Theory of High-Energy Deuteron Scattering*

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The scattering of high-energy deuterons from spin-zero targets is treated in the framework of the impulse approximation, using the polarization formalism of Wolfenstein and Ashkin, which is here extended to the case of spin-1 particles. The contributions of the deuteron D-state are included and are found to be important in large-angle scattering. The contributions to the deuteron scattering due to the simultaneous scattering of both particles of the deuteron are also included. These contributions are treated by a multi-time formalism similar to that used in the Bethe-Salpeter and Lévy-Klein approach to the relativistic two-body wave equation, but here a slightly different assumption regarding the relative time dependence is made. It is found that these contributions are important at both large and small scattering angles, and account for the large disparity between the experimental and theoretical values of the differential cross section obtained in previous calculations.

INTRODUCTION

In recent experiments the differential cross sections and the polarization effects in the scattering of deuterons by carbon and various other nuclei have been measured.[1] An attempt to interpret the experimental results on the basis of an impulse approximation has been made by Baldwin.[1,2] For a typical case of 157-Mev deuterons on carbon, the differential cross section he obtains is larger than the measured value by a factor of about 2.5 for small scattering angles, and at large angles it becomes smaller than the measured value by a factor of about 7. The predicted polarization reaches a maximum of about 5%, whereas the experimental value rises to about 50%. It has been suggested[1] that the discrepancy at large angles may be due, in part, to the effects of the deuteron D-state contributions, which were not considered in Baldwin's treatment. The D-state contributions are, of course, suppressed by a factor of the D-state amplitude, which is $\sim$20%, but at large angles they might be expected to become important for the following reason. A dominating factor in the large-angle differential cross section predicted on the basis of the impulse approximation is the sticking factor.[3] At large angles this factor, which is essentially the Fourier transform of the square of the deuteron wave function, becomes quite small when only the S-state part of the deuteron wave function is included. Since the D-state wave function is sharply peaked, compared with the S-state wave function, the D-state parts of the sticking factor might be expected to have larger high-momentum components than the pure S-state contribution. As high-momentum components correspond to large scattering angles, it is possible that at large angles the D-state parts of the sticking factor may become large enough to compensate for the small D-state amplitude.

In order to investigate this possibility, the impulse approximation for the scattering of deuterons by carbon has been extended to include the D-state contributions. The calculations, which are carried out in Sec. II, show that the D-state contributions at large angles are almost equal in importance to the S-state contributions but that they are not sufficiently large to produce by themselves the large changes required to obtain agreement with the experimental results.

A second process that would evidently contribute significantly at large angles is the simultaneous scattering of both particles of the deuteron. In Baldwin's treatment, which includes only the effects of processes in which a single particle of the deuteron is scattered, the sharp decrease in the large-angle differential cross section caused by the sticking factor reflects the large probability that the deuteron will become disassociated if one particle of the deuteron receives a large impulse. However, if both particles receive large impulses of approximately equal magnitudes this tendency to break apart should be reduced, and at sufficiently large angles this type of contribution might be expected to predominate over those in which only a single particle is scattered. The theory for the simultaneous scattering is developed in Sec. III, and it is shown that for large scattering angles the effects of the simultaneous scattering indeed becomes the dominant contribution.

The effects of the simultaneous scattering are important also in the small-angle region. In Baldwin's treatment the particle that is not scattered remains, in effect, undisturbed in some plane-wave state. The fact that Baldwin's result is too large in the small-angle region can be explained, qualitatively, by noting that the amplitude for the "unscattered" particle should evidently be reduced to account for the fact that some of these particles will be scattered and hence removed from the unscattered beam. Simple estimates show that Baldwin's results should be reduced to approximately the experimental values when this effect is considered.

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[1] Baldwin, Chamberlain, Segrè, Tripp, Wiegand, and Ypsilantis, Phys. Rev. **103**, 1502 (1956).

[2] John A. Baldwin, Jr., thesis, University of California Radiation Laboratory Report UCRL–3412, May, 1956 (unpublished); see also W. Lakin, reference 8.

[3] Goeffrey F. Chew, Phys. Rev. **80**, 196 (1950); **74**, 809 (1948).

