Two-Component Spinors in General Relativity*

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In this paper we develop a general-relativistic theory of two-component spinors somewhat along the lines pioneered more than twenty years ago by Weyl and by Infeld and van der Waerden. This formalism is in some respects more natural than the theory of four-component spinors. We begin by introducing into a four-dimensional manifold as our basic geometric structure a set of four 2×2 Hermitian matrices, σ^μ, and we show that these matrices by themselves define uniquely a Riemannian metric with the usual signature of a space-time manifold. It turns out that one can describe the gravitational field and its laws very conveniently in this matrix formalism; at the same time the σ^μ enable one to construct invariant two-component and four-component spinor wave equations. We use these formal possibilities to define local Lorentz transformations and, in particular, the transformations corresponding to time reversal and to space inversion.

1. INTRODUCTION

In recent months physicists have become interested increasingly in two-component spinors, because of their possible bearing on the nature of neutrinos.[1] It appears worth while to review the formal relationship of these quantities to general-relativistic theories, which was originally examined by Infeld and van der Waerden.[2] Not unnaturally, their work was concerned primarily with Dirac theory and hence with four-component spinors, though they developed a remarkably complete general-relativistic two-component formalism.

In this paper we shall take a point of view somewhat at variance with that of Infeld and van der Waerden. These authors started with a Riemannian manifold, into which they placed spinor fields. We shall begin with a field of spin matrices and generate from them the Riemannian manifold. We shall take special pains to demonstrate that the introduction of four (linearly independent) spin matrices is completely equivalent to the introduction of a metric tensor, in that the spin matrices define the tensor uniquely, while the metric tensor in turn defines the spin matrices, except for a group of transformations which may be described as local Lorentz transformations.

Aside from this different point of view we shall contribute to Infeld's and van der Waerden's work a few additional results, and we shall examine in some detail the role of parity and time reversal in our formalism.

2. SPINOR ALGEBRA WITH UNIMOLECULAR TRANSFORMATIONS

We shall begin with the algebra of spinors that are defined with respect to unimodular transformations. At each point of physical space-time we define a two-dimensional complex linear vector space. With its help we may describe a spinor field, such as

$$\psi(x^\rho)=\begin{pmatrix}\psi^1(x^\rho)\\ \psi^2(x^\rho)\end{pmatrix}. \tag{2.1}$$

We can describe the same field in terms of a different base system by forming the linear combination of the two components ψ^1 and ψ^2,

$$\psi'(x)=a(x)\psi(x). \tag{2.2}$$

The coefficients of this transformation are arbitrary complex functions of space-time but are restricted by the requirement that their determinant be unity,

$$\|a\|\equiv a^1{}_1a^2{}_2-a^1{}_2a^2{}_1=1. \tag{2.3}$$

All relations to be set up are required to reproduce themselves under this group of transformations as well as under arbitrary curvilinear coordinate transformations. In what follows we shall use matrix notation without the use of the spinor indices α, $\dot\alpha$ originally suggested by van der Waerden.[3] These indices permit the visual identification of certain (though not all) types of transformation properties in spin space but otherwise are cumbersome. Needless to say, the choice between index notation and one like ours is a matter of convenience and not of principle.

In addition to the transformation law (2.2), a vector in spin space may obey any of the following three relations:

$$\bar\psi'=\bar a\bar\psi,\quad \psi^{\dagger\prime}=\psi^\dagger a^\dagger, \tag{2.4}$$

$$\chi'=\chi a^{-1}, \tag{2.5}$$

or

$$\chi^{\dagger\prime}=a^{\dagger-1}\chi^\dagger, \tag{2.6}$$

where $\bar a$, $a^\dagger$, and a^{-1} are the complex conjugate, Hermitian conjugate, and inverse of a, respectively. In van der Waerden's notation the four types (2.3) through (2.6) are written ψ^α, $\psi^{\dot\alpha}$, χ_α, and $\chi_{\dot\alpha}$, respectively. In spin space we do not introduce any symmetric or Hermitian matrix that would correspond to the metric

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[1] T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956); **105**, 1671 (1957).

[2] L. Infeld and B. L. van der Waerden, Sitzber. preuss. Akad. Wiss., Physik.-math. Kl., p. 380 (1933). Reviewed in W. L. Bade and H. Jehle, Revs. Modern Phys. **25**, 714 (1953).

