nucleus. If the target is very fissionable it is to be expected that fission will compete more effectively against neutron emission during the early stages of boil-off and thus contributions from (p,xnf) reactions will disappear more quickly with increasing values of x than is the case in less fissionable species. Glass *et al.*[14] have concluded that in the reaction of Pu^{239} with helium ions (compound nucleus Cm^{243}) the chain of successive neutron emission is very quickly interrupted by competition from fission and that the excess neutrons must therefore be emitted from the fission fragments.

Comparatively small changes in the nature of the compound nucleus might well result in large changes in the pattern of reactions. For example, in the simple case of the competitive reactions (γ,f) and (γ,n) with γ rays of 17–20 Mev, the probability of fission is 6% for Th^{232} and 60% for Pu^{239}.[30] Such differences would be magnified in cases where a succession of competitive reactions occurred. We are therefore extending our measurements to other target nuclei.

ACKNOWLEDGMENTS

We wish to thank the operating crew of the Harwell cyclotron for their help in bombarding the thorium targets. Two of us, F. Brown and J. P. Butler, wish to thank Atomic Energy of Canada Limited, Chalk River and the Atomic Energy Research Establishment, Harwell, for arranging the exchanges of personnel which enabled us to work at Harwell. We also wish to thank Mr. G. C. Hanna and Mr. T. D. Newton of Atomic Energy of Canada Limited for many helpful discussions.

[30] Huizenga, Grindler, and Duffield, Phys. Rev. 95, 1009 (1954).

PHYSICAL REVIEW VOLUME 107, NUMBER 3 AUGUST 1, 1957

Coulomb Effects in Inner Bremsstrahlen*

R. R. Lewis, Jr., and G. W. Ford
Department of Physics, University of Notre Dame, Notre Dame, Indiana
(Received November 19, 1956)

The Coulomb correction to the photon spectrum accompanying beta decay is calculated, treating the Coulomb field as a perturbation. It is shown that for allowed and unique first forbidden transitions, the result differs from that of Knipp-Uhlenbeck-Bloch only by the appearance of an extra factor, related to the Sommerfeld factor in ordinary bremsstrahlen. Analytic formulas are presented for these two selection rules, and a comparison made with recent experiments.

I. INTRODUCTION

THE original calculations[1] of the intensity of the photon spectrum accompanying beta decay have long been successful in explaining the observed data,[2] in spite of the expected inaccuracy due to the use of plane wave functions rather than Coulomb wave functions for the electrons. Recently, the first deviations from these predictions have been reported[3] in the spectrum of photons emitted by P^{32}, S^{35}, and Y^{90}. It is the purpose of this note to report the result of a derivation of the correction to the photon spectrum due to the Coulomb field of the nucleus, and to compare this result with these experiments. The Coulomb field is treated using perturbation theory; that is, only these additional terms proportional to Z are retained. Both allowed and forbidden transitions will be treated.

A few words in justification of this method of calculation are appropriate. It is not immediately obvious why the original calculation is so successful, or why additional terms in the perturbation theory would be expected to adequately treat the effect of the Coulomb field. One might expect for example, that a first order calculation would add terms of order $Z/137$, which is not a small correction in moderately heavy nuclei. In fact, our results indicate that the necessary corrections to the KUB formula are generally smaller than this. The fundamental reason for the accuracy of the KUB formula is that it has been used to predict only the photon intensity *relative to the beta intensity*, and not the absolute photon intensity. The number of photons per beta decay is a quantity which is comparatively independent of the atomic number, owing to a partial cancellation of the Coulomb effects on the photon and beta intensities. Thus, it is hoped that if we calculate this quantity to first order in Z, a similar partial cancellation of the higher ordered terms will considerably extend the validity of the perturbation theory, which would otherwise be expected to be valid only for light nuclei and high-energy decays.

As an example of this cancellation, we can consider the effect of attempting to correct for the Coulomb field

* This work supported in part by the U. S. Atomic Energy Commission and in part by the Office of Naval Research.

[1] J. K. Knipp and G. E. Uhlenbeck, Physica 3, 425 (1936); F. Bloch, Phys. Rev. 50, 272 (1936). We shall refer to the expression for the photon intensity given in these papers as the KUB formula.

[2] For a recent review see C. S. Wu, in *Beta and Gamma Ray Spectroscopy* (Interscience Publishers, New York, 1955).

[3] K. Liden and N. Starfelt, Phys. Rev. 97, 419 (1955); N. Starfelt and N. L. Svantesson, Phys. Rev. 97, 708 (1955); H. Langevin-Joliot, Compt. rend. 241, 872 (1955); M. Goodrich (private communication).

by inserting, without justification, a factor $F(Z,W_e)$ into the KUB formula. [$F(Z,W_e)$ is the Fermi function, which gives the Coulomb correction to the allowed beta decay transition probability for emission of an electron with energy W_e.] This means of accounting for the Coulomb effects has been used, for example, by Starfeld and Svantesson,[4] and by Nilsson.[5] Since the same factor appears in the beta decay probability, the effect of inserting such a factor is not large, even though the factor itself becomes quite large for heavy nuclei or low energies. Furthermore, the replacement of $F(Z,W_e)$ by its first order approximation, $(1+\pi\alpha ZW_e/p_e)$ does not significantly change the resultant correction to the KUB formula, even when $F(Z,W_e)$ and $(1+\pi\alpha ZW_e/p_e)$ differ considerably. This can be traced to the fact that $F(Z,W_e)$ and $(1+\pi\alpha ZW_e/p_e)$ differ mainly by a multiplicative factor, which is independent of the electron energy, and therefore has the same effect on the electron and photon spectrum, cancelling out in the ratio of their intensities.

In Sec. 2, we shall consider the first order Coulomb correction to an allowed transition. It will be shown that, to the desired order, the result differs from that of KUB only by a factor which is independent of the directions of the emitted particles. Thus the resultant formulas are obtained by simple substitution from those of KUB. In Sec. 3, we will prove a similar result to be true for the unique first forbidden transition as well. For other forbidden transitions the appearance of "extraordinary Coulomb effects," proportional to $\alpha Z/R$, will be discussed. In Sec. 4, analytic formulas will be quoted for the rather tedious integrals which arise in Secs. 2 and 3. In Sec. 5, a discussion of these results is presented, and a comparison made with several experiments.

2. ALLOWED TRANSITIONS

Since the final result differs little in form from the KUB result, we shall use a formalism and notation similar to that of Knipp and Uhlenbeck,[1] and later papers.[6] We are concerned with transitions which result in the emission by a nucleus of an electron, a neutrino, and a photon. These transitions are generated by the beta interaction, H_β, the interaction with the electromagnetic field, H_γ, and will also involve the Coulomb interaction between the electrons and nucleus, V. The diagrams in Fig. 1 denote the various processes considered.