The quantitative treatment of this effect is obtained by considering the interference between the processes in which a single particle and those in which both particles are scattered. The evaluation of the interference term requires a knowledge of phase of the scattering amplitude for the nucleon-carbon scattering. In the forward direction this may be determined by the use of the optical theorem. The differential cross section obtained if one assumes this phase to persist at all angles is in good agreement with the experimental results for angles less than 14°. At larger angles the predicted values become considerably too small, owing to the large destructive interference. If, as a more realistic approximation, the phase angle predicted by the Fernbach-Serber-Taylor model of nucleon-nucleon scattering is used, the interference becomes constructive at large angles, and the theoretical and experimental differential cross sections at all angles are brought into reasonable agreement; for angles less than 14° the experimental and theoretical cross sections are in virtually perfect agreement, whereas at large angles the predicted value is about 50% larger than the experimental value. The model of Fernbach, Serber, and Taylor is not a completely reliable basis for detailed considerations at large angles because, for one thing, polarization effects are not included. The results demonstrate, however, the importance of the simultaneous scattering processes in the scattering of deuterons both at large and at small angles. Results obtained by using more realistic models of the nucleon-nucleus interaction will be discussed in a subsequent paper.[4]

I. POLARIZATION FORMALISM

In this section the general formalism for the description of the nonrelativistic scattering of spin-1 particles by spin-0 targets is developed. The treatment is along the same general lines as that used by Wolfenstein and Ashkin[5] in their treatment of spin-$\frac{1}{2}$ particles, and is based upon the use of the density matrix and the M-matrix.

The M matrix that describes the scattering of a spin-1 particle by a target of zero spin will be three by three, and may be written in the following form:

$$M(\theta,\phi)=A(\theta,\phi)+B_i(\theta,\phi)S_i+C_{ij}(\theta,\phi)S_{ij}. \qquad (1)$$

A summation convention is to be understood, and i and j run over x, y, and z. The S_i are the usual matrices,

$$S_x=\frac{1}{\sqrt{2}}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix},\quad S_y=\frac{1}{\sqrt{2}}\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix},$$

$$S_z=\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}, \qquad (2)$$

while

$$S_{ij}\equiv\tfrac{1}{2}(S_iS_j+S_jS_i)-\tfrac{2}{3}I\delta_{ij}. \qquad (3)$$

These matrices, together with the unit matrix, form a complete set in the space of three-by-three matrices. The $CC_{ij}(\theta,\phi)$ are made unique by imposition of the condition that the matrix $C(\theta,\phi)$ with elements $C_{ij}(\theta,\phi)$ by symmetric and traceless. With the definitions

$$P_i(\theta,\phi)=\mathrm{Tr}[\tfrac{1}{3}M(\theta,\phi)\bar{M}(\theta,\phi)S_i]/\mathrm{Tr}[\tfrac{1}{3}M(\theta,\phi)\bar{M}(\theta,\phi)], \qquad (4)$$

$$T_{ij}(\theta,\phi)=\mathrm{Tr}[\tfrac{1}{3}M(\theta,\phi)\bar{M}(\theta,\phi)S_{ij}]/\mathrm{Tr}[\tfrac{1}{3}M(\theta,\phi)\bar{M}(\theta,\phi)], \qquad (5)$$

$$\tilde{P}_i'(\theta',\phi')\equiv\mathrm{Tr}[\tfrac{1}{3}\bar{M}'(\theta',\phi')M'(\theta',\phi')S_i]/\mathrm{Tr}[\tfrac{1}{3}\bar{M}'(\theta',\phi')M'(\theta',\phi')], \qquad (6)$$

$$\tilde{T}_{ij}'(\theta',\phi')\equiv\mathrm{Tr}[\tfrac{1}{3}\bar{M}'(\theta',\phi')M'(\theta',\phi')S_{ij}]/\mathrm{Tr}[\tfrac{1}{3}\bar{M}'(\theta',\phi')M'(\theta',\phi')], \qquad (7)$$

the differential cross section after the second scattering of a double scattering experiment is given by[6]

$$I'(\theta',\phi')=3I_0'(\theta',\phi')[\tfrac{1}{3}+\tfrac{1}{2}P_i(\theta,\phi)\tilde{P}_i'(\theta',\phi')+T_{ij}(\theta,\phi)\tilde{T}_{ij}'(\theta',\phi')], \qquad (8)$$

where

$$I_0'(\theta',\phi')\equiv\tfrac{1}{3}\,\mathrm{Tr}\bar{M}'(\theta',\phi')M'(\theta',\phi'). \qquad (9)$$

The primed quantities refer to the second scattering.