[3] B. L. van der Waerden, Nachr. Akad. Wiss. Göttingen, Math.-physik. Kl. 100 (1929).

tensor in a Riemannian space. However, there exists the Levi-Civita object ϵ,

$$\epsilon=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.7}$$

which reproduces itself under spin transformations according to any one of the following schemes:

$$\begin{aligned} \epsilon&=a\epsilon a^T, \quad \epsilon=a^{-1T}\epsilon a^{-1}, \\ \epsilon&=\bar{a}\epsilon a^\dagger, \quad \epsilon=a^{\dagger-1}\epsilon\bar{a}^{-1}, \end{aligned} \tag{2.8}$$

where a^T is the transpose of a.

Up to this point we have not introduced into our scheme any geometric object that could serve as the carrier of the basic geometric structure, such as the metric tensor does in Riemannian geometry. For this purpose we shall now postulate the existence of a field of four Hermitian matrices σ^ρ which are to be linearly independent of each other, i.e., the equation

$$a_\rho\sigma^\rho=0 \tag{2.9}$$

(a_ρ ordinary numbers) implies $a_\rho=0$; under coordinate and spin transformations the σ^ρ are assumed to transform according to the law:

$$\sigma^{\rho'}=\left(\frac{\partial x^{\rho'}}{\partial x^\mu}\right)a^{\dagger-1}\sigma^\mu a^{-1}. \tag{2.10}$$

The product of two σ matrices will not possess any simple transformation law. However, we may construct a second set of matrices, τ^ρ, by means of the operation

$$\tau^\rho=\epsilon\bar{\sigma}^\rho\epsilon. \tag{2.11}$$

The transformation law

$$\tau^{\rho'}=\left(\frac{\partial x^{\rho'}}{\partial x^\mu}\right)a\tau^\mu a^\dagger \tag{2.12}$$

is a direct consequence of the defining Eq. (2.11). If we give the individual elements of the spin matrices the designations

$$\sigma^\rho=\begin{pmatrix} u^\rho+\alpha^\rho, & \beta^\rho-i\gamma^\rho \\ \beta^\rho+i\gamma^\rho, & u^\rho-\alpha^\rho \end{pmatrix}, \tag{2.13}$$

then the τ matrices have the components

$$\tau^\rho=\begin{pmatrix} -u^\rho+\alpha^\rho, & \beta^\rho-i\gamma^\rho \\ \beta^\rho+i\gamma^\rho, & -u^\rho-\alpha^\rho \end{pmatrix}. \tag{2.14}$$

The variables u^ρ, α^ρ, β^ρ, and γ^ρ are real. They are coordinate vectors but are not covariant under spin transformations. The product of a σ matrix by a τ matrix has a simple transformation law. The combination $-\frac{1}{2}(\tau^\mu\sigma^\nu+\tau^\nu\sigma^\mu)$ in particular is proportional to the unit matrix and hence is invariant with respect to spin transformations. Under coordinate transformations it is a contravariant tensor:

$$\begin{aligned} -\tfrac{1}{2}(\tau^\mu\sigma^\nu+\tau^\nu\sigma^\mu)&\equiv u^\mu u^\nu-(\alpha^\mu\alpha^\nu+\beta^\mu\beta^\nu+\gamma^\mu\gamma^\nu) \\ &\equiv g^{\mu\nu}. \end{aligned} \tag{2.15}$$

This expression has all the formal properties of a c number contravariant symmetric tensor. We shall call it the contravariant metric tensor. By constructing such quantities as

$$\begin{aligned} u_\mu&=\Delta^{-1}\delta_{\mu\iota\kappa\lambda}\alpha^\iota\beta^\kappa\gamma^\lambda, \\ \alpha_\mu&=\Delta^{-1}\delta_{\mu\iota\kappa\lambda}u^\mu\beta^\kappa\gamma^\lambda, \quad \text{etc.}, \\ \Delta&=\delta_{\mu\iota\kappa\lambda}u^\mu\alpha^\iota\beta^\kappa\gamma^\lambda, \end{aligned} \tag{2.16}$$

we can easily obtain the components of the covariant metric tensor:

$$g_{\mu\nu}=u_\mu u_\nu-(\alpha_\mu\alpha_\nu+\beta_\mu\beta_\nu+\gamma_\mu\gamma_\nu). \tag{2.17}$$

In terms of this metric, u^μ, α^μ, β^μ, and γ^μ are unit vectors, the first with positive norm and the remainder with negative norm.

