parameters the matrix elements of Table I have been evaluated and are exhibited in Table II. (All entries are again measured in units of Mev.) The (1,0) 3×3 matrix has the eigenvalues -13.14, -10.00, and 2.50. The lowest of these corresponds to the N^{14} ground-state energy, and agrees quite well with the experimental value of -13.21 Mev. (Such close agreement was not expected and can only be regarded as fortuitous.) This value is arrived at by noting that the preparation of the ground state of N^{14} by removal of a neutron and a proton from the O^{16} core requires 22.94 Mev.[32] But the energy required to create a proton or neutron hole individually is 16.33 Mev and 19.82 Mev, respectively. (Here we have averaged over spin orientation, so as to eliminate the spin-orbit interaction with the core.) Thus the total energy of the two holes interacting with the core via the central forces (the tensor forces contribute nothing) with the core is 36.15 Mev. Since this energy has been omitted from the diagonal matrix elements of Tables I and II, one must also subtract it from the experimental energy, thus obtaining the above value.[33] It might be mentioned in this connection that a simple j-j coupling theory involving no interaction between the holes would put the ground state at -8.44 Mev, owing simply to the spin-orbit interaction of each hole with the core. The additional lowering of 4.77 Mev can therefore be assigned to interaction and correlation of the two holes within the p-shell.

TABLE II. Calculated values (in Mev) of the Hamiltonian matrix elements for the s^4p^{10} configurations.

(J,T)	$^{(2S+1)}L_J$			
(1,0)	$^3S_1$	−9.23	3.45	−2.58
	$^1P_1$	...	0	−3.86
	$^3D_1$	...	...	−11.16
(2,0)	$^3D_2$	−9.26		
(3,0)	$^3D_3$	−1.44		
(0,1)	$^1S_0$	−5.85	−5.97	
	$^3P_0$	...	−1.65	
(1,1)	$^1P_1$	−3.40		
(2,1)	$^3P_2$	2.37	2.99	
	$^1D_2$	...	−3.80	

All the other energy eigenvalues are readily obtained from Table II, and their numerical values need not be listed here. They are exhibited, however, in Fig. 2, which shows the $T=0$ levels on the right and the $T=1$ levels on the left. For comparison, the experimental levels appear in the center and are labeled by their excitation energies (again in Mev). N^{14} has a multitude of excited states, most of which are no doubt due to excitation of the O^{16} core. Only those have been shown which can reasonably be selected and identified as s^4p^{10} configurations. This is clearly limited to the low-lying levels, and only the ground state and the first two excited states can be considered identified with a high degree of confidence. The other two identifications are quite tentative. It should be explained that the excitation energy of 9.03 Mev, which is identified as the lower of the two (2,1) levels, has been calculated by adding the 2.31-Mev excitation energy of the lowest N^{14} $T=0$ level to the 6.72-Mev excitation energy of C^{14}. (The

[30] A. Feingold, Phys. Rev. **105**, 944 (1957). For another approach to the correlation problem see E. Feenberg and P. Goldhammer, Phys. Rev. **105**, 750 (1957), and references to earlier work given there.

[31] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949). See also R. G. Sachs, reference 17, pp. 84–85.

[32] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1956).

[33] We wish to thank Dr. C. Levinson for valuable discussion on this point.

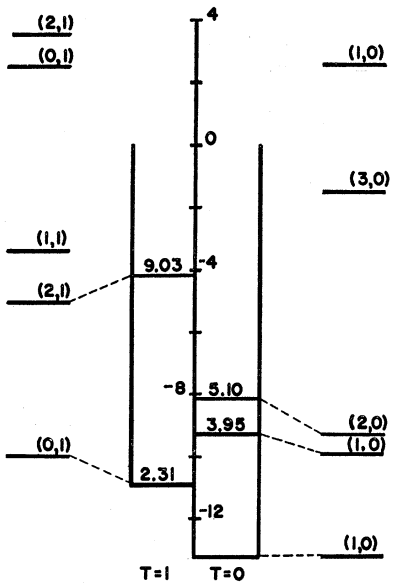

FIG. 2. Energy levels of N^{14}. Experimentally observed levels are shown in the center and are labeled by their excitation energy (in Mev) above the ground state. The ground state lies at -13.21 Mev (see text for normalization of the energy scale). Calculated $T=0$ and $T=1$ levels are exhibited on the right and left, respectively. Although five identifications are made, only the three lowest are certain. The agreement is regarded as satisfactory for the "first principles" calculation (see text).

other levels of C^{14} at 6.09 Mev and 6.89 Mev seem to have odd parity and do not come into consideration.)

It will be noted that all of the calculated levels fall within about a Mev of the experimental values, which is as good agreement as can be expected from the present rather crude "first principles" approach. The only apparent discrepancy is with the (1,1) level, which falls at -3.40 Mev and seems to have no experimental analog in C^{14}. The threshold for free neutron emission lies just above, however, at -2.73 Mev. The level might actually exist, but happen to be sufficiently above threshold as to be broadened and thus elude direction. The other four calculated levels of the s^4p^{10} configurations can similarly be explained away, or alternatively be considered as lost in the confusion which the dense N^{14} spectrum exhibits at these high excitation energies.

V. WAVE FUNCTIONS

The N^{14} and C^{14} ground state wave functions calculated from the Hamiltonian matrix elements of Table II are $(C_S, C_P, C_D)=(0.460, 0.137, 0.877)$ and $(C_S', C_P')=(0.816, 0.577)$. The terms appearing in the left-hand member of Eq. (5) are consequently $C_SC_S'=0.375$ and $C_PC_P'/\sqrt{3}=0.046$. Their difference is 0.329, or about four hundred times larger than the experimental value. But the discrepancy is not serious. The important point to note is that the relative sign of the two terms is now correct for cancellation. The tensor force has, via the off-diagonal element H_{SD}, not only been able to reduce the S-component to zero and change its sign; in addition it has been strong enough to build the S-wave back up to a quite large amplitude, but now with the desired sign relative to the P-component. Our theory is only confronted with the quantitative difficulty of providing larger values for the P-components, rather than with any serious qualitative discrepancy. A serious discrepancy might have occurred, for example, if the tensor force had been too weak to change the sign of C_S. The situation here, however, is just the opposite —we are in the comfortable position of having a tensor force somewhat more effective than required.