Let us consider the processes in the order of their appearance in Fig. 1. The first diagram describes ordinary beta decay of the nucleus; the transition rate is related to the matrix element M_a,

$$M_a=(f|H_\beta|0)=G_sM_su^\dagger(\mathbf{p})O_su(\mathbf{q}), \qquad (1)$$

where $u(p)$ and $u(q)$ are the plane-wave amplitudes of

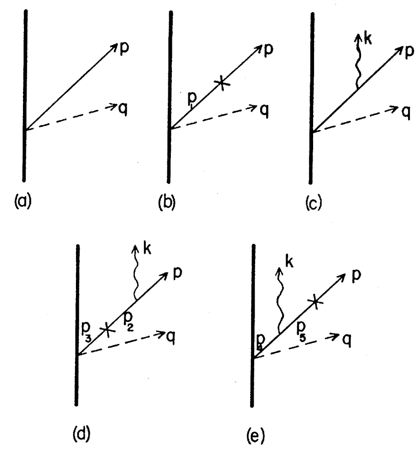

FIG. 1. Diagrams for the various beta decay and inner bremsstrahlen processes considered.

the electron and neutrino respectively. We have written the matrix element in the form appropriate for an allowed transition generated by one of the five possible beta interactions. Thus, G_s is the coupling constant, M_s is the nuclear matrix element, and O_s the appropriate operator for the particular interaction chosen. For example, for an allowed scalar interaction, M_s is $\int\beta$ and O_s is β, while for an allowed tensor interaction, M_s is $\int\beta\boldsymbol{\sigma}$ and O_s is $\beta\boldsymbol{\sigma}$. The generalization of our results to an arbitrary mixture of the five interactions is trivial if we neglect the possibility of the Fierz interference terms, and so, to avoid needless complication of the notation, we will always assume a single pure beta interaction.

The second diagram denotes the first order Coulomb correction to the beta decay. The matrix element corresponding to this diagram, M_b, which must be added to M_a, is

$$M_b=\sum_i\frac{(f|V|i)(i|H_\beta|0)}{E_0-E_i}. \qquad (2)$$

For V, we assume the pure Coulomb interaction between the electrons and the nucleus, neglecting the effects of shielding by the atomic electrons and the finite size of the nucleus. To avoid divergence difficulties due to the long range of the Coulomb interaction, we shall use the shielded Coulomb potential $V(r)=-\alpha Ze^{-\lambda r}/r$, and then let the shielding constant approach zero at the end of the calculation. The matrix elements of V for plane-wave electrons are

$$(j|V|i)=-4\pi\alpha Zu^\dagger(\mathbf{p}_j)u(\mathbf{p}_i)[|\mathbf{p}_j-\mathbf{p}_i|^2+\lambda^2]^{-1},$$

and so (2) becomes[7]

$$M_b=4\pi\alpha ZG_sM_s\left(\frac{1}{2\pi}\right)^3 \times\sum\int d\mathbf{p}_1\frac{u^\dagger(\mathbf{p})u(\mathbf{p}_1)u^\dagger(\mathbf{p}_1)O_su(\mathbf{q})}{(W+i\epsilon-W_1)[|\mathbf{p}_1-\mathbf{p}|^2+\lambda^2]}. \qquad (3)$$

[4] N. Starfelt and N. L. Svantesson, reference 3, Sec. III(A).
[5] S. B. Nilsson, Arkiv Fysik **10**, 467 (1956).
[6] C. S. Wang Chang and D. L. Falkoff, Phys. Rev. **76**, 365 (1949); Madansky, Lipps, Bolgiano, and Berlin, Phys. Rev. **84**, 596 (1951); Bolgiano, Madansky, and Rasetti, Phys. Rev. **89**, 679 (1953).

[7] We shall always denote the energy associated with a momentum p_a by W_a, i.e., $W_a=(p_a^2+1)^{\frac{1}{2}}$.

If we carry out the sum over the four plane wave states with momentum p_1 in the usual manner, we find

$$M_b=4\pi\alpha ZG_sM_s\left(\frac{1}{2\pi}\right)^3\times\int d\mathbf{p}_1\frac{u^\dagger(\mathbf{p})[\mathcal{H}(\mathbf{p}_1)+W]O_su(\mathbf{q})}{(p_1^2-p^2-i\epsilon)[|\mathbf{p}_1-\mathbf{p}|^2+\lambda^2]}. \quad (4)$$

Here ϵ is a small positive parameter which is to vanish at the end of the calculation, and $\mathcal{H}$ is the Dirac Hamiltonian for momentum $\mathbf{p}_1$.

The third diagram denotes the inner bremstrahlung process, generated by the matrix element M_c,

$$M_c=\sum_i\frac{(f|H_\gamma|i)(i|H_\beta|0)}{E_0-E_i}. \quad (5)$$

If we call $\mathbf{p}_2=\mathbf{p}+\mathbf{k}$, the matrix element of the interaction with the electromagnetic field becomes

$$(j|H_\gamma|i)=\left(\frac{2\pi\alpha}{k}\right)^{\frac{1}{2}}u^\dagger(\mathbf{p}_j)\boldsymbol{\alpha}\cdot\hat{e}u(\mathbf{p}_i),$$

and we have

$$M_c=\left(\frac{2\pi\alpha}{k}\right)^{\frac{1}{2}}G_sM_s\sum\frac{u^\dagger(\mathbf{p})\boldsymbol{\alpha}\cdot\hat{e}u(\mathbf{p}_2)u^\dagger(\mathbf{p}_2)O_su(\mathbf{q})}{E_0-E_i}.$$

Inserting the correct energy denominators and carrying out the sum over the spins of the intermediate state, we obtain

$$M_c=-\left(\frac{2\pi\alpha}{k}\right)^{\frac{1}{2}}G_sM_s\frac{u^\dagger(\mathbf{p})\boldsymbol{\alpha}\cdot\hat{e}[\mathcal{H}(\mathbf{p}_2)+W_e]O_su(\mathbf{q})}{W_2^2-W_e^2}, \quad (6)$$

where $W_e=W+k$ and $W_2^2=p_2^2+1$.

The last two graphs denote the first-order Coulomb corrections to the inner bremsstrahlung process. Their matrix elements M_d and M_e, which must be added to M_c, are

$$M_d=\sum_{i,j}\frac{(f|H_\gamma|i)(i|V|j)(j|H_\beta|0)}{(E_0-E_i)(E_0-E_j)}, \quad (7a)$$

$$M_e=\sum_{i,j}\frac{(f|V|i)(i|H_\gamma|j)(j|H_\beta|0)}{(E_0-E_i)(E_0-E_j)}, \quad (7b)$$

which can be written as

$$M_d=-\left(\frac{2\pi\alpha}{k}\right)^{\frac{1}{2}}4\pi\alpha ZG_sM_s\frac{1}{(2\pi)^3}\int d\mathbf{p}_3\times\frac{u^\dagger(\mathbf{p})\boldsymbol{\alpha}\cdot\hat{e}[\mathcal{H}(\mathbf{p}_2)+W_e][\mathcal{H}(\mathbf{p}_3)+W_e]O_su(\mathbf{q})}{(W_2^2-W_e^2)(p_3^2-p_e^2-i\epsilon)[|\mathbf{p}_3-\mathbf{p}_2|+\lambda^2]}, \quad (8a)$$

$$M_e=-\left(\frac{2\pi\alpha}{k}\right)^{\frac{1}{2}}4\pi\alpha ZG_sM_s\frac{1}{(2\pi)^3}\int d\mathbf{p}_4\times\frac{u^\dagger(\mathbf{p})[\mathcal{H}(\mathbf{p}_5)+W]\boldsymbol{\alpha}\cdot\hat{e}[\mathcal{H}(\mathbf{p}_4)+W_e]O_su(\mathbf{q})}{(p_5^2-p^2-i\epsilon)(p_4^2-p_e^2-i\epsilon)[|\mathbf{p}_5-\mathbf{p}|^2+\lambda^2]}, \quad (8b)$$

where $\mathbf{p}_4=\mathbf{p}_5+\mathbf{k}$, and $p_e^2=W_e^2-1$. The above signs are all chosen correctly for emission of negative electrons; for positive electrons, one must change the sign of Z everywhere.