It is easy to verify the following results, which are both coordinate- and spin-covariant:

$$\sigma_\rho\tau^\rho=-4, \quad \sigma_\rho\sigma^\rho=-2, \quad \sigma_\rho A\tau^\rho=-2\,\mathrm{tr}(A), \tag{2.18}$$

and the following, which deal with quantities that are coordinate-covariant but not spin-covariant:

$$u^\rho=\tfrac{1}{2}\,\mathrm{tr}(\sigma^\rho), \quad u_\rho\sigma^\rho=1, \quad \tau^\rho=\sigma^\rho-2u^\rho. \tag{2.19}$$

For comparison with the algebraic properties of Pauli's original three σ matrices, it is convenient to introduce the projections of various quantities into the three-space perpendicular to the unit vector u^ρ. To this end we define:

$$\begin{aligned} \boldsymbol{\sigma}^\rho&=\sigma^\rho-u^\rho=\tau^\rho+u^\rho, \\ \boldsymbol{\delta}_\mu{}^\nu&=\delta_\mu{}^\nu-u_\mu u^\nu, \\ \mathbf{g}_{\mu\nu}&=g_{\mu\nu}-u_\mu u_\nu, \quad \text{etc.}, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} \boldsymbol{\delta}^{\iota\kappa\lambda}&=\delta^{\iota\kappa\lambda\alpha}\delta_{\rho\sigma\tau\alpha}\alpha^\rho\beta^\sigma\gamma^\tau \\ &=\alpha^\iota(\beta^\kappa\gamma^\lambda-\beta^\lambda\gamma^\kappa)+\beta^\iota(\gamma^\kappa\alpha^\lambda-\gamma^\lambda\alpha^\kappa) \\ &\qquad+\gamma^\iota(\alpha^\kappa\beta^\lambda-\alpha^\lambda\beta^\kappa). \end{aligned} \tag{2.21}$$

Here $\boldsymbol{\delta}^{\iota\kappa\lambda}$ is a tensor with respect to coordinate transformations, is completely skew-symmetric, and is orthogonal to u_ρ. The product of two $\boldsymbol{\sigma}$ is given by the relatively simple expression

$$\boldsymbol{\sigma}^\mu\boldsymbol{\sigma}^\nu=-\mathbf{g}^{\mu\nu}+i\boldsymbol{\delta}^{\mu\nu\rho}\boldsymbol{\sigma}_\rho, \tag{2.22}$$

which incorporates both the commutator and the anti-commutator of the $\boldsymbol{\sigma}$ matrices. Finally, for some calculations, the following formula is useful:

$$\begin{aligned} -\boldsymbol{\delta}_{\alpha\beta\gamma}\boldsymbol{\delta}^{\iota\kappa\lambda}&=\boldsymbol{\delta}_\alpha{}^\iota(\boldsymbol{\delta}_\beta{}^\kappa\boldsymbol{\delta}_\gamma{}^\lambda-\boldsymbol{\delta}_\beta{}^\lambda\boldsymbol{\delta}_\gamma{}^\kappa) \\ &\quad+\boldsymbol{\delta}_\alpha{}^\kappa(\boldsymbol{\delta}_\beta{}^\lambda\boldsymbol{\delta}_\gamma{}^\iota-\boldsymbol{\delta}_\beta{}^\iota\boldsymbol{\delta}_\gamma{}^\lambda) \\ &\quad+\boldsymbol{\delta}_\alpha{}^\lambda(\boldsymbol{\delta}_\beta{}^\iota\boldsymbol{\delta}_\gamma{}^\kappa-\boldsymbol{\delta}_\beta{}^\kappa\boldsymbol{\delta}_\gamma{}^\iota). \end{aligned} \tag{2.23}$$

3. LOCAL LORENTZ TRANSFORMATIONS

Equation (2.15), along with (2.11), demonstrates that to any chosen field of σ^μ there belongs one, and only one, Riemannian metric, which has the signature ordinarily assumed in general relativity. We shall now examine the set of σ matrices that may be associated, at any one world point, with a given Riemannian metric of the same signature. Because of Eq. (2.13), the members of this set are in a one-to-one relationship to the set of quadruplets of mutually perpendicular unit vectors that may be constructed at that world point. Thus each set of four σ matrices at a world point may be associated uniquely with a local Lorentz frame. Relations between two possible sets of σ matrices compatible with the same metric (always at one world point) must be representable by Lorentz transformations.