The agreement between the calculated and experimental level schemes shown in Fig. 2 can be considered satisfactory for such a "first principles" calculation. By this phrase we mean only that all of the experimental data used in the calculation have come from other sources than the mass-fourteen system itself. No data pertaining to the latter have been used in obtaining its theoretical energy-level spectrum. The purpose of this approach has been to give confidence that the properties of all the nuclei, and of the mass-fourteen system in particular, can be understood in terms of the basic two-nucleon interactions. But because of many uncertainties, the accuracy of such an approach is definitely limited at the present time. Assumption 3 of Sec. IV C above contains perhaps the greatest of these uncertainties. It is by no means definitely established that the Serber condition holds for low-energy central-force interactions. But even if the basic interaction satisfied this condition, it does not necessarily follow that the effective central force also does. Therefore, we drop this assumption, with the result that no *a priori* values can be assigned to the P-diagonal Hamiltonian matrix elements, H_{PP} and H'_{PP}. [Throughout this paper we use a prime to distinguish the (0,1) matrix elements and wave function coefficients from the corresponding (1,0) quantities.] Abandoning the "first principles" approach which we have followed up to now, we adopt an empirical procedure, using information from the mass-fourteen system itself for determining these additional unknowns. H'_{PP} is immediately fixed by $E'=-10.90$ Mev, the C^{14} ground-state energy (corrected, of course, for the $C^{14}-N^{14}$ Coulomb energy difference). The resulting diagonal element is $H'_{PP}=-3.83$ Mev, or about 2 Mev lower than the "first principles" value. The corresponding eigenfunction has consequently a larger P-component than before. The empirical C^{14} ground state coefficients are given by

$$(C_S', C_P')=(0.764, 0.646). \tag{21}$$

Cancellation in the beta-decay matrix element is now guaranteed provided the N^{14} ground-state coefficients satisfy the condition

$$C_S/C_P=C_P'/(\sqrt{3}C_S')=0.488. \tag{22}$$

If the components of the matrix form of Schrödinger's equation for the (1,0) case are written out separately,

one obtains three equations for the three unknowns: E (the eigenvalue), C_D/C_P (the only remaining undetermined ratio in the wave function), and H_{PP}. Consequently it is an easy matter to solve for H_{PP}. When this is done, one has a 3×3 matrix which is not only adjusted to yield beta-decay cancellation, but whose lowest eigenvalue still agrees well with the N^{14} ground-state energy. The next eigenvalue will be found, however, to lie about 2 Mev too low to correspond to the 3.95-Mev N^{14} excited level. This somewhat unsatisfactory result can be alleviated by further departing from the "first principles" calaculation. The exact value of H_{SS} cannot be considered to be known *a prior*, because of correlation effects in the triplet state. Therefore we now take this as an additional unknown which is to be empirically determined by requiring the calculated and the 3.95-Mev experimental levels to agree. [Note that, according to Assumption 1 of Sec. IV C above, we are not free to treat the singlet diagonal element H'_{SS} in this way.] This procedure not only fits the cancellation and the first excited (1,0) level, but surprisingly enough still leaves the lowest (1,0) eigenvalue within a few hundredths of an Mev of the experimental ground state energy. The effects on the ground state of lowering H_{PP} and raising H_{SS} cancel out and leave a negligible net shift. Replacing H_{DD} by a slightly lower value fits the ground-state energy exactly. The resulting empirical (1,0) and (0,1) matrices are exhibited in Table III. It should be pointed out that the off-diagonal elements as well as the H'_{SS} (singlet) diagonal element are still derived from "first principles." Only the remaining four diagonal elements have been empirically fitted to the four pieces of data on the mass-fourteen system (i.e., cancellation and the three lowest energy levels).

From Eq. (22) and Table III, the ratio C_D/C_P can immediately be calculated. The resulting ground-state components are given by

$$(C_S, C_P, C_D)=(0.173, 0.355, 0.920). \tag{23}$$

The ground-state properties of N^{14} are the magnetic moment (measured in units of the nuclear magneton),

$$\mu=0.879C_S^2+0.500C_P^2+0.310C_D^2, \tag{24}$$

and the electric quadrupole moment,

$$Q=\frac{5}{2\gamma}\left(\frac{4}{5\sqrt{5}}C_SC_D-\frac{1}{5}C_P^2+\frac{7}{50}C_D^2\right). \tag{25}$$

Substitution from Eq. (23) yields $\mu=0.351$ and $Q=1.06\times10^{-26}$ cm^2, which compare satisfactorily with the experimental values[34] of $\mu_{\text{exp}}=0.404$ and $Q_{\text{exp}}\approx1\times10^{-26}$ cm^2. The small deviation in the magnetic moment can be ascribed to meson exchange and configuration mixing. A further test of the wave functions is the half-life of the first excited state of N^{14}. An upper limit has been set by Sherr *et al.*[35] at 3.5×10^{-13} sec. From their Eq. (3) we calculate $T_{\frac{1}{2}}=2.0\times10^{-14}$ sec, which is an order of magnitude smaller than and clearly consistent with their upper limit.

TABLE III. Empirical values (in Mev) of the Hamiltonian matrix elements for the s^4p^{10} configurations having the same symmetry as the N^{14} and C^{14} ground states.

(J,T)	$^{(2S+1)}L_J$			
(1,0)	$^3S_1$	−6.60	3.45	−2.58
	$^1P_1$	...	−4.91	−3.86
	$^3D_1$	...	...	−11.23
(0,1)	$^1S_0$	−5.85	−5.97	
	$^3P_3$	...	−3.83	

A more positive test is the cross-over to cascade ratio for the decay of the N^{14} second excited state.[36] The cross-over transition gives a mixture of magnetic dipole and electric quadrupole radiation. The contribution of the magnetic transition rate is relatively small, however, since its matrix element contains only diagonal terms in the LS representation. The excited state is predominantly S just as the ground state is predominantly D, so that the square of the overlap of the wave functions is only about 1%, as contrasted with good overlap for the cascade transition. The remaining energy and magnetic moment factors in the ratio of the cross-over to cascade magnetic transitions are $(3.95/1.64)^3=14.0$ and $[(\mu_p+\mu_n)/(\mu_p-\mu_n)]^2=(0.88/4.70)^2=0.037$, where μ_p and μ_n denote the proton and neutron magnetic moments, respectively. These factors tend to compensate one another, so that the ratio of the magnetic transition rates is of the order of the ratio of the overlaps, or less than 1%. This can be neglected compared to the contribution of the electric transition. In evaluating the latter we make the approximation of neglecting the small components in the wave functions and take the ground and excited states to be pure $^3D_1$ and pure $^3S_1$, respectively. (This can be expected to produce about a 10% error in the results.) On the basis of this approximation we find a cross-over rate of 1.9×10^{12} sec^{-1}. Equation (3) of Sherr *et al.*,[35] can be used, with due allowance for a spin-degeneracy factor of three, to calculate the cascade rate and yields 2.0×10^{14} sec^{-1}. The cross-over to cascade ratio is therefore predicted to be 0.9%, which is much smaller than the experimental value of (3.7 ± 0.6)% determined by Bromley *et al*,[37] due presumably, as suggested by Elliott,[36] to core deformation.

There are in addition other experimental methods of investigating the N^{14} wave functions, and we shall briefly mention those which have come to our attention.

[34] J. E. Mack, Revs. Modern Phys. **22**, 64 (1950).

[35] Sherr, Gerhart, Horie, and Hornyak, Phys. Rev. **100**, 945 (1955).

[36] J. P. Elliott (see reference 10) seems to be the first to have studied this matter.

[37] Bromley, Almqvist, Gove, Litherland, Paul, and Ferguson, Phys. Rev. **105**, 957 (1957).