The relations between these matrix elements and the photon and electron intensities are as follows: defining $S(k)$ as the number of photons per second per mc^2 energy interval, and N as the total number of electrons per second, we find

$$S(k)=\frac{k^2}{(2\pi)^8}\int_1^{W_0-k}dWpW(W_0-W-k)^2\int d\Omega_p\times\int d\Omega_q\int d\Omega_k|M_c+M_d+M_e|^2, \quad (9a)$$

$$N=\frac{1}{(2\pi)^5}\int_1^{W_0}dWpW(W_0-W)^2\times\int d\Omega_p\int d\Omega_q|M_a+M_b|^2. \quad (9b)$$

The integrals represent integration over directions and summation over spins (polarizations) of the electron, neutrino, and photon.

The most appropriate order for carrying out these operations is to first discuss the integrals over intermediate momenta. We intend to prove next that if we discard those portions of the matrix elements which will cancel out during the subsequent sums over spins, a particularly simple result is obtained for these matrix elements; namely, M_b is proportional to M_a, and M_d+M_e is proportional to M_c. Having proved this result, the remainder of the derivation need not be repeated, since it is exactly the same as the calculations in the absence of the Coulomb field.

To prove this result, notice first that, in keeping with our stated program, we must discard the terms in (9) which are quadratic in M_b, M_d, and M_e, since these will be quadratic in Z. Furthermore, *if* we could consider M_a and M_c as real numbers, then in forming the square of the absolute magnitude we would need only the *real parts* of M_b, M_d, and M_e; that is, we could keep only the *principal values* of the integrals over intermediate momenta. Since M_a and M_c involve complex matrices, and are not in general real numbers, this result is not immediately obvious. Nevertheless, it can be shown that if we sum over the spins of the electron and neutrino, the above situation results; only the principal values of the integrals will appear in the result. This is proved in Appendix A.

Thus in calculating the integrals which appear in (4) and (8) we have the following prescription: keep only the principal value of the integrals at the poles in the denominator, and take the limit $\lambda \to 0$ and $\epsilon \to 0$. In Appendix B the various integrals which occur are performed with this prescription. To facilitate comparison with the results of the Appendix, let us rewrite these integrals, by translating the origin of our momentum coordinates:

$$M_b = 4\pi\alpha Z G_s M_s \left(\frac{1}{2\pi}\right)^3 \int d\mathbf{Q} \times \frac{u^\dagger(\mathbf{p})[\mathcal{H}(\mathbf{Q}+\mathbf{p})+W]O_s u(\mathbf{q})}{(Q^2+\lambda^2)[|\mathbf{Q}+\mathbf{p}|^2-p^2-i\epsilon]}, \quad (4')$$

$$M_d = -\left(\frac{2\pi\alpha}{k}\right)^{\frac{1}{2}} 4\pi\alpha Z_s G M_s \left(\frac{1}{2\pi}\right)^3 \int d\mathbf{Q} \times \frac{u^\dagger(\mathbf{p})\alpha\cdot\hat{e}[\mathcal{H}(\mathbf{p}_2)+W_e][\mathcal{H}(\mathbf{Q}+\mathbf{p}_2)+W_e]O_s u(\mathbf{q})}{(W_2{}^2-W_e{}^2)(Q^2+\lambda^2)[|\mathbf{Q}+\mathbf{p}_2|^2-p_e{}^2-i\epsilon]}, \quad (8a')$$

$$M_e = -\left(\frac{2\pi\alpha}{k}\right)^{\frac{1}{2}} 4\pi\alpha Z G_s M_s \left(\frac{1}{2\pi}\right)^3 \int d\mathbf{Q} \times \frac{u^\dagger(\mathbf{p})[\mathcal{H}(\mathbf{Q}+\mathbf{p})+W]\alpha\cdot\hat{e}[\mathcal{H}(\mathbf{Q}+\mathbf{p}_2)+W_e]O_s u(\mathbf{q})}{(Q^2+\lambda^2)[|\mathbf{Q}+\mathbf{p}|^2-p^2-i\epsilon][|\mathbf{Q}+\mathbf{p}_2|^2-p_e{}^2-i\epsilon]}. \quad (8b')$$

Using the results of Appendix B, and the relation $\mathcal{H}(\mathbf{p})u(\mathbf{p}) = Wu(\mathbf{p})$ we find,

$$M_b \to \pi\alpha Z \frac{W}{2p} G_s M_s u^\dagger(\mathbf{p}) O_s u(\mathbf{q}),$$

$$M_d \to 0,$$

$$M_e \to -\pi\alpha Z \frac{W}{2p}\left(\frac{2\pi\alpha}{k}\right)^{\frac{1}{2}} G_s M_s \times \frac{u^\dagger(\mathbf{p})\alpha\cdot\hat{e}[\mathcal{H}(\mathbf{p}_2)+W_e]O_s u(\mathbf{q})}{W_2{}^2-W_e{}^2}.$$

Therefore we have the result stated before,

$$M_b \to \pi\alpha Z(W/2p)M_a, \quad (10a)$$

$$M_d + M_e \to \pi\alpha Z(W/2p)M_c. \quad (10b)$$

The only effect of the additional terms M_b, M_d, M_e, is to add a factor $1+\pi\alpha Z(W/2p)$ to the matrix element, therefore a factor $1+\pi\alpha Z(W/p)$ to the square of the matrix element. (This factor is readily recognized as the first order term in the expansion of the Sommerfeld-Elwert factor, $2\pi\alpha Z(W/p)/(1-\exp\{-2\pi\alpha ZW/p\})$, or its relativistic analogue, the Fermi function.) Our final result[8] can be obtained without further derivation by inserting this factor in the KUB result:

$$S(k) = \frac{\alpha}{k\pi}\frac{1}{2\pi^3}|G_s M_s|^2 \times \int_1^{W_0-k} dW(W_0-k-W)^2\left(1+\pi\alpha Z\frac{W}{p}\right) \times \{(W^2+W_e{}^2)\log(W+p)-2pW_e\}, \quad (11a)$$

$$N = \frac{1}{2\pi^3}|G_s M_s|^2 \int_1^{W_0} dW pW(W_0-W)^2 \times \left(1+\pi\alpha Z\frac{W}{p}\right). \quad (11b)$$

We should emphasize that we have proved that the factor $(1+\pi\alpha ZW/P)$ will appear after the sum over electron and neutrino spins is carried out but before the angular integrals are performed. The angular correlation between the electron and photon is therefore unchanged, to first order in Z.