Given two such sets of possible spin matrices, designated by σ^μ and by $\sigma^{\mu\prime}$, respectively, we can always express one set linearly in terms of the other,

$$\sigma^{\mu\prime}=\gamma^\mu{}_\nu\sigma^\nu. \tag{3.1}$$

By assumption, both sets are Hermitian, and the coefficients $\gamma^\mu{}_\nu$ accordingly real. They satisfy the relationship

$$\gamma^\mu{}_\rho\gamma^\nu{}_\sigma g^{\rho\sigma}=g^{\mu\nu}, \tag{3.2}$$

which is typical of the coefficients of a Lorentz transformation. The two sets σ^μ and $\sigma^{\mu\prime}$ may (but need not) be related to each other by a spin transformation of the type (2.10) (without coordinate transformation). In that event we have

$$\gamma^\mu{}_\nu\sigma^\nu=a^{\dagger-1}\sigma^\mu a^{-1},\quad \gamma^\mu{}_\nu\tau^\nu=a\tau^\mu a^\dagger. \tag{3.3}$$

With the help of the relationship (2.15) we may solve for the "Lorentz coefficients":

$$\gamma^\mu{}_\nu=-\tfrac{1}{2}(a^{\dagger-1}\sigma^\mu a^{-1}\tau_\nu+\sigma_\nu a\tau^\mu a^\dagger). \tag{3.4}$$

If the transformations are infinitesimal, i.e., if a differs from the unit matrix only by an infinitesimal matrix δa then this relationship simplifies. If we write δa in the form

$$\delta a=\delta a_\rho\boldsymbol{\sigma}^\rho,\quad \delta a_\rho u^\rho=0, \tag{3.5}$$

(the trace of δa vanishes, because a itself is unimodular), then we obtain for the infinitesimal Lorentz coefficients the expressions

$$\delta\gamma^\mu{}_\nu=2\,\mathrm{Re}(\delta a_\rho)(\mathbf{g}^{\rho\mu}u_\nu-\boldsymbol{\delta}_\nu{}^\rho u^\mu)-2\,\mathrm{Im}(\delta a_\rho)\boldsymbol{\delta}^{\rho\mu}{}_\nu. \tag{3.6}$$

These expressions can be readily solved for the components of the two real vectors $\mathrm{Re}(\delta a_\rho)$, $\mathrm{Im}(\delta a_\rho)$. It is well known, from the special relativistic theory of two-component spinors, that Eqs. (3.4) possess solutions if, and only if, the Lorentz coefficients on the left represent a "proper" Lorentz transformation, that is, if the two conditions

$$u_\mu u^\nu\gamma^\mu{}_\nu>0,$$
$$\mathrm{Det}\begin{Vmatrix}\alpha_\mu\alpha^\nu\gamma^\mu{}_\nu, & \alpha_\mu\beta^\nu\gamma^\mu{}_\nu, & \alpha_\mu\gamma^\nu\gamma^\mu{}_\nu\\ \beta_\mu\alpha^\nu\gamma^\mu{}_\nu, & \beta_\mu\beta^\nu\gamma^\mu{}_\nu, & \beta_\mu\gamma^\nu\gamma^\mu{}_\nu\\ \gamma_\mu\alpha^\nu\gamma^\mu{}_\nu, & \gamma_\mu\beta^\nu\gamma^\mu{}_\nu, & \gamma_\mu\gamma^\nu\gamma^\mu{}_\nu\end{Vmatrix}>0 \tag{3.7}$$

are satisfied. These solutions are ambiguous, in that with any matrix a the matrix $-a$ is also a solution of Eqs. (3.4) with given left-hand side. Among the proper Lorentz transformations we shall call those *rotations* for which $u_\mu u^\nu\gamma^\mu{}_\nu=1$. The corresponding spin transformation matrices are unitary. Under rotations the trace vector u^μ is invariant.

Two sets of spin matrices σ^μ and $\sigma^{\mu\prime}$ that belong to the same metric may also be related by an *improper* Lorentz transformation. We shall in particular consider the possible relationships

$$\sigma^{\mu\prime}=\tau^\mu\quad (T), \tag{3.8}$$

"time reversal," and

$$\sigma^{\mu\prime}=-\tau^\mu\quad (P), \tag{3.9}$$

"space inversion" or "parity." The transformation T reverses the sign of u^μ, whereas P leaves u^μ unchanged and instead reverses the signs of α^μ, β^μ, and γ^μ. The product of these two transformations, PT,

$$\sigma^{\mu\prime}=-\sigma^\mu \tag{3.10}$$

is also an improper Lorentz transformation. T, P, and PT all commute with the rotations; that is to say, because under unitary spin transformations the transformation laws for σ^μ, (2.10), and for τ^μ, (2.12), are identical, the performance of a unitary spin transformation U is equivalent to the transformations TUT, PUP, and $PTUPT$.