It has been noted by Banerjee *et al.*[38] that the inelastic low-energy proton cross section is abnormally small for the first excited state of N^{14} (the $T_z=0$ member of the same isotopic spin triplet as the ground state of C^{14}). This they attribute to the same cancellation as in the C^{14} β-decay, since the matrix element for direct-interaction S-wave scattering is identical up to a factor with the β-decay matrix element. The same explanation applies to the abnormally small inelastic neutron cross section for this state, as measured by Day.[39] For the same reason the charge exchange reactions $C^{14}(p,n)N^{14}$, $N^{14}(p,n)O^{14}$, and $N^{14}(n,p)C^{14}$ can be expected to have very small low-energy cross sections. In addition to the inelastic nucleon scattering experiments, there has been considerable interest in the extent to which the N^{14} ground-state wave function can be determined from stripping and pickup reactions. French's[40] theoretical analysis has been applied by Warburton and McGruer[41] to their experimental results on the $N^{14}(d,p)N^{15}$ stripping reaction. It is found that agreement cannot be obtained for any wave function even approximately given by Eq. (23). Considerably more P- and S-component is required. The reason for the discrepancy is not clear at the present time. It is possibly due to an energy dependence of the single-particle reduced widths,[42] which would prevent a direct comparison of the stripping cross sections to the ground state and 6.33-Mev excited state of N^{15}. The single-particle reduced width might be expected to be greater for the excited state, since the last neutron is more weakly bound and its wave function extends farther from the nuclear surface. This effect is in the right direction to decrease the discrepancy.

Note added in proof.—The second excited state wave function coefficients have been found to be

$$(C_S'', C_P'', C_D'')=(0.813, -0.580, 0.073).$$

If k' is the wave number of the 1.64-Mev radiation, the cascade rate is given by the dimensional factor

$$\frac{e^2\hbar k'^3}{3M^2c^2}=1.86\times10^{13}\ \text{sec}^{-1}$$

times the square of the dimensionless quantity obtained by replacing the ground state by the second excited state coefficients in the expression appearing between braces in Eq. (3) of Sherr *et al.* The cascade rate is accordingly 2.11×10^{14} sec^{-1}. The $M1$ cross-over rate contains the same dimensional factor, in which, however, k' is replaced by k, the wave number for the 3.95-Mev radiation. The associated dimensionless factor is twice the square of

$$(\mu_p+\mu_n-\tfrac{1}{2})(C_SC_S''-\tfrac{1}{2}C_DC_D'').$$

The $M1$ cross-over rate amounts thus to 0.86×10^{12} sec^{-1}, or 0.41% of the cascade rate. The $E2$ cross-over rate is

$$\frac{e^2k^5}{6\hbar\gamma^2}=9.28\times10^{12}\ \text{sec}^{-1}$$

times the square of

$$\frac{1}{\sqrt{5}}(C_SC_D''+C_DC_S'')-\frac{1}{2}C_PC_P''+\frac{7}{20}C_DC_D'',$$

which is closely related to the diagonal $E2$ matrix element appearing in the right hand member of Eq. (25). The $E2$ rate equals 2.03×10^{12} sec^{-1}, or 0.96% of the stop-over rate. The total cross-over rate amounts to 1.37% of the stop-over rate. The remaining 2.3% found experimentally is about what would be expected for the collective participation of the O^{16} core, judging from the already familiar enhanced $E2$ decays of the first excited states of C^{12}, N^{16}, and O^{17}.

VI. BETA DECAY OF O^{14}

The N^{14} and C^{14} wave functions have been determined in the preceding section so that the left hand member of Eq. (5) vanishes. The individual terms in this quantity are, according to Eqs. (21) and (23), individually equal to $C_SC_S'=C_PC_P'/\sqrt{3}=0.132$. A very slight change in the coefficients will change the difference from zero to the experimental value of 8.3×10^{-4}. Thus, the cancellation of the two terms is complete up to one part in 160. Jancovici and Talmi[9] have noted that this delicate cancellation will be perturbed by the Coulomb interaction of the two holes, and that it will therefore not be as complete in O^{14} as in C^{14}. The Coulomb repulsion of the two negatively charged proton holes in C^{14} is slightly greater in the $^1S_0$ than in the $^3P_0$ state by the amount

$$E_C=\tfrac{1}{2}e^2(\gamma/2\pi)^{\frac{1}{2}}=0.172\ \text{Mev}. \tag{26}$$

The diagonal Hamiltonian matrix elements for O^{14} relative to C^{14} therefore contain a perturbation of this amount in the H'_{PP} element. If we denote the excited (0,1) eigenvalue and eigenfunction by $E'+\Delta E$ and $C_P'(^1S_0)-C_S'(^3P_0)$, respectively, it follows from a simple application of perturbation theory that in passing from C^{14} to O^{14} the wave-function coefficients undergo the fractional changes

$$\frac{\Delta C_S'}{C_S'}=\frac{E_C}{\Delta E}(C_P')^2 \quad\text{and}\quad \frac{\Delta C_P'}{C_P'}=-\frac{E_C}{\Delta E}(C_S')^2. \tag{27}$$

Consequently the quantity inside the absolute value signs in Eq. (5) is increased by

$$\Delta\equiv\Delta(C_S'C_S-C_PC_P'/\sqrt{3})=C_SC_S'(E_C/\Delta E)$$
$$=2.27\times10^{-2}\ \text{Mev}/\Delta E. \tag{28}$$

[38] Banerjee, Levinson, Albright, and Tobocman, Bull Am. Phys. Soc. Ser. II, **1**, 194 (1956).
[39] R. B. Day, Phys. Rev. **102**, 767 (1956).
[40] J. B. French, Phys. Rev. **103**, 1391 (1956).
[41] E. K. Warburton and J. N. McGruer, Phys. Rev. **105**, 369 (1957).
[42] A. M. Lane, Atomic Energy Research Establishment, Harwell Report T/R 1289 (unpublished).

This equation is equivalent to Eq. (6) of Jancovici and Talmi. ΔE has been regarded by Jancovici and Talmi and by Sherr *et al.* as an undetermined parameter which might be identified with any 0+ excited state of C^{12}, with no *a priori* restriction on its excitation energy. This is clearly not permitted, since it follows immediately from Table III that $\Delta E=12.12$ Mev, giving $\Delta=1.88\times10^{-3}$. This can be compared with the experimental value of Δ which follows from the measured O^{14} *ft*-value.[35] From the latter we find

$$(C_S'C_S-C_PC_P'/\sqrt{3})_{O^{14}}=(6.3\pm0.5)\times10^{-3}, \quad (29)$$

so $\Delta_{exp}=(5.5\pm0.5)\times10^{-3}$ or $(7.1\pm0.5)\times10^{-3}$, depending on whether the β-decay matrix element for C^{14} happens to be positive or negative. If, since it is easier to explain, we adopt the first possibility, we are still left with a discrepancy by a factor of about three between the theoretical and experimental values of Δ.