3. FORBIDDEN TRANSITIONS

Since several of the commonly studied inner bremsstrahlung sources (RaE,Y^{90}) undergo forbidden transitions, there is some interest in extending this discussion of Coulomb effects to include them. We will restrict ourselves to first forbidden transitions, however, since at present there appear to be no examples of more highly forbidden transitions suitable for experiment.

We must carefully reconsider the validity of a perturbation calculation of forbidden transitions. One readily sees that it will no longer be possible to neglect the effect of the finite size of the nucleus, as in the first-order corrections to allowed transitions. This is evident from the fact that since the operators will, for forbidden transitions, depend on the momenta of the particles created (linearly for first forbidden transitions, quadratically for second, etc.), the integrals over intermediate momenta would diverge at large momenta for a pure Coulomb field. This divergence is a consequence of the singularity of the Coulomb field at small radii, and can be removed by the introduction of a nuclear charge distribution of radius R. The resultant matrix elements will then, in general, contain terms proportional to $\alpha Z/R$, which were absent in allowed transitions. These "extraordinary" Coulomb terms are well known to be present in the beta decay matrix elements.[9] Moreover, terms proportional to $(\alpha Z/R)^2$ will also occur, and will play a dominant role in most transitions. Since the $\alpha Z/R$ terms appear in the radiative beta decay, there is no reason to doubt that $(\alpha Z/R)^2$ terms will also be present, and provide a non-negligible contribution to the matrix elements. Our first order calculation will not contain these latter terms, since they are quadratic in Z, and

[8] After completing this work, it came to our attention that this result had been obtained previously by R. E. Cutkosky, Dissertation, Carnegie Institute of Technology, 1953 (unpublished). By a method different from ours, the same result was obtained.

[9] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. 60, 308 (1941).

therefore cannot be considered adequate to describe most forbidden transitions. If, as in beta decay, we assume that the $(\alpha Z/R)^2$ terms are negligible only for $\alpha Z/R \ll W_0$ then only in this rather restrictive limit can we trust the perturbation result.

However, as in beta decay there are special selection rules for which these "extraordinary" Coulomb terms are absent; these are the so-called "unique" first forbidden transitions with $\Delta J=2$ (yes). Since Y^{90} is an example of such a transition which has been studied experimentally, we will consider only this selection rule in detail.

The only changes necessary in our derivation in Sec. 1 are the modification of the potential by introduction of a nuclear charge distribution, and the appropriate change in M_sO_s. For simplicity the charge distribution is chosen to be that of a Yukawa charge distribution $\rho(r)=e^{-\Lambda r}/r$, so that

$$(j|V|i)=-4\pi\alpha Z\frac{u^\dagger(\mathbf{p}_j)u(\mathbf{p}_i)}{|\mathbf{p}_j-\mathbf{p}_i|^2+\lambda^2}\frac{\Lambda^2}{|\mathbf{p}_j-\mathbf{p}_i|^2+\Lambda^2}. \quad (12)$$

We will identify Λ^{-1} with R. The appropriate substitution for M_sO_s in the case of the tensor unique first forbidden selection rule is

$$M_sO_s\rightarrow\tfrac{1}{2}iB_{ij}(\mathbf{p}+\mathbf{q})_i\sigma_j, \quad (13)$$

with

$$B_{ij}=\int(x_i\sigma_j+x_j\sigma_i-\tfrac{2}{3}\delta_{ij}\boldsymbol{\sigma}\cdot\mathbf{r}).$$

Here $\mathbf{p}$ must be set equal to p, $\mathbf{p}_1$, $\cdots$ depending on whether we are considering graphs (a), (b), $\cdots$ in Fig. 1.

Now the derivation is very similar to the previous one; the necessary integrals are discussed in Appendix C. The most significant difference is, as we have mentioned, the appearance of the "extraordinary" Coulomb terms. These occur only in M_b and in M_d, and there only in the terms involving two (or more) factors, q_iq_j, in the numerator. However, for the unique first forbidden transition, this term is proportional to $\alpha_iB_{ij}\sigma_j$ which is easily shown to *vanish identically* by virtue of the symmetry and zero trace of B_{ij}. In this case, one readily sees that the same result is obtained as for an allowed transition: $M_b\rightarrow\pi(\alpha ZW/\alpha p)M_a$ and $M_d+M_e\rightarrow\pi(\alpha ZW/\alpha p)M_c$, leading to the appearance of a Sommerfeld factor. As before we can obtain the first order result by simple substitution of a factor $(1+\pi\alpha ZW/p)$ into the $Z=0$ result.[10] Thus we obtain

$$S(k)=\frac{\alpha}{\pi k}\frac{|G_T|^2|B_{ij}|^2}{24\pi^3}\int_1^{W_0-k}dW(W_0-W-k)^2$$
$$\times\{[(W^2+W_e^2)(W_e^2+q^2-1)+2kW_e]$$
$$\times\log(W+p)-2p[W_e(W_e^2+q^2-1)$$
$$+k(W^2-WW_e+W_e^2)]\}\left(1+\pi\alpha Z\frac{W}{p}\right), \quad (14a)$$

$$N=\frac{|G_T|^2|B_{ij}|^2}{24\pi^3}\int_1^{W_0}dWpW(W_0-W)^2$$
$$\times(p^2+q^2)\left(1+\pi\alpha Z\frac{W}{p}\right). \quad (14b)$$

4. ANALYTIC FORMULAS

The final results of our derivations have been expressed in (11) and (14) as rather complicated single integrals. The evaluation of these integrals is tedious but straightforward. For the sake of completeness we quote here the results of the evaluation of four of the integrals which arise in our work.