A relationship may be considered invariant under T or P if along with the σ and τ matrices the remaining variables occurring in the equations may be transformed so that the equations reproduce their form. For instance, if two spinors ψ and χ^μ are related by the equation

$$\chi^\mu=\sigma^\mu\psi, \tag{3.11}$$

then this relationship will reproduce itself under the transformation T if

$$\chi^{\mu\prime}=\epsilon\bar{\chi}^\mu,\quad \psi'=\epsilon\bar{\psi}. \tag{3.12}$$

Under P the corresponding relationship is

$$\chi^{\mu\prime}=-\epsilon\bar{\chi}^\mu,\quad \psi'=\epsilon\bar{\psi} \tag{3.13}$$

and under PT

$$\chi^{\mu\prime}=-\chi^\mu,\quad \psi'=\psi. \tag{3.14}$$

From the equivalence of spin transformations and local proper Lorentz transformations we derive the following *lemma on locally geodesic frames:* given an arbitrary field of σ^μ matrices, $\sigma^\mu(x^\rho)$; at any chosen world point x^0 we can bring it about that the four spin matrices take the values of ± 1, $\pm\boldsymbol{\sigma}$ (where $\boldsymbol{\sigma}$ denotes the three customary Pauli spin matrices) and that the partial derivatives of all their components with respect to the coordinates vanish.

For proof we remark first that the corresponding lemma for the metric tensor is well known (i.e., that the components of the metric tensor assume the values of the Minkowski metric and that all Christoffel symbols vanish at a chosen world point). Accordingly we may carry out a coordinate transformation so that the metric tensor (2.15) takes the desired form. Then the spin matrices will differ from the standard form at most by a unimodular spin transformation, plus possibly the transformation P, T, or PT. In all these cases we may achieve the standard form (*modulo* the two signs as indicated above) by a unimodular spin transformation. Any remaining derivatives may be removed by an infinitesimal spin transformation.

4. SPIN-AFFINE CONNECTION

We shall define the spin-affine connection as a set of four 2×2 matrices, Γ_ρ, which will help us to define a covariant derivative of a spinor,

$$\psi_{;\rho}\equiv\psi_{,\rho}+\Gamma_\rho\psi,\quad \psi_{;\rho}'=a\psi_{;\rho}. \tag{4.1}$$

Under a coordinate transformation Γ_ρ is a vector. Under a spin transformation the Γ_ρ change as follows:

$$\Gamma_\rho'=(a\Gamma_\rho-a_{,\rho})a^{-1}. \tag{4.2}$$

Depending on the transformation laws of various types of spinor fields, we may require also the Hermitian adjoint, as well as the transpose and the conjugate complex, of Γ_ρ.

We shall make the covariant derivative of ϵ vanish. We have

$$\epsilon=a\epsilon a^T,\quad \epsilon_{;\rho}=\Gamma_\rho\epsilon+\epsilon\Gamma_\rho{}^T=0,\quad \Gamma_\rho{}^T=\epsilon\Gamma_\rho\epsilon. \tag{4.3}$$

Hence the trace of Γ_ρ must vanish.

Next we shall require that the covariant derivatives of σ^μ and τ^μ vanish. That this is indeed possible follows from the lemma on locally geodesic spin frames proved at the end of the last section. In such a locally geodesic spin frame we set the components of the affine connection (locally) equal to zero. In any other spin frame Γ_ρ is then determined by the transformation law (4.2).

These requirements are not only compatible with any set of σ^ρ and their derivatives, the spin-affine connection is thereby uniquely determined. For proof we examine the weaker condition

$$\begin{aligned}0=\tau_\mu\sigma^\mu{}_{;\rho}&=\tau_\mu\left(\sigma^\mu{}_{,\rho}+\begin{Bmatrix}\mu\\ \alpha\rho\end{Bmatrix}\sigma^\alpha-\Gamma_\rho{}^\dagger\sigma^\mu-\sigma^\mu\Gamma_\rho\right)\\ &=\tau_\mu\left(\sigma^\mu{}_{,\rho}+\begin{Bmatrix}\mu\\ \alpha\rho\end{Bmatrix}\sigma^\alpha\right)+2\operatorname{tr}\Gamma_\rho{}^\dagger+4\Gamma_\rho.\end{aligned} \tag{4.4}$$

Accordingly we find for Γ_ρ the expression

$$\begin{aligned}\Gamma_\rho&=-\tfrac{1}{4}\tau_\mu\left(\sigma^\mu{}_{,\rho}+\begin{Bmatrix}\mu\\ \alpha\rho\end{Bmatrix}\sigma^\alpha\right)\\ &=\tfrac{1}{4}\left(\tau^\mu{}_{,\rho}+\begin{Bmatrix}\mu\\ \alpha\rho\end{Bmatrix}\tau^\alpha\right)\sigma_\mu.\end{aligned} \tag{4.5}$$