Thus we see that the cancellation is by no means as delicate as might have been supposed. The Coulomb effect pointed out by Jancovici and Talmi is too weak to perturb the wave function sufficiently and produce the observed difference between O^{14} and C^{14}. There is evidently an additional perturbation present of amount 0.33 Mev, or about twice the Coulomb shift. One does not have to look far for the origin of such an additional shift. A proton moving in the Coulomb field of the O^{16} core experiences an electromagnetic spin-orbit splitting which is much smaller than but of the same sign as the nuclear spin-orbit splitting. Since the electromagnetic interaction depends on the magnetic moment of the nucleon, it differs both in sign and magnitude for a proton and a neutron. A comparison of the 6.33-Mev splitting in N^{15} with the 6.14-Mev splitting in O^{15} seems to reveal this effect. Thus the value of the one-nucleon spin-orbit strength decreases by $\Delta a=-0.13$ Mev[43] in passing from C^{14} to O^{14}. According to Table I this not only shifts H'_{PP} upwards by 0.13 Mev, but also decreases the off-diagonal elements by 3.1%. Since only the relative magnitudes of the matrix elements determine the eigenfunctions, this change in the off-diagonal elements is equivalent to a 3.1% increase in $H'_{PP}-H'_{SS}$. Thus there is an additional upward shift of H'_{PP} by 0.06 Mev, giving a total magnetic shift of 0.19 Mev. The magnetic effect is hence of about the same order of magnitude as the Coulomb effect. The sum of the two is 4.0×10^{-3} and is nearly sufficient to account for the experimental value of $\Delta_{exp}=(5.5\pm0.5)\times10^{-3}$.

In connection with the O^{14} β decay, it should be mentioned that, barring experimental difficulties, its spectrum may be more apt to reveal slight deviations from a straight Fermi-Kurie plot than that of C^{14}. This is because the "second-forbidden" terms in the β-decay matrix element (the "first-forbidden" terms are opposite in parity and do not contribute) are proportional to the square of the electron momentum which, at the end of the spectrum, is one hundred and twenty times as great for O^{14} as it is for C^{14}. The allowed term tends, of course, to mask the nonlinearity. It is seven times larger for O^{14} than C^{14}, but this still leaves O^{14} favored by a factor of seventeen, as far as the nonlinearity is concerned. The ratio of the nuclear radius to the reduced De Broglie wavelength, (i.e., $\lambda\!\!\!^{-}=\hbar/$momentum), for electrons at the end of the O^{14} spectrum is 3.9×10^{-2}. The "second-forbidden" terms are of the order of the square of this, or 1.56×10^{-3}, which is about one-fourth of the allowed term. Other numerical factors enter, of course, but an appreciable nonlinearity, perhaps of the order of 10%, ought to be expected in the O^{14} Fermi-Kurie plot. It might even amount to a few percent in C^{14} and eventually be detectable.[44] This would be of considerable interest, since it would probably be possible to determine the sign of the allowed matrix element from the interference term. This would provide a check on our work in Sec. VI, where we were forced to assume the sign to be positive.

VII. SUMMARY AND ACKNOWLEDGMENTS

In the preceding work we have shown how the accidental cancellation in the C^{14} β-decay matrix element is brought about by the tensor force which follows from meson theory. By adopting this "first principles" approach to the problem we hope to have given plausibility to what might otherwise be considered an *ad hoc* explanation. Because of the uncertainties in the central-force parameters, we have been forced to determine empirically the diagonal elements of the Hamiltonian matrix. Perhaps nuclear theory will have eventually advanced to a point where this is no longer necessary. In any case, we think it likely that the result of such a more basic calculation would be close to the matrix elements exhibited in Table III, which we feel must provide about as accurate a fit to the true Hamiltonian as can be obtained, at least within the framework of the shell model. The same applies to the corresponding N^{14} ground-state wave function [Eq. (23)], which has been checked against various experimental properties in Sec. V. The only serious discrepancy is with the stripping experiment of Warburton and McGruer,[41] and may possibly be attributed to stripping theory in its present somewhat rudimentary form. To clear up this question it will be necessary for someone to attack the difficult theoretical problem of calculating absolute values of single-particle reduced widths.

In Sec. VI it has been shown that the mechanism of Jancovici and Talmi inadequately explains the $C^{14}-O^{14}$ difference in *ft*-values. In addition to the Coulomb repulsion of the proton holes in C^{14}, it is necessary to

[43] It is planned to calculate this difference directly, since the $N^{15}-O^{15}$ comparison may be subject to error due to the Ehrman-Thomas shift (reference 24). A similar calculation of the electromagnetic spin-orbit splitting in the mass seven nuclei has been made by Inglis, [D. R. Inglis, Phys. Rev. **82**, 181 (1951)].

[44] A theoretical investigation into these nonlinearities is being carried out by David T. Goldman of this university.

take into account the slight difference in spin-orbit coupling strength for protons and neutrons. The O^{14} β-decay has further interest because of the expected nonlinearity in the Fermi-Kurie plot due to interference with the higher-order "forbidden" terms in the β-decay operator. This can only happen when the lowest-order "allowed" term is drastically reduced by an accidental cancellation. Thus O^{14} may be one of the rare cases where such an interference between different orders is detectable. (The cancellation in O^{14} is not, however, excessively unlikely from an *a priori* point of view, being complete only to about one part in twenty.)

One of us (RAF) is grateful to Professor W. Heisenberg for suggesting this problem to him, to the Max-Planck Institut für Physik for its generous hospitality, and to the U. S. Atomic Energy Commission for a postdoctoral fellowship during the tenure of which initial investigations were carried out. We also wish to thank Professor I. Talmi for helpful discussions.

APPENDIX I. HOLE CONJUGATION[45]

States of nuclei having nearly filled shells are most conveniently labeled by specifying the one-nucleon states which are vacant. In the formalism of second-quantization the annihilation and creation operators for the states κ and λ, a_κ and $a_\lambda^\dagger$, satisfy the anticommutation relations[46]

$$\{a_\kappa, a_\lambda^\dagger\} = \delta_{\kappa\lambda}, \tag{30}$$

and a nuclear closed-shell state is represented by $\Psi_0 = \Pi_\lambda a_\lambda^\dagger \Phi_0$. The product is over the set of one-nucleon basis states in the shell, and Φ_0 represents the vacuum state. The state of the nucleus in which all the one-nucleon states are filled except κ is thus $\Pi_{\lambda\neq\kappa} a_\lambda^\dagger \Phi_0$, which may be shown by using Eq. (30) to be $\pm a_\kappa \Psi_0$. Therefore a_κ and $a_\kappa^\dagger$ are hole creation and annihilation operators, respectively. Let ψ_k be the one-nucleon wave function with the same quantum numbers (i.e., z component of angular momentum, etc.) as the hole produced by the vacancy in the state φ_κ. The one-to-one correspondence between such pairs of wave functions is expressed by

$$\varphi_\kappa = \tau_y \sigma_y \psi_k^*, \tag{31}$$

and permits us to introduce explicitly the hole operators $b_k^\dagger \equiv a_\kappa$ and $b_k \equiv a_\kappa^\dagger$. The complex conjugation changes the sign of the z component of orbital angular momentum, while the σ_y and τ_y factors flip the spin and isotopic spin, respectively. Thus, as required for quantum numbers which add algebraically, the holes are opposite in sign to the one-nucleon states which are vacant. That the correspondence must be antiunitary can be seen by considering a transition involving as initial state hole k. This can also be looked upon as a nucleon transition into final state κ. A phase factor of $\exp(i\alpha)$ in ψ_k must appear as $\exp(-i\alpha)$ in φ_κ for the transition matrix element to be given correctly. A completely rigorous derivation of Eq. (31) requires considerations similar to those employed by Wigner[47] to reduce ray representations to group representations, but for the sake of brevity we prefer here not to enter further into this question.