$$I_1=\int_1^x dW(W-x)^2\{(W^2+W_e^2)\log(W+p)-2W_ep\}$$
$$=\left[W_0^2\left(\frac{1}{3}x^3+\frac{1}{2}x\right)-\frac{1}{2}W_0\left(x^4+x^2-\frac{1}{8}\right)+\left(\frac{7}{30}x^5-\frac{3}{16}x\right)\right]\log(x+s)$$
$$-\left[W_0^2\left(\frac{11}{18}x^2+\frac{2}{9}\right)-W_0\left(\frac{7}{8}x^3+\frac{1}{16}x\right)+\left(\frac{689}{1800}x^4-\frac{1021}{3600}x^2-\frac{4}{75}\right)\right]s,$$

$$I_2=\int_1^x dW(W-x)^2\frac{W}{p}\{(W^2+W_e^2)\log(W+p)-2W_ep\}$$
$$=\left[-\frac{1}{2}xW_0^2+W_0\left(\frac{3}{2}x^2+\frac{3}{8}\right)-\left(x^3+\frac{9}{8}x\right)\right]\log^2(x+s)$$
$$+\left[W_0^2\left(\frac{1}{3}x^2+\frac{2}{3}\right)-W_0\left(\frac{1}{2}x^3+\frac{13}{4}x\right)+\left(\frac{7}{30}x^4+\frac{59}{20}x^2+\frac{16}{15}\right)\right]s\log(x+s)$$
$$+\left[W_0^2\left(-\frac{11}{18}x^3+x^2+\frac{7}{6}x-\frac{7}{9}\right)+W_0\left(\frac{7}{8}x^4-2x^3+\frac{49}{8}x^2-6x+1\right)+\left(-\frac{689}{1800}x^5+x^4-\frac{2059}{360}x^3+\frac{67}{9}x^2-\frac{61}{15}x+\frac{388}{225}\right)\right],$$

[10] L. Madansky *et al.*, Phys. Rev. 84, 596 (1951).

$$I_3=\int_1^x dW(W-x)^2\{[(W^2+W_e^2)(W_e^2+q^2-1)+2kW_e]\log(W+p)-2[W_e(W_e^2+q^2-1)+k(W^2-WW_e+W_e^2)]p\}$$

$$=\left[W_0^4\left(\frac{1}{3}x^3+\frac{1}{2}x\right)-W_0^3\left(x^4+x^2-\frac{1}{8}\right)+W_0^2\left(\frac{43}{30}x^5+\frac{4}{3}x^3-\frac{3}{16}x\right)\right.$$

$$\left.-W_0\left(\frac{31}{30}x^6+\frac{7}{6}x^4+\frac{3}{16}x^2+\frac{1}{24}\right)+\left(\frac{32}{105}x^7+\frac{4}{15}x^5+\frac{1}{24}x\right)\right]\log(x+s)$$

$$-\left[W_0^4\left(\frac{11}{18}x^2+\frac{2}{9}\right)-W_0^3\left(\frac{7}{4}x^3+\frac{1}{8}x\right)+W_0^2\left(\frac{4481}{1800}x^4+\frac{97}{1200}x^2+\frac{2}{225}\right)\right.$$

$$\left.-W_0\left(\frac{363}{200}x^5+\frac{1337}{3600}x^3+\frac{437}{1800}x\right)+\left(\frac{653}{1225}x^6-\frac{769}{44\,100}x^4+\frac{7733}{88\,200}x^2+\frac{12}{1225}\right)\right]s,$$

$$I_4=\int_1^x dW(W-x)^2\frac{W}{p}\{[(W^2+W_e^2)(W_e^2+q^2-1)+2kW_e]\log(W+p)-2[W_e(W_e^2+q^2-1)+k(W^2-WW_e+W_e^2)]p\}$$

$$=\left[-\frac{1}{2}W_0^4x+W_0^3\left(3x^2+\frac{3}{4}\right)-W_0^2\left(7x^3+\frac{49}{8}x\right)+W_0\left(\frac{15}{2}x^4+\frac{107}{8}x^2+\frac{5}{4}\right)-\left(3x^5+\frac{19}{2}x^3+\frac{19}{8}x\right)\right]\log^2(x+s)$$

$$+\left[W_0^4\left(\frac{1}{3}x^2+\frac{2}{3}\right)-W_0^3\left(x^3+\frac{13}{2}x\right)+W_0^2\left(\frac{43}{30}x^4+\frac{1193}{60}x^2+\frac{74}{15}\right)\right.$$

$$\left.-W_0\left(\frac{31}{30}x^5+\frac{1511}{60}x^3+\frac{541}{30}x\right)+\left(\frac{32}{105}x^6+\frac{2377}{210}x^4+\frac{2431}{140}x^2+\frac{16}{21}\right)\right]s\log(x+s)$$

$$-\left[W_0^4\left(\frac{11}{18}x^3-x^2+\frac{7}{6}x-\frac{7}{9}\right)-W_0^3\left(\frac{7}{4}x^4-4x^3+\frac{49}{4}x^2-12x+2\right)\right.$$

$$+W_0^2\left(\frac{4481}{1800}x^5-7x^4+\frac{13\,451}{360}x^3-\frac{406}{9}x^2+\frac{299}{15}x-\frac{1727}{225}\right)$$

$$-W_0\left(\frac{363}{200}x^6-6x^5+\frac{17\,147}{360}x^4-\frac{596}{9}x^3+\frac{3107}{60}x^2-\frac{7214}{225}x+\frac{55}{18}\right)$$

$$\left.+\left(\frac{653}{1225}x^7-2x^6+\frac{54\,989}{2520}x^5-\frac{311}{9}x^4+\frac{108\,323}{2520}x^3-\frac{7703}{225}x^2+\frac{419}{63}x-\frac{2644}{2205}\right)\right].$$

In the above expressions W_0 is the end-point energy of the beta spectrum, $x=W_0-k$, and $s=(x^2-1)^{\frac{1}{2}}$.

5. CONCLUSIONS

The main contributions of this work are the formulas (11) and (14) for the first order Coulomb corrections in allowed and unique first forbidden transitions. These are of rather limited generality, since they include only first-order terms, and only consider three different selection rules ($\Delta J=0$, 1, no and $\Delta J=2$, yes). Fortunately, however, most of the commonly used sources of inner bremsstrahlen spectra are satisfactorily described by these formulas. We shall conclude this work with a comparison of our results with experiments in P^{32}, S^{35}, and Y^{90}.

The first significant deviations from the KUB results were reported by Liden and Starfelt[3] in P^{32}, and by Starfelt and Svantesson[3] in S^{35}, both of which are classed as allowed transitions.[11] In both cases, the experiments showed an increase in the photon spectrum above the predictions of the KUB result, especially for high photon energies. Their results are shown in Figs. 2 and 3, along with the KUB and the first order theoretical curves. In P^{32}, the experiment and theory are in clear disagreement; in S^{35} the disagreement is not as unequivocal, yet still unsatisfactory. Since S^{35} has a low upper end point energy (167 kev), the validity of the first order correction is more uncertain than in P^{32}. As an attempt to

[11] Mayer, Moszkowski, and Nordheim, Revs. Modern Phys. **23**, 315 (1951); L. Nordheim, Revs. Modern Phys. **23**, 322 (1951).