The quantities that appear on the right in Eq. (8) may be expressed in terms of the parameters $A(\theta,\phi)$, $B_i(\theta,\phi)$, and $C_{ij}(\theta,\phi)$ which determine the M matrix. It is convenient, however, to first reduce these parameters to the forms that are imposed upon them by the requirements of invariance under spatial rotations and time reversal. Arguments similar to those used by Wolfenstein and Ashkin[5] show that one may write

$$A(\theta,\phi)=a(\theta),\quad B_i(\theta,\phi)=b(\theta)N_i, \qquad (10)$$

while $C_{ij}(\theta,\phi)$ must be a linear combination of the terms

$$C_N(\theta)(N_iN_j-\tfrac{1}{3}\delta_{ij}), \qquad (11)$$

$$C_P(\theta)(D_iD_j-\tfrac{1}{3}\delta_{ij}), \qquad (12)$$

$$C_K(\theta)(E_iE_j-\tfrac{1}{3}\delta_{ij}). \qquad (13)$$

Here

$$\mathbf{N}=\mathbf{k}_{\mathrm{in}}\times\mathbf{k}_{\mathrm{out}}/|\mathbf{k}_{\mathrm{in}}\times\mathbf{k}_{\mathrm{out}}|, \qquad (14)$$

$$\mathbf{D}=\mathbf{k}_{\mathrm{out}}+\mathbf{k}_{\mathrm{in}}/|\mathbf{k}_{\mathrm{out}}+\mathbf{k}_{\mathrm{in}}|, \qquad (15)$$

$$\mathbf{E}=\mathbf{k}_{\mathrm{out}}-\mathbf{k}_{\mathrm{in}}/|\mathbf{k}_{\mathrm{out}}-\mathbf{k}_{\mathrm{in}}|, \qquad (16)$$

where the vectors $\hbar\mathbf{k}_{\mathrm{in}}$ and $\hbar\mathbf{k}_{\mathrm{out}}$ are the incident and final momenta. Using the relation $(\mathbf{N}_i\mathbf{N}_j+\mathbf{D}_i\mathbf{D}_j+\mathbf{E}_i\mathbf{E}_j)=\delta_{ij}$, one may write the matrix C_{ij} as

$$C_{ij}=c(\theta)(N_iN_j-\tfrac{1}{3}\delta_{ij})+d(\theta)(D_iD_j-E_iE_j), \qquad (17)$$

[4] W. Heckrotte and H. P. Stapp (to be published).

[5] L. Wolfenstein and J. Ashkin, Phys. Rev. 85, 947 (1952).

[6] Henry P. Stapp, University of California Radiation Laboratory Report UCRL-3657, January, 1957 (unpublished).

and the M matrix is reduced to the form

$$M(\theta,\phi)=a(\theta)+b(\theta)N_iS_i +[c(\theta)(N_iN_j-\tfrac{1}{3}\delta_{ij})+d(\theta)(D_iD_j-E_iE_j)]S_{ij}. \quad (18)$$

The scalar coefficients $a(\theta)$, $b(\theta)$, $c(\theta)$, and $d(\theta)$ give a complete description of the scattering, and the cross section and polarizations may be expressed in terms of them. Carrying out the required matrix multiplications, one obtains

$$I_0=aa^*+\tfrac{2}{3}bb^*+(2/9)cc^*+\tfrac{2}{3}dd^*, \quad (19)$$

$$I_0P_i=I_0\tilde{P}_i=\tfrac{2}{3}\{2\,\mathrm{Re}[b(a+\tfrac{1}{3}c)^*]\}N_i, \quad (20)$$