As usual, we define the spin curvature by means of the commutator of mixed covariant derivatives:

$$\begin{aligned}\psi_{;\iota\kappa}-\psi_{;\kappa\iota}&=\mathrm{P}_{\iota\kappa}\psi,\\ \mathrm{P}_{\iota\kappa}&=\Gamma_{\iota,\kappa}-\Gamma_{\kappa,\iota}-\Gamma_\iota\Gamma_\kappa+\Gamma_\kappa\Gamma_\iota.\end{aligned} \tag{4.6}$$

If the covariant derivatives of the spin matrices are to vanish, we obtain the following relationship between the Riemannian curvature and the spin curvature:

$$R_{\iota\kappa\lambda\mu}\sigma^\lambda-\mathrm{P}_{\iota\kappa}{}^\dagger\sigma_\mu-\sigma_\mu\mathrm{P}_{\iota\kappa}=0, \tag{4.7}$$

and, likewise:

$$R_{\iota\kappa\lambda\mu}\tau^\lambda+\mathrm{P}_{\iota\kappa}\tau_\mu+\tau_\mu\mathrm{P}_{\iota\kappa}{}^\dagger=0. \tag{4.8}$$

With their help, we may express the two curvatures explicitly in terms of each other. We have

$$\begin{aligned}R_{\iota\kappa\lambda\mu}&=\tfrac{1}{2}(\sigma_\lambda\tau_\mu\mathrm{P}_{\iota\kappa}{}^\dagger-\mathrm{P}_{\iota\kappa}{}^\dagger\sigma_\mu\tau_\lambda+\sigma_\lambda\mathrm{P}_{\iota\kappa}\tau_\mu-\sigma_\mu\mathrm{P}_{\iota\kappa}\tau_\lambda)\\ &=\tfrac{1}{4}\operatorname{tr}[\mathrm{P}_{\iota\kappa}{}^\dagger(\sigma_\lambda\tau_\mu-\sigma_\mu\tau_\lambda)+(\tau_\mu\sigma_\lambda-\tau_\lambda\sigma_\mu)\mathrm{P}_{\iota\kappa}],\end{aligned} \tag{4.9}$$

and

$$\mathrm{P}_{\iota\kappa}=\tfrac{1}{4}\tau^\lambda\sigma^\mu R_{\iota\kappa\lambda\mu},\quad \operatorname{tr}(\mathrm{P}_{\iota\kappa})=0. \tag{4.10}$$

The contracted forms of the Riemannian curvature, the Ricci tensor and the curvature scalar, hence take the forms

$$\begin{aligned}R_{\kappa\lambda}&=\tfrac{1}{4}\operatorname{tr}[\sigma_\lambda\tau^\rho\mathrm{P}_{\rho\kappa}{}^\dagger-\mathrm{P}_{\rho\kappa}{}^\dagger\sigma^\rho\tau_\lambda+\sigma_\lambda\mathrm{P}_{\rho\kappa}\tau^\rho-\sigma^\rho\mathrm{P}_{\rho\kappa}\tau_\lambda),\\ R&=\tfrac{1}{2}\operatorname{tr}(\tau^\sigma\mathrm{P}_{\sigma\rho}{}^\dagger\sigma^\rho+\sigma^\rho\mathrm{P}_{\sigma\rho}\tau^\sigma).\end{aligned} \tag{4.11}$$

This last expression, multiplied by $(-g)^{\frac{1}{2}}$, may be adopted as the Lagrangian of the gravitational field.

It remains to examine the transformation properties of the affine connection and of the spin curvature under improper Lorentz transformations. Substitution of Eqs. (3.8) and (3.9) into the expression for the spin-affine connection (4.5) yields under both T and P

$$\Gamma_\rho'=-\Gamma_\rho{}^\dagger, \tag{4.12}$$

and

$$\mathrm{P}_{\iota\kappa}'=-\mathrm{P}_{\iota\kappa}{}^\dagger. \tag{4.13}$$

Under the transformation PT the spin-affine connection remains unchanged. Inspection shows that such relations as Eqs. (4.7) with (4.8) and (4.11) are invariant under time reversal and space inversion.