We now express the operators of interest in terms of the hole operators. First, a general one-body operator,

$$\mathfrak{B} = \sum_{\kappa,\lambda} B_{\lambda\kappa} a_\lambda^\dagger a_\kappa, \tag{32}$$

with $B_{\lambda\kappa} \equiv (\varphi_\lambda, B\varphi_\kappa)$, can be rewritten, by using Eq. (30), as

$$\mathfrak{B} = \sum_\kappa B_{\kappa\kappa} - \sum_{\kappa\lambda} B_{\lambda\kappa} a_\kappa a_\lambda^\dagger. \tag{33}$$

But with the one-to-one correspondence of the preceding paragraph, we have $a_\kappa a_\lambda^\dagger = b_k^\dagger b_l$ and

$$\begin{aligned} B_{\lambda\kappa} &= (\tau_y\sigma_y\psi_l^*, B\tau_y\sigma_y\psi_k^*) \\ &= (\sigma_y\tau_y B^\dagger \tau_y\sigma_y\psi_l^*, \psi_k^*), \\ &= (\psi_k, \sigma_y\tau_y B^{\dagger *}\tau_y\sigma_y\psi_l), \end{aligned}$$

where a dagger denotes Hermitian conjugation and an asterisk complex conjugation. Thus the one-body operator becomes

$$\mathfrak{B} = \sum_\kappa B_{\kappa\kappa} + \sum_{kl} B'_{kl} b_k^\dagger b_l, \tag{34}$$

where

$$B' = -\tau_y \tilde{B} \tau_y \tag{35}$$

and

$$\tilde{B} = \sigma_y B^{\dagger *} \sigma_y. \tag{36}$$

$\tilde{B}$ is the time-inverted operator[48] corresponding to B. Thus, aside from the constant $\sum_\kappa B_{\kappa\kappa}$, the effect of the one-nucleon operator B acting on the nucleons is expressed by the action of the one-hole operator B^1 acting on the holes. In the case of operators which do not depend explicitly on isotopic spin, *the one-hole operator is the negative of the time inverted operator.*

The angular momenta and the one-nucleon Hamiltonian illustrate the above result. The orbital and spin angular momenta are so-called "imaginary" operators and reverse sign under time inversion. Therefore the one-hole operators are identical to the one-nucleon operators (the constant $\sum_\kappa B_{\kappa\kappa}$ even vanishes in these cases), so that holes couple their angular momenta in exactly the same way as do nucleons. The same holds true for isotopic spin, since $-\tau_y \tilde{B} \tau_y = B$, when $B = \tau_x$, τ_y, or τ_z. The one-nucleon Hamiltonian, however, behaves

[45] Part of this presentation follows closely that given by W. Heisenberg, Ann. Physik **10**, 888 (1931).

[46] P. Jordan and E. Wigner, Z. Physik **47**, 631 (1928).

[47] E. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Friedrich Vieweg und Sohn, Braunschweig, 1931), pp. 251–254.

[48] E. P. Wigner, Nachr. Ges. Wiss. Göttingen **32**, 35 (1932), and R. G. Sachs, reference 17, p. 353. Note the Hermitian conjugation which is to be included when the operator is not itself Hermitian. This combination of Hermitian and complex conjugation is, of course, simply the transpose operation.

differently. If it is invariant with respect to time inversion and is independent of isotopic spin, we have $-\tau_y\tilde{B}\tau_y=-\tau_y B\tau_y=-B$. Thus, for example, in the case of the one-nucleon spin-orbit force, the hole operator is simply the negative of the nucleon operator. Interactions of nucleons with electromagnetic fields fall into a separate category. The interaction of the protons in an almost closed shell with an electric field set up either externally or by the protons in another completely closed shell contains the isotopic spin factor $(1+\tau_z)/2$. The hole operator is then simply

$$-\tau_y[(1+\tau_z)/2]^{\dagger *}\tau_y=-\tau_y[(1+\tau_z)/2]\tau_y=-(1-\tau_z)/2,$$

which has sign opposite to that for nucleons, and is nonzero only for $\tau_z=-1$, corresponding to a proton hole. On the other hand, the coupling of an external magnetic field, $\mathbf{H}$, with the intrinsic nucleonic magnetic moments is described by the interaction $-[\mu_p(1+\tau_z)/2+\mu_n(1-\tau_z)/2]\boldsymbol{\sigma}\cdot\mathbf{H}$, where μ_p and μ_n are the proton and the neutron magnetic moments, respectively. The corresponding hole operator is easily found from Eqs. (35) and (36) to be $-[\mu_p(1-\tau_z)/2+\mu_n(1+\tau_z)/2]\boldsymbol{\sigma}\cdot\mathbf{H}$. Thus the magnetic moment of a proton or a neutron hole is the same as that of a proton or a neutron, respectively. Similarly, the orbital magnetic moment is the same for a proton hole as for a proton. The beta-decay interactions can also be expressed as one-hole operators. We illustrate with the Gamow-Teller operator $(\sigma_x+i\sigma_y)(\tau_x+i\tau_y)$, which describes the C^{14} beta-decay. The hole operator is

$$\begin{aligned}&-\tau_y\sigma_y(\sigma_x+i\sigma_y)^{\dagger *}(\tau_x+i\tau_y)^{\dagger *}\sigma_y\tau_y\\&\quad=-\tau_y\sigma_y(\sigma_x-i\sigma_y)(\tau_x-i\tau_y)\sigma_y\tau_y\\&\quad=-(-\sigma_x-i\sigma_y)(-\tau_x-i\tau_y)=-(\sigma_x+i\sigma_y)(\tau_x+i\tau_y)\end{aligned}$$

or, aside from a trivial sign factor, simply the one-nucleon operator itself. This result, which was used in deriving Eq. (3), has also been obtained by Inglis,[49] in a different way.