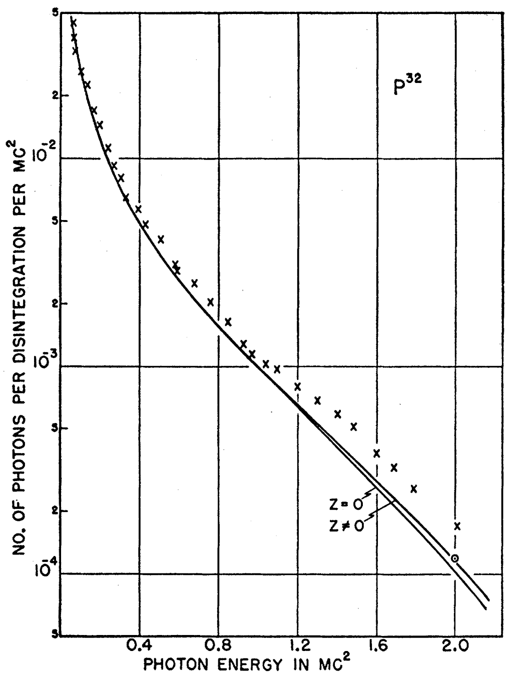

FIG. 2. The inner bremsstrahlen spectrum for P^{32}. The curves are the theoretical results; the crosses are the experimental points of Liden and Starfelt; the open circle is a theoretical result computed with $F(Z,W)$ replacing $1+\pi\alpha ZW/p$ in the $Z\neq 0$ formula.

check the effect of higher order terms, one can replace the factor $(1+\pi\alpha ZW/p)$ by the Fermi function $F(Z,W)$.

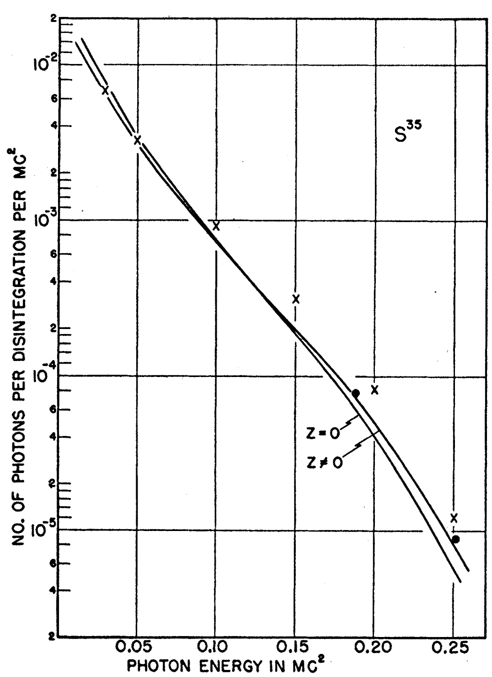

FIG. 3. The inner bremsstrahlen spectrum for S^{35}. The curves are the theoretical results; the crosses are the experimental points of Starfelt and Svantesson; the open circles are theoretical results computed with $F(Z,W)$ replacing $1+\pi\alpha ZW/p$ in the $Z\neq 0$ formula.

As discussed in the introduction, this replacement has relatively little effect on the photon spectrum; in this case, the computation was carried out at two points in the S^{35} spectrum and at one point in P^{32}, marked with circles in Figs. 2 and 3. The resultant deviation from the first order theory was at most 10% in S^{35}, and 2% in P^{32}. Later experiments by Goodrich[12] seem to disagree with this result for P^{32}, and show satisfactory agreement with first order theory (see Fig. 2). However, Goodrich obtains large deviations in Y^{90}, which is classed as a unique first forbidden transition.[11] See Fig. 4. Although Y^{90} is rather high Z for comparison with a first order theory, the disagreement is probably larger than can be explained by a more accurate Coulomb correction. Langevin-Joliot[13] reports large deviations in S^{35}, in approximate agreement with Starfelt and Svantesson.

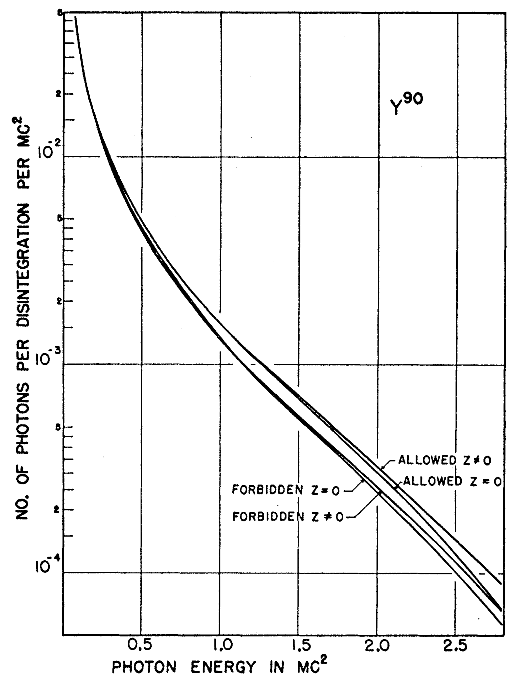

FIG. 4. The theoretical inner bremsstrahlen spectrum for Y^{90}.

It should be emphasized that there are other possible origins of deviations from the KUB result than corrections for Coulomb effects. One of these is the presence of nuclear radiation accompanying beta decay.[14] The present authors intend to publish a discussion of these effects as well.

ACKNOWLEDGMENTS

The authors wish to thank Professor G. E. Uhlenbeck for his interest in this work, as well as Dr. Starfelt, Dr. Svantesson, and Dr. Goodrich for communication of their experimental results. Thanks are also due Mr. J.

[12] M. Goodrich (private communication).

[13] H. Langevin-Joliot (private communication).

[14] C. Longmire, Phys. Rev. **75**, 15 (1949); J. Horowitz, J. phys. radium **13**, 429 (1952).

Waldron for his assistance with the numerical work and with the tedious integrals in Sec. 4.

APPENDIX A

The fact that we need retain only the principal part of the integrals over intermediate momenta, so long as we average over the spin directions of the electron and neutrino, can be proven by recourse to a theorem on the reality of traces of Dirac matrices.[15] Let us first prove the following theorem:

Theorem.—The trace of a product of Dirac matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ is real or imaginary, depending on whether the number (n) of times $\boldsymbol{\sigma}$ appears is even or odd.

Consider the most general matrix formed from the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$ real vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\cdots$

$$M=(\boldsymbol{\alpha}\cdot\mathbf{a})(\boldsymbol{\alpha}\cdot\mathbf{b})\cdots(\boldsymbol{\sigma}\cdot\mathbf{l})(\boldsymbol{\sigma}\cdot\mathbf{m})\cdots. \qquad \text{(A1)}$$

Then we have

$$T^*=\{\mathrm{tr}M\}^*=\mathrm{tr}M^*$$
$$=\mathrm{tr}\{(\boldsymbol{\alpha}^*\cdot\mathbf{a})(\boldsymbol{\alpha}^*\cdot\mathbf{b})\cdots(\boldsymbol{\sigma}^*\cdot\mathbf{l})(\boldsymbol{\sigma}^*\cdot\mathbf{m})\cdots\}. \qquad \text{(A2)}$$

Although the theorem is true independently of the representation of the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\sigma}$, we shall use for simplicity the "standard" representation in which

$$\boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \boldsymbol{\sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix};$$

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Then

$$\boldsymbol{\alpha}^*=(\alpha_1,-\alpha_2,\alpha_3),\quad \boldsymbol{\sigma}^*=(\sigma_1,-\sigma_2,\sigma_3). \qquad \text{(A3)}$$

If we transform these matrices by a unitary transformation generated by $S=\beta\sigma_2$, then

$$S\boldsymbol{\alpha}^*S^\dagger=\boldsymbol{\alpha},\quad S\boldsymbol{\sigma}^*S^\dagger=-\boldsymbol{\sigma}.$$

However, the trace is invariant under a unitary transformation and so we can write

$$T^*=\mathrm{tr}(SM^*S^\dagger)=\mathrm{tr}\{(\boldsymbol{\alpha}\cdot\mathbf{a})(\boldsymbol{\alpha}\cdot\mathbf{b})\cdots(-\boldsymbol{\sigma}\cdot\mathbf{l})(-\boldsymbol{\sigma}\cdot\mathbf{m})\cdots\}$$
$$=(-)^nT, \qquad \text{(A4)}$$

and so the theorem is proven. Of course, many of these traces vanish also: those matrices with an odd number of terms containing $\boldsymbol{\alpha}$ will all vanish.