5. GAUGE TRANSFORMATIONS

Weyl[4] has called attention to the possibility of enlarging the group of spin transformations (2.2) by permitting the determinant of a to be a quantity of *magnitude* 1. This enlargement of the transformation

[4] H. Weyl, Z. Physik **56**, 330 (1929).

group would have no effect on the transformation law of the spin matrices σ^ρ and τ^ρ as shown by the form of Eqs. (2.4) and (2.5). The matrix ϵ, on the other hand, would remain invariant only if its transformation law were changed to

$$\epsilon' = \|a\|^{-1} a \epsilon a^T \equiv \epsilon, \quad \|a\| = e^{2i\theta}, \tag{5.1}$$

i.e., if its density character were explicitly recognized. In that case the requirement that its covariant derivative vanish would reduce to

$$\epsilon_{;\iota} = \Gamma_\iota \epsilon + \epsilon \Gamma_\iota{}^T - \epsilon\, \mathrm{tr}(\Gamma_\iota) = 0, \tag{5.2}$$

and this requirement is empty. Hence one would no longer require that the trace of Γ_ρ vanish, but one may require that $\mathrm{tr}(\Gamma_\rho)$ be imaginary, as

$$\mathrm{tr}(\Gamma_\rho') = \mathrm{tr}(\Gamma_\rho) - 2i\theta_{,\rho}. \tag{5.3}$$

In this slightly generalized formalism it is possible to introduce wave-function spinors corresponding to differently charged particles. We may introduce the electromagnetic potentials by setting

$$\frac{e}{\hbar c}\phi_\rho = -\frac{i}{2}\,\mathrm{tr}(\Gamma_\rho), \tag{5.4}$$

where e signifies the elementary unit of charge (not necessarily the charge of the particle whose wave function is being considered). A spinning particle of arbitrary electric charge and zero rest mass would then be represented by a wave function whose transformation law under spin transformations would include as an extra factor an appropriate power of the determinant of a. A neutral particle, in particular, would correspond to the transformation law

$$\psi' = \|a\|^{-\frac{1}{2}} a \psi. \tag{5.5}$$

The expression (4.5) for the affine connection would be changed by the addition of the term $-(ie/\hbar c)\phi_\rho$ on the right. The value of the spin curvature (4.6) would be changed by the addition of the c-number term $-(ie/\hbar c)\phi_{\iota\kappa}$. The equalities (4.7), (4.8), and (4.9) would remain unchanged, whereas Eq. (4.10) would add the electromagnetic field tensor, $-(ie/\hbar c)\phi_{\iota\kappa}$, on the right.

Under the improper Lorentz transformations the values of $e\phi_\rho$ as defined by Eq. (5.4) remain unchanged.

The fusion of spin transformations and gauge transformations brought about by the generalization of the transformation group is, of course, quite superficial. Any nonsingular matrix may be represented as the product of a unimodular matrix and a (complex) number. The principal virtue of the approach sketched here, as compared with other possibilities, is that it leads naturally to Dirac's momentum operator.

Infeld and van der Waerden[2] have also considered the further generalization of the spin transformation that consists of permitting a matrices to have any determinant whatsoever. In that case wave functions must possess two distinct densities, one with respect to the phase of the determinant (as sketched out in this section) and one with respect to the norm. Accordingly, particles represented by such wave functions would possess two scalar properties, one presumably the electric charge and the other some other quantum number, such as total isotopic spin, strangeness, or the like. Such a formal possibility would have similarity with the suggestions advanced by Yang and Mills,[5] Bludman,[6] and Utiyama.[7] The analog of the b field advanced by Yang and Mills would be the real part of $\mathrm{tr}(\Gamma_\rho)$, which could no longer be set equal to zero and which under T and P would change sign, in contrast to the imaginary part. Because the complex numbers (multiplied by unit matrices) form part of the center of any transformation group of matrices, this extension of the original unimodular group also has little formal rigidity and should not be interpreted or accepted as a step toward a unified field theory.

6. WAVE EQUATIONS. FOUR-COMPONENT SPINORS

The wave equation of a two-component spinning particle would take either one or the other of the two forms

$$\sigma^\rho \psi_{;\rho} = 0, \quad \psi' = \|a\|^{\frac{1}{2}(N-1)} a \psi \tag{6.1}$$

or

$$\tau^\rho \chi_{;\rho} = 0, \quad \chi' = \|a\|^{-\frac{1}{2}(N+1)} a^{\dagger -1} \chi. \tag{6.2}$$

These two wave equations go over into each other under either P or T. The effect of an incident electromagnetic potential on the wave function and the value of the electric charge of the particle (Ne) are included in the form of the covariant derivative.