We proceed now to deal with two-nucleon operators, and take specifically an interaction Hamiltonian of the form

$$\mathcal{H}=\tfrac{1}{2}\sum_{\kappa\lambda\mu\nu}H_{\lambda\nu;\mu\kappa}a_\lambda^\dagger a_\nu^\dagger a_\mu a_\kappa, \tag{37}$$

where

$$H_{\lambda\nu;\mu\kappa}\equiv(\varphi_\lambda(1)\varphi_\nu(2),H(1,2)\varphi_\mu(2)\varphi_\kappa(1)), \tag{38}$$

and $H(1,2)$ is the interaction between nucleons 1 and 2 in configuration space. Clearly

$$H_{\lambda\nu;\mu\kappa}=H_{\nu\lambda;\kappa\mu}. \tag{39}$$

Again making use of Eq. (30) to put the hole-annihilation operators to the right of the hole-creation operators, we find that

$$\begin{aligned}a_\lambda^\dagger a_\nu^\dagger a_\mu a_\kappa=a_\kappa a_\mu a_\nu^\dagger a_\lambda^\dagger&-\delta_{\lambda\kappa}a_\mu a_\nu^\dagger\\&-\delta_{\nu\mu}a_\kappa a_\lambda^\dagger+\delta_{\lambda\mu}a_\kappa a_\nu^\dagger+\delta_{\kappa\nu}a_\mu a_\lambda^\dagger\\&+\delta_{\lambda\kappa}\delta_{\nu\mu}-\delta_{\lambda\mu}\delta_{\nu\kappa}.\end{aligned} \tag{40}$$

[49] D. R. Inglis, reference 2, pp. 447–449.

The last two terms, when substituted into Eq. (37), contribute $\frac{1}{2}\sum_{\kappa\mu}(H_{\kappa\mu;\mu\kappa}-H_{\mu\kappa;\mu\kappa})$ to the total energy of the filled shell. When the first term of Eq. (40) is substituted into Eq. (37) and use is made of Eq. (31), there results the two-hole operator

$$\tfrac{1}{2}\sum_{klmn}H'_{km,nl}b_k^\dagger b_m^\dagger b_n b_l,$$

where the matrix element is taken, in the notation of Eq. (38), of the configuration space operator $H'(1,2)=\tau_y^{(1)}\tau_y^{(2)}\tilde{H}(1,2)\tau_y^{(2)}\tau_y^{(1)}$. Since the two-nucleon interaction is invariant under time inversion, we can replace $\tilde{H}(1,2)$ by $H(1,2)$. If the interaction is also charge-independent it contains the isotopic spin operators, if at all, in the form $\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}$, a combination which is left unchanged by hole conjugation. Thus, the charge-independent *two-hole interaction is identical to the two-nucleon interaction*. The Coulomb interaction, on the other hand, contains the factors $(1+\tau_z^{(1)})(1+\tau_z^{(2)})$. These become replaced, under hole conjugation, by $(1-\tau_z^{(1)})(1-\tau_z^{(2)})$, corresponding to the Coulomb interaction of proton holes.

When the remaining terms in Eq. (40) are substituted into Eq. (37) and use is made of Eqs. (39) and (31), there results the operator

$$\begin{aligned}-\sum_{\kappa\lambda\mu}(H_{\lambda\mu;\mu\kappa}-H_{\mu\lambda;\mu\kappa})a_\kappa a_\lambda^\dagger&\\=-\sum_{klm}(H'_{km;ml}-H'_{mk;ml})b_k^\dagger b_l.&\end{aligned} \tag{41}$$

The direct and exchange matrix elements are of the operator $H'(1,2)$, which has already been defined above. Thus, there is present an additional one-hole energy which results from the interaction of the hole with the filled shell. To summarize, the interaction operator written in terms of its effect on the holes is, after some trivial changes in subscript notation,

$$\begin{aligned}\mathcal{H}=\tfrac{1}{2}\sum_{ijkl}H'_{jl;ki}b_j^\dagger b_l^\dagger b_k b_i-\sum_{ijk}(H'_{jk;ki}-H'_{kj;ki})b_j^\dagger b_i&\\+\tfrac{1}{2}\sum_{ik}(H_{ik;ki}-H_{ki;ki}).&\end{aligned} \tag{41a}$$

We illustrate by calculating the second term on the right-hand side of Eq. (41) for the case of the p shell. Omitting the Coulomb force, which can be treated separately, we replace H' by H. The sum over spins eliminates the tensor force since

$$\mathrm{Tr}_\sigma{}^{(1)}S_{12}=\mathrm{Tr}_\sigma{}^{(2)}S_{12}=0.$$

The central part of H gives a contribution to the coefficient of $b_j^\dagger b_i$ which is proportional to δ_{ji}. Since this amounts to the addition of a multiple of the unit matrix to the Hamiltonian, it does not interest us here. We therefore concentrate our attention on the spin-orbit force.

The hole in the p shell may have $j=\frac{1}{2}$ or $j=\frac{3}{2}$. The energy in the $j=\frac{1}{2}$ state can be found if the $j=\frac{3}{2}$ energy

is known, so it suffices to determine only the latter. [In other words, the spin-orbit one-hole operator of Eq. (41) is describable by the single-particle form, Eq. (6).] Therefore, upon inserting Eq. (10) (the spin-orbit potential) into the second term of the right hand side of Eq. (41), we find

$$E_p(3/2)=V_0''\sum_k(\varphi_{\frac{3}{2}}(1)\varphi_k(2),(1-\mathbf{P}_{12})(w''+m''P_{12})\times(\mathbf{s}_1+\mathbf{s}_2)\cdot(\mathbf{r}_{12}\times\mathbf{p}_{12})U_{\text{s.o.}}(r_{12})\varphi_k(2)\varphi_{\frac{3}{2}}(1)),$$

where the sum is to be carried out over the filled p shell, and $\mathbf{P}_{12}=P_{12}(1+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)(1+\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)/4$ interchanges the coordinates of nucleons 1 and 2. The spin and isotopic spin sums may be evaluated as traces, yielding

$$\begin{aligned}E_p(\tfrac{3}{2})=&(2w''-m'')V_0''\sum_m\int u_1^*(\mathbf{r}_1)u_m^*(\mathbf{r}_2)(\mathbf{r}_{12}\times\mathbf{p}_{12})_z\\&\times U_{\text{s.o.}}(r_{12})u_m(\mathbf{r}_2)u_1(\mathbf{r}_1)d^3\mathbf{r}_1d^3\mathbf{r}_2\\&-(w''-2m'')V_0''\sum_m\int u_m^*(\mathbf{r}_1)u_1^*(\mathbf{r}_2)(\mathbf{r}_{12}\times\mathbf{p}_{12})_z\\&\times U_{\text{s.o.}}(r_{12})u_m(\mathbf{r}_2)u_1(\mathbf{r}_1)d^3\mathbf{r}_1d^3\mathbf{r}_2,\end{aligned}\tag{42}$$

where u_m is the p-shell orbital with z component of angular momentum equal to m. The methods of Talmi[13] greatly facilitate the evaluation of the integrals appearing in Eq. (42). One finds

$$E_p(3/2)=-\tfrac{3}{2}(w''-m'')I_1''-\tfrac{5}{2}I_2''.\tag{43}$$

This represents only the energy shift due to interaction with all the nucleons in the p shell. The interaction with the s shell nucleons produces an additional spin-orbit shift, which is found in Appendix III to be

$$E_s(3/2)=-3(w''-m'')I_1''.\tag{44}$$

(Here we have included a minus sign, in accordance with Eq. (35), since we want the shift in energy of a hole in the p shell.) The total determines the spin-orbit strength parameter of Eq. (6):

$$\begin{aligned}a&=2E_s(3/2)+2E_p(3/2)\\&=-9(w''-m'')I_1''-5I_2''.\end{aligned}\tag{45}$$

APPENDIX II. IMPOSSIBILITY OF CANCELLATION IN THE BETA-DECAY MATRIX ELEMENT WITH AN ARBITRARY SPIN-ORBIT FORCE, BUT NO TENSOR FORCE

We present here a generalization of Inglis' theorem already referred to in Sec. II, and prove that it still holds if a general two-body spin-orbit force is assumed. For the latter we take

$$V_{\text{s.o.}}=\mathbf{V}_1\cdot\boldsymbol{\sigma}_1+\mathbf{V}_2\cdot\boldsymbol{\sigma}_2,$$

which is the most general form linear in the nucleon spins for the two particle or two hole case. $\mathbf{V}_1$ and $\mathbf{V}_2$ are arbitrary vector functions of the positions and momenta of the two holes.