Now to see that the contributions from the poles in the energy denominators cancel in the average over spins, we must simply write the first order correction terms in the form appropriate for the average over spins. For example, in the case of beta decay, the first order term is, when averaged over spins of the electron and neutrino

$$\sum(M_a{}^*M_b+M_aM_b{}^*)=\mathrm{Rl}\int d\mathbf{p}_1\frac{\mathrm{Tr}\{\ \}}{(\ )(\ )}. \qquad \text{(A5)}$$

The operators O_s are the only terms which can contain a $\boldsymbol{\sigma}$, but in this case they will always appear an even number of times, since O_s appears twice in the above trace. Therefore, by our theorem, the traces are real, and only the contribution from the principal value of the integral is nonvanishing in (A5). Note that had we carried out the traces for specified directions of the spins, the traces would have contained extra projection operators for these spin directions, such as $P=\frac{1}{2}(1+\boldsymbol{\sigma}\cdot\hat{n})$. In this case the traces would in general be complex, and the contributions from the pole could not be discarded.

The argument in the case of the radiative beta decay is precisely the same; for an average over spins, the only $\boldsymbol{\sigma}$ matrices which appear will be contained in the beta interaction, and will therefore appear an even number of times. All the resulting traces will be real.

APPENDIX B

The integrals required in Sec. 2 can be defined as follows:

$$\{I,I_i\}=\int d\mathbf{Q}\frac{\{1,Q_i\}}{(Q^2+\lambda^2)[|\mathbf{Q}+\mathbf{P}|^2-s^2-i\epsilon]}, \qquad \text{(B1)}$$

$$\{J,J_i,J_{ij}\}=\int d\mathbf{Q}$$
$$\times\frac{\{1,Q_i,Q_iQ_j\}}{(Q^2+\lambda^2)[|\mathbf{Q}+\mathbf{P}|^2-s^2-i\epsilon][|\mathbf{Q}+\mathbf{p}|^2-p^2-i\epsilon]}. \qquad \text{(B2)}$$

The prescriptions for computing these integrals are that we keep only the real parts, and take the limits $\lambda\to 0$ and $\epsilon\to 0$. These integrals are all properly convergent, even though it may appear at first glance that I_i diverges logarithmically at large momenta. For large Q, the integrand of I_i approaches Q_i/Q^2, which averages to zero in the angular integration.

Our technique for computing these integrals follows closely that of Dalitz,[16] who showed that the integrals involving extra numerator factors Q_i, Q_iQ_j, etc., could be derived from I and J by differentiation and integration in the parameters $\mathbf{P}$ and s. For example, the formal expression of J_i in terms of J is

$$J_i=\int_s^\infty ds'\left\{P_i\frac{\partial}{\partial s'}+s'\frac{\partial}{\partial P_i}\right\}J(\mathbf{P},s'). \qquad \text{(B3)}$$

The integrals I and J are presented in general form elsewhere[17]; we shall simply quote the final results:

$$I(\lambda,\mathbf{P},s)=\frac{\pi^2 i}{P}\ln\left[\frac{\lambda-is-iP}{\lambda-is+iP}\right],$$
$$J(\lambda,\mathbf{P},\mathbf{p},s)=\frac{\pi^2}{(\beta^2-\alpha\gamma)^{\frac{1}{2}}}\ln\left[\frac{\beta+(\beta^2-\alpha\gamma)^{\frac{1}{2}}}{\beta-(\beta^2-\alpha\gamma)^{\frac{1}{2}}}\right], \qquad \text{(B4)}$$

[15] This result can also be seen to be a consequence of invariance under time inversion.

[16] R. H. Dalitz, Proc. Roy. Soc. (London) A206, 509 (1951).
[17] R. R. Lewis, Jr., Phys. Rev. 102, 537 (1956).

where

$$\beta = ip(s^2-P^2)-\lambda[(s+p)^2-|\mathbf{P}-\mathbf{p}|^2]-i\lambda^2(s+p),$$
$$\alpha\gamma = -\lambda[(s+p)^2-|\mathbf{P}-\mathbf{p}|^2]\times[s^2-P^2+2is\lambda-\lambda^2][2ip-\lambda].$$

It is clear that the operations (B3) will commute with taking the real part, and with the limit $\epsilon\to 0$. For $s^2>P^2$, we can also let $\lambda\to 0$ before performing (B3), since the expansion in powers of λ is uniformly convergent in s in this case. We find

$$I\to\lambda\frac{2\pi^2}{s^2-P^2},\quad J\to\frac{\pi^3}{2p(s^2-P^2)}. \tag{B5}$$

Notice that both of these results depend only on s^2-P^2, so that the operations (B3) will give zero. Furthermore, I vanishes as $\lambda\to 0$, and so we have the result that J is the *only* nonvanishing integral in this limit.

For $s^2=P^2$, we cannot carry out the limit $\lambda\to 0$ first, but must evaluate I_i by performing (B3); the necessary integrals are readily done however, and it is found that I_i vanishes as $\lambda\to 0$, and I becomes

$$I\to\pi^3/2P. \tag{B6}$$

APPENDIX C

In Sec. 3, we need further integrals similar to those in Appendix B, but differing in the important respect that they do not converge for a pure Coulomb field. The appearance of such terms can be seen quite generally, if we remember that, before making the expansion into different degrees of forbiddenness, the integrals of Appendix B will contain an extra factor $\exp(i\mathbf{Q}\cdot\mathbf{r})$. They will all converge except I_i, which will now diverge at large momenta. As mentioned in the text, we will correct this by introducing a nuclear charge distribution of Yukawa shape, which produces an extra factor $(\Lambda^2/Q^2+\Lambda^2)$ in each of the integrals. Most of the integrals converge without this factor, and so we expect these integrals to be independent of Λ for $\Lambda\to\infty$. On the other hand, I_i will depend on Λ, and will in fact give rise to the "extraordinary" terms proportional to $\Lambda=1/R$.