Let us consider the first Eq. (6.1) and attempt to separate the individual components of this wave equation by continued differentiation. We form the two equations

$$\tau^\alpha \sigma^\beta \psi_{;\beta\alpha} = 0, \quad \tau^\beta \sigma^\alpha \psi_{;\alpha\beta} = 0, \tag{6.3}$$

and obtain the second-order wave equation,

$$g^{\alpha\beta}\psi_{;\alpha\beta} + \tfrac{1}{2}\tau^\beta \sigma^\alpha \left[\mathrm{P}_{\alpha\beta} - \frac{ie}{\hbar c}(N-1)\phi_{\alpha\beta} \right] \psi = 0. \tag{6.4}$$

The second term contains references both to the electromagnetic field and to the space-time curvature.

If we wish to introduce particles with nonvanishing rest mass, we are led naturally to four-component spinors. The only manner in which we can construct first-order spin-covariant wave equations is by writing, for instance,

$$\sigma^\rho \psi_{;\rho} + m\chi = 0, \quad \tau^\rho \chi_{;\rho} + m\psi = 0. \tag{6.5}$$

The fusion of the two two-component spinors ψ and χ into one field with four components results in the

[5] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954).
[6] S. A. Bludman, Phys. Rev. **100**, 372 (1955).
[7] R. Utiyama, Phys. Rev. **101**, 1597 (1956).

formation of a Dirac-type spinor. This procedure is exactly the same that is employed in special-relativistic treatments. The proper Lorentz transformations (L), and the transformations T, P, and PT correspond to 4×4 matrices having the form

$$L=\begin{pmatrix} a, & 0 \\ 0, & a^{\dagger -1} \end{pmatrix}, \quad T=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$P=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad PT=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.6}$$

where each symbol on the right stands itself for a 2×2 matrix.

With these considerations in mind, we shall first construct a Lagrangian density that leads to two-component spinor wave equations. Such a Lagrangian takes the form

$$L_n=-i\gamma(\psi^\dagger\sigma^\rho\psi_{;\rho}+\chi^\dagger\tau^\rho\chi_{;\rho}), \quad \gamma=(-g)^{\frac{1}{2}}. \tag{6.7}$$

With respect to coordinate transformations L_n is certainly a scalar density. Each term is invariant, too, with respect to proper spin transformations as well as with respect to the transformation PT. With respect to the transformations P and T, there is an ambiguity concerning the density character of the spinors with respect to (proper) spin transformations. If ψ and χ have the weight $\frac{1}{2}$, which corresponds to neutral particles, then each of the two terms of L_n goes over into itself. Otherwise the density character of either function changes under P or under T so that the original and the transformed weight satisfy the equality

$$W+W'=1. \tag{6.8}$$

Hence for charged particles L_n remains invariant under P and T only if each term is assumed to go over into the other and if the two wave functions ψ and χ are given such weights that they correspond to opposite electric charges.

7. ACTION PRINCIPLES. CONCLUSION

To be relativistically invariant a Lagrangian must be a scalar density of weight 1 with respect to coordinate transformations. With respect to proper spin transformations the density weight should be zero (in the formalism with gauge transformations; with respect to strictly unimodular transformations the weight is meaningless). Finally with respect to the improper transformations all members of the Lagrangian density must have the same transformation properties if the theory is to be invariant with respect to these transformations. The coordinate weight of the Lagrangian is usually achieved by the multiplication of a scalar by the metric density $(-g)^{\frac{1}{2}}$. In terms of the spin matrices this density is

$$(-g)^{\frac{1}{2}}=i\delta_{\iota\kappa\lambda\mu}\sigma^\iota\tau^\kappa\sigma^\lambda\tau^\mu\equiv\gamma. \tag{7.1}$$

The expression on the right lacks the ambiguity of sign of the square root. Under a coordinate transformation it multiplies by the Jacobian of the transformation. Under the transformations T and P it changes sign. If we consider the gravitational and the electromagnetic field Lagrangians to be integral parts of L, then the transformation properties of any additional terms are fully determined to the extent that we require invariance of L under these various transformations.

This situation differs from the special-relativistic theory in that the present formalism contains the Lorentz group as a local transformation group quite apart from the group of (curvilinear) coordinate transformations and that the forms which in the special theory are numerics (the metric tensor, the Pauli and Dirac spin matrices) are here field variables. Our formalism contains the additional elements required for the representation of the gravitational field; it thereby acquires a rigidity with respect to transformation properties which the usual theory apparently does not possess.

As our next step we shall attempt to develop an algorithm for the routine derivation of transformation properties of various physical variables under time reversal, space inversion, and particle conjugation, particularly if the particle fields are hyperquantized.

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