For this interaction plus arbitrary central forces, the secular Eqs. (7) are simplified because $H_{SD}=0$. H_{SD} is a matrix element between states which differ by two units of orbital angular momentum, and $\mathbf{V}_i$ is capable of inducing a change of not more than one unit. Inglis solved Eqs. (7) for the ratios of coefficients, and showed that the β-decay matrix element is proportional to

$$C_SC_S'-\frac{C_PC_P'}{\sqrt{3}}=C_SC_S'\left(1-\frac{1}{\sqrt{3}}\frac{E-H_{SS}}{H_{SP}}\frac{H'_{SP}}{E'-H'_{PP}}\right).\tag{46}$$

He then proceeded to show that H_{SP} and H'_{SP} are always opposite in sign for a one-body spin-orbit force, rendering cancellation impossible.

Using Inglis' notation [see Eq. (2)], we find that the critical matrix elements are given by

$$\begin{aligned}H_{SP}&=H_{PS}=2(^1P_1,\mathbf{V}_1\cdot\boldsymbol{\sigma}_1\,^3S_1)\\&=2(\chi_0^0,\boldsymbol{\sigma}_1\chi_1^1)\cdot(P^1,\mathbf{V}_1S^0),\end{aligned}\tag{47a}$$

$$\begin{aligned}H'_{SP}&=H'_{PS}=2(^3P_0,\mathbf{V}_1\cdot\boldsymbol{\sigma}_1\,^1S_0)\\&=2\times3^{-\frac{1}{2}}[(\chi_1^1,\boldsymbol{\sigma}_1\chi_0^0)\cdot(P^{-1},\mathbf{V}_1S^0)\\&\quad-(\chi_1^0,\boldsymbol{\sigma}_1\chi_0^0)\cdot(P^0,\mathbf{V}_1S^0)\\&\quad+(\chi_1^{-1},\boldsymbol{\sigma}_1\chi_0^0)\cdot(P^1,\mathbf{V}_1S^0)].\end{aligned}\tag{47b}$$

The factor of two arises from the equal contribution of $\mathbf{V}_2\cdot\boldsymbol{\sigma}_2$ to the matrix elements. (The wave functions are eigenfunctions of the hole-interchange operator, with eigenvalues $(-1)^{J+1}$.) If one evaluates the spin matrix elements, introduces $V^0\equiv V_z$ and $V^\pm\equiv2^{-\frac{1}{2}}(\mp V_x-iV_y)$, the above equations become

$$H_{SP}=2(P^1,V_1{}^+S^0),\tag{48a}$$

$$\begin{aligned}H'_{SP}=&-2\times3^{-\frac{1}{2}}[(P^1,V_1{}^+S^0)\\&+(P^0,V^0S^0)+(P^{-1},V^-S^0)].\end{aligned}\tag{48b}$$

Simple group-theoretic considerations suffice to show that the three terms in H'_{SP} are all equal, from which it follows

$$H'_{SP}=-\sqrt{3}H_{SP}.\tag{49}$$

Thus the coefficient in the right hand member of Eq. (46) reduces to

$$1+(E-H_{SS})/(E'-H'_{PP}),$$

and is always positive. Consequently the C^{14} β-decay matrix element cannot vanish if the two-nucleon force is restricted to be linear in the spins.

APPENDIX III. SPIN-ORBIT SPLITTING IN THE MASS FIVE SYSTEM

By the methods illustrated in the last paragraph of Appendix I, it is a straightforward calculation to verify that a $j=\frac{3}{2}$ p-nucleon interacting via the two-nucleon spin-orbit force with the four nucleons in a filled s shell

has its energy shifted by

$$E_s(3/2)=(2w''-m'')I_{\rm dir}''-(w''-2m'')I_{\rm ex}'', \quad (50)$$

where

$$I_{\rm dir}''=-V_0''(1+\beta''/\gamma_s+\beta''/\gamma_p)^{-\frac{5}{2}}, \quad (51)$$

$$I_{\rm ex}''=V_0''[4\gamma_s\gamma_p(\gamma_s+\gamma_p)^{-1}(\gamma_s+\gamma_p+4\beta)^{-1}]^{\frac{5}{2}}. \quad (52)$$

Here the oscillator wave functions have different Gaussian factors for the s and p shells.[50] In the special case when $\gamma_s=\gamma_p=\gamma$ Eqs. (51) and (52) reduce to $I_{\rm dir}''=I_1''$ and $I_{\rm ex}''=-I_1''$, from which Eq. (44) of Appendix I follows.

To apply the above equations to the mass five system it is necessary to obtain estimates of γ_s and γ_p. The elastic electron scattering measurements on He^4 provide a determination of γ_s. According to McAllister and Hofstadter[51] and Blankenbecler and Hofstadter,[52] the experimental form factor implies an rms radius of 1.61×10^{-13} cm. Subtracting from the square of this the square of the proton rms radius (0.72×10^{-13} cm —see reference 15) yields an intrinsic rms radius for He^4 of 1.44×10^{-13} cm. (The neutron is known from independent experiments[53] to couple electrically with electrons only very weakly. That this poses a considerable difficulty for meson theory need not concern us here.[54]) The theoretical root-mean-square radius with which this is to be identified must be calculated with respect to the center of mass of the He^4 nucleus. In general, in a nucleus of mass A, where the center-of-mass coordinate is $\mathbf{R}=A^{-1}\sum_i \mathbf{r}_i$ and the individual nucleon coordinates measured from the center of mass are $\mathbf{r}_i'=\mathbf{r}_i-\mathbf{R}$, the mean square radius is the expectation value of the operator