A further complication arises when we make the forbiddenness expansion by replacing $e^{iQ\cdot r}$ by $1+iQ\cdot r+\cdots$, since now the individual terms will again diverge for large momenta. This is a much more trivial difficulty, which clearly has its origin in an improper interchange of orders of integration and expansion. As is customary, we shall treat these divergences by introducing convergence factors; for convenience we shall use algebraic convergence factors, $(\Lambda^2/Q^2+\Lambda^2)^n$, wherever necessary. We should emphasize that the appearance of one factor $(\Lambda^2/Q^2+\Lambda^2)$ is necessary and significant, while the extra factors are essentially mathematical artifacts.

Then we intend to consider the integrals

$$\{I^{(n)},I_i^{(n)},I_{ij}^{(n)}\cdots\} = \int d\mathbf{Q}\frac{\{1,Q_i,Q_iQ_j,\cdots\}}{(Q^2+\lambda^2)[|\mathbf{Q}+\mathbf{P}|^2-s^2-i\epsilon]}\left[\frac{\Lambda^2}{Q^2+\Lambda^2}\right]^n, \tag{C1}$$

$$\{J^{(n)},J_i^{(n)},\cdots\} = \int d\mathbf{Q}\frac{\{1,Q_i,\cdots\}}{(Q^2+\lambda^2)[|\mathbf{Q}+\mathbf{P}|^2-s^2-i\epsilon][|\mathbf{Q}+\mathbf{p}|^2-p^2-i\epsilon]}\times\left[\frac{\Lambda^2}{Q^2+\Lambda^2}\right]^n, \tag{C2}$$

where in $I^{(n)}$ the number of indices is at the most $2n$, and in $J^{(n)}$ at most $2n+1$. These integrals can be obtained recursively as follows. We can express $I^{(n)}$ and $J^{(n)}$ in terms of I and J, utilizing the partial fractions expansion

$$\Lambda^2(Q^2+\lambda^2)^{-1}(Q^2+\Lambda^2)^{-1}=A[(Q^2+\lambda^2)^{-1}-(Q^2+\Lambda^2)^{-1}],$$

with $A=\Lambda^2/\Lambda^2-\lambda^2$. This gives

$$\begin{aligned} I^{(1)} &= A[I(\lambda)-I(\Lambda)],\\ J^{(1)} &= A[J(\lambda)-J(\Lambda)]. \end{aligned} \tag{C3}$$

From these, we can form the additional integrals $I^{(2)}$, $I^{(3)}$, etc., by using the recursion formula

$$I^{(n+1)}=I^{(n)}-\frac{1}{2n}\Lambda\frac{\partial}{\partial\Lambda}I^{(n)}. \tag{C4}$$

The integrals with additional indexes can then be obtained using (B3). Of course we shall content ourselves with carrying out this program in the limit $\lambda\to 0$ and $\Lambda\to\infty$.

Let us consider the evaluation of $I^{(1)}$, $I_j^{(1)}$, $I_{ij}^{(1)}$ in detail, since the "extraordinary" terms appear in them. We will take $s^2>P^2$; that is, we shall evaluate the integrals necessary for radiative beta decay first. Keeping only real parts, and with $\epsilon\to 0$, we have

$$I^{(1)}(\lambda,\Lambda)=\frac{\pi^2A}{P}\left\{\arctan\frac{2\lambda P}{\lambda^2+s^2-P^2}-\arctan\frac{2\Lambda P}{\Lambda^2+s^2-P^2}\right\}. \tag{C5}$$

Now for $s^2>P^2$ the arguments of both arctangents are small *uniformly* in s, as $\lambda\to 0$ and $\Lambda\to\infty$, and so we can interchange the expansions with the integration (B3).

Keeping terms of order λ^3, and Λ^{-3}, we have

$$I^{(1)}\to\frac{\pi^2}{P}\left\{\frac{2\lambda P}{\lambda^2+s^2-P^2}-\frac{2\Lambda P}{\Lambda^2+s^2-P^2}-\frac{8\lambda^3P^3}{3(\lambda^2+s^2-P^2)^3}+\frac{8\Lambda^3P^3}{3(\Lambda^2+s^2-P^2)^3}\right\}, \tag{C6}$$

and so

$$\left(s\frac{\partial}{\partial P_i}+P_i\frac{\partial}{\partial s}\right)I^{(1)}\to\pi^2\left\{-\frac{16\lambda^3 sP_i}{3(\lambda^2+s^2-P^2)^3}+\frac{16\Lambda^3 sP_i}{3(\Lambda^2+s^2-P^2)^3}\right\},$$

which gives on integration

$$I_i^{(1)}\to\pi^2\left\{-\frac{4\lambda^3 P_i}{3(\lambda^2+s^2-P^2)^2}+\frac{4\Lambda^3 P_i}{3(\Lambda^2+s^2-P^2)^2}\right\}. \quad \text{(C7)}$$

Continuing the process

$$\left(s\frac{\partial}{\partial P_j}+P_j\frac{\partial}{\partial s}\right)I_i^{(1)}\to\pi^2\left\{-\frac{4\lambda^3\delta_{ij}}{3(\lambda^2+s^2-P^2)^2}+\frac{4\Lambda^3\delta_{ij}}{3(\Lambda^2+s^2-P^2)^2}\right\},$$

and so

$$I_{ij}^{(1)}\to\pi^2\left\{-\frac{2\lambda^3\delta_{ij}}{3(\lambda^2+s^2-P^2)}+\frac{2\Lambda^3\delta_{ij}}{3(\Lambda^2+s^2-P^2)}\right\}. \quad \text{(C8)}$$

Therefore we find that

$$I^{(1)}\to I\to 0,\quad I_i^{(1)}\to I_i\to 0,\quad I_{ij}^{(1)}\to(2\pi^2/3)\Lambda\delta_{ij}. \quad \text{(C9)}$$

This last term, proportional to $\Lambda=1/R$, is the "extraordinary" term. Note the interesting fact that had we ignored the convergence difficulty and performed the operation (B3) on I_i to form I_{ij}, we would have found a finite, but incorrect answer, namely zero.

For completeness, we must still discuss this procedure for the integrals I with $s^2=P^2$, and the integrals J. In the former case, we can *not* expand first in λ, although we can expand in Λ. The process can be carried out however, with the results

$$I^{(1)}\to I\to\pi^3/2P,\quad I_i^{(1)}\to I_i\to 0,\quad I_{ij}^{(1)}\to(2\pi^2/3)\Lambda\delta_{ij}. \quad \text{(C10)}$$

In the case of the integrals J we can again expand first and then differentiate, etc., but the work becomes rather tedious and will not be given in detail. The result is, as we expected, that no "extraordinary" terms appear:

$$J^{(1)}\to J\to -\frac{\pi^3}{2p(s^2-P^2)},\quad J_i^{(1)}\to J_i\to 0,\quad J_{ij}^{(1)}\to J_{ij}\to 0,\quad J_{ijk}^{(1)}\to 0. \quad \text{(C11)}$$

Although the integrals with $n>1$ are not needed in first forbidden transitions, one can easily carry out the recusion (C4) in n, and show that the extraordinary terms will also appear in more highly forbidden transitions, but only in the integrals I.