$$A^{-1}\sum_i r_i'^2=A^{-1}\sum_i r_i^2-R^2. \quad (53)$$

This operator is invariant with respect to displacements of the center of mass, as should also be the case with the nuclear wave function over which the expectation value is calculated. This is not, however, the case in the ordinary shell model, where the nucleons are assumed to move independently and without the required correlation. For the special case of oscillator wave functions and normal shell filling, the necessary correction in the shell model is easily made, as pointed out by Elliott and Skyrme.[55] This is because the wave function then factors into the Gaussian $\exp(-A\gamma R^2/2)$ times a function of the relative variables only. Thus multiplication of the shell model wave function by $\exp(A\gamma R^2/2)$ introduces the required nucleon-nucleon correlations and yields a new wave function with the requisite displacement invariance. It is clear that it is not actually necessary to carry out this multiplication provided, as will always be the case, expectation values are computed only for displacement-invariant operators. For example, the expectation value of the first terms of the right hand member of Eq. (53) can be computed as a conventional shell-model calculation, while the expectation value of the second term is readily obtained by virtue of the factorization of the wave function into center of mass and relative variable parts. Only the center-of-mass Gaussian factor contributes, and gives A^{-1} times the mean square radius of an s nucleon, or $3/(2\gamma)$. Thus this correction is practically negligible except for the lightest of the nuclei. For the deuteron it amounts to 50%, while in the present case of $A=4$ it is a 25% correction. Taking the square root of the expectation value of Eq. (53) and equating it to the above experimental value yields

$$\gamma_s^{-\frac{1}{2}}=(8/9)^{\frac{1}{2}}\times1.44\times10^{-13}\text{ cm}=1.35\times10^{-13}\text{ cm}. \quad (54)$$

The p-shell size can be determined from the Li^5-He^5 Coulomb energy difference,

$$E_C=\frac{8e^2}{3}\left[\frac{\gamma_s x}{\pi(1+x)}\right]^{\frac{1}{2}}\left[1+\frac{x(1+x^2)}{2(1+x)^3}\right], \quad (55)$$

where $x=\gamma_p/\gamma_s$. Equating to the experimental value of 0.85 Mev and solving for x by iteration gives $\gamma_p^{-\frac{1}{2}}=2.38\times10^{-13}$ cm. The dame procedure applied to the Li^6-He^6 Coulomb energy difference of 0.80 Mev yields $\gamma_p^{-\frac{1}{2}}(Li^6)=2.55\times10^{-13}$ cm, and a radius value of

$$r_{\rm rms}(Li^6)=(\gamma_s^{-1}+5/6\gamma_p^{-1})^{\frac{1}{2}}=2.69\times10^{-13}\text{ cm}. \quad (56)$$

This agrees very well with Streib's value of 2.78×10^{-13} cm,[56] which, when corrected for proton size, becomes just 2.69×10^{-13} cm. The agreement is better than should be expected, particularly since the center-of-mass corrections discussed in the preceding paragraph cannot be applied to Eq. (56), due to the difference in shell sizes. It should also be pointed out that the same procedure applied to Li^7 leads to a discrepancy. The Be^7-Li^7 Coulomb energy difference is significantly larger than for the mass six nuclei, and leads to the smaller rms charge radius of about 2.3×10^{-13} cm. Streib, on the other hand, finds a value not much smaller than that for Li^6. Corrected for proton size, it amounts to 2.61×10^{-13} cm. The cause of this discrepancy is not known but is perhaps to be attributed to a closer correlation in the positions of the protons in Be^7

[50] In this respect our treatment differs from that of Elliott and Lane (see reference 23).

[51] R. W. McAllister and R. Hofstadter, Phys. Rev. **102**, 851 (1956).

[52] R. Blankenbecler and R. Hofstadter, Bull. Am. Phys. Soc. Ser. II, **1**, 10 (1956). See also R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).

[53] Melkonian, Rustad, and Havens, Bull. Am. Phys. Soc. II, **1**, 62 (1956).

[54] G. Salzman, Phys. Rev. **99**, 973 (1955). S. B. Treiman and R. G. Sachs, Phys. Rev. **103**, 435 (1956). See also Yennie, Levy, and Ravenhall, Revs. Modern Phys. **29**, 144 (1957).

[55] J. P. Elliott and T. H. R. Skyrme, Proc. Roy. Soc. (London) **232**, 561 (1955).

[56] J. F. Streib, Phys. Rev. **100**, 1797(A) (1955). See also R. Hofstadter, reference 52.

than can be accounted for with s^4p^8 configurations. Such intensified correlation might be brought about by the attractive nucleon-nucleon interaction, and might be described in the shell model by the admixture of higher configurations.

Although the mass-five system can only be treated rigorously as a scattering problem, a rough picture seems to be that there are two virtual states split by about 3 Mev. This requires $E_s(3/2)\approx -1$ Mev. Substitution of this and the numerical values of γ_s and γ_p determined above into Eq. (50) gives

$$V_0''-1.89m''V_0''\approx 5.71 \text{ Mev}, \tag{57}$$

in rough agreement with Eq. (14) of Sec. IV.

APPENDIX IV. ALPHA-PARTICLE BINDING ENERGY

The spin-independent central force strength quoted in Sec. IV as Eq. (20) is derived in this appendix. A simple variational treatment similar to that of Wigner,[57] Feenberg,[58] and Heisenberg,[59] is followed. The shell model is used, with all four s orbitals filled and approximated by oscillator wave functions of the form $\exp(-\gamma r^2/2)$. The expectation value of the total potential energy is easily found to be $6(w+m)I_0$, while the total kinetic energy is given by $9\hbar\gamma/4M$. (Here we have corrected for center-of-mass motion, as explained in Appendix III. The uncorrected value has been reduced by a factor of three-fourths.) The variational calculation is facilitated by introducing the quantities

$$E_0\equiv\frac{\hbar^2\beta}{2M}=6.90 \text{ Mev},$$

$$B\equiv(w+m)V_0/E_0,$$

$$b\equiv\gamma/\beta.$$

The expectation value of the total Hamiltonian is then

$$\langle H\rangle=6E_0\left[\tfrac{3}{4}b-B\left(\frac{b}{b+2}\right)^{\frac{3}{2}}\right]. \tag{58}$$

Differentiating Eq. (58) with respect to b and equating to zero yields

$$B=4^{-1}b^{-\frac{1}{2}}(b+2)^{\frac{5}{2}}, \tag{59}$$

which when substituted back into Eq. (58) gives

$$\langle H\rangle_{\min}=-\tfrac{3}{2}b(b-1)E_0=-28.11 \text{ Mev}. \tag{60}$$

By equating the minimum expectation value of the Hamiltonian to the negative of the α-particle binding energy, we have arrived at a quadratic equation for b, which may be solved to give $b=2.22$, or $\gamma_s^{-\frac{1}{2}}=1.16\times10^{-13}$ cm. Although this value is 19% smaller than that deduced in Appendix III from electron scattering data, we do not regard the disagreement as serious—at least for the present purposes. The discrepancy does not seem to be quite as bad as that discussed by Hofstadter[60] in relation to the wave functions of Clark.[61] Returning to Eq. (59) we find $B=6.14$, or $(w+m)V_0=42.4$ Mev.

[57] E. Wigner, Phys. Rev. **43**, 252 (1933).
[58] E. Feenberg, Phys. Rev. **47**, 850 (1935). See also E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937).
[59] W. Heisenberg, *Theorie des Atomkerns* (Dokumentationsstelle der Max-Planck-Ges., Göttingen, 1951), p. 67.
[60] Reference 52, p. 238.
[61] A. C. Clark, Proc. Phys. Soc. (London) **A67**, 323 (1954).