$$I_0T_{ij}=\tfrac{1}{3}\{[(a+\tfrac{1}{3}c)c^*+(a+\tfrac{1}{3}c)^*c-cc^*+dd^*+bb^*] \times(N_iN_j-\delta_{ij})+[(a+\tfrac{1}{3}c)d^*+(a+\tfrac{1}{3}c)^*d](D_iD_j-E_iE_j) +2\,\mathrm{Im}(db^*)(D_iE_j+E_iD_j)\}, \quad (21)$$

$$I_0\tilde{T}_{ij}=\text{same except for sign of last term.} \quad (22)$$

These equations, when substituted into Eq. (8), will give the differential cross section after the second scattering; this is the quantity measured in the polarization experiments. The result of this substitution may be reduced to the form[7]

$$I'(\theta',\phi')=I_0'(\theta') \times[1+\tfrac{1}{2}tt'+\tfrac{2}{3}(uu'-vv')\cos\phi'+\tfrac{1}{6}ww'\cos 2\phi'], \quad (23)$$

where ϕ' is the azimuthal angle for the second scattering in the right-handed coordinate system in which the intermediate beam moves in the z direction and the normal to the first scattering is along the y axis. The parameters t, u, v, and w are functions of the scattering angle θ, and are given in terms of the scattering parameters by the equations

$$I_0=aa^*+\tfrac{2}{3}bb^*+(2/9)cc^*+\tfrac{2}{3}dd^*, \quad (24)$$

$$I_0t=2\cos\theta\,\mathrm{Re}[d(a+\tfrac{1}{3}c+ib\tan\theta)^*] -\tfrac{2}{3}\,\mathrm{Re}[c(a+\tfrac{1}{3}c)^*]-\tfrac{1}{3}dd^*-\tfrac{1}{3}bb^*+\tfrac{1}{3}cc^*, \quad (25)$$

$$I_0u=2\,\mathrm{Re}[b(a+\tfrac{1}{3}c)^*], \quad (26)$$

$$I_0v=2\cos\theta\,\mathrm{Re}[d(-ib+a\tan\theta+\tfrac{1}{3}c\tan\theta)^*], \quad (27)$$

$$I_0w=-2\cos\theta\,\mathrm{Re}[d(a+\tfrac{1}{3}c+ib\tan\theta)^*] -2\,\mathrm{Re}[c(a+\tfrac{1}{3}c)^*]-dd^*-bb^*+cc^*. \quad (28)$$

The primed parameters are given by the same equations, but with I_0 and the quantities on the right replaced by the corresponding quantities for the second scattering.

An expression for $I'(\theta',\phi')$ having the same general form as Eq. (23) has also been deduced by Lakin.[8] In Lakin's expression the parameters t, u, v, and w are expressed as expectation values of certain spin-space operators in the intermediate beam, and the t', u', v', and w' are defined in a similar way. The expressions for these parameters given in Eqs. (24) through (28) are more complicated than Lakin's, but they are expressed directly in terms of the scattering matrix amplitudes. The latter are the quantities obtained directly from particular models for the interaction. In the following sections and the subsequent paper[4] these expressions are used to obtain the cross section and the asymmetry parameters predicted on the basis of the impulse approximation.

II. IMPULSE APPROXIMATION WITH DEUTERON D-STATE INCLUDED

The transition matrices T_1 and T_2 for the individual scatterings of the two particles of the deuteron by the target nucleus, which is assumed to have zero spin, are defined by

$$M_i(\theta,\phi)=\frac{-2m_i}{4\pi\hbar^2}(\mathbf{k}|T_i|\mathbf{k}')=\frac{1}{n_i}(\mathbf{k}|T_i|\mathbf{k}'),\ (i=1,2) \quad (29)$$

where the m_i are the masses of the two particles and the $M_i(\theta,\phi)$ are the corresponding M matrices. The quantity $(\mathbf{k}|T_i|\mathbf{k}')$ is the matrix element of T_i between the single-particle initial and final momentum eigenstates. In the first Born approximation the T_i may be identified with that part of the Hamiltonian which represents the interaction between the target nucleus and the individual particle of the deuteron. The transition matrix T for the scattering of the entire deuteron is defined, analogously, by

$$M(\theta,\phi)=\frac{-2m}{4\pi\hbar^2}(\mathbf{K}|T|\mathbf{K}')=\frac{1}{n}(\mathbf{K}|T|\mathbf{K}'), \quad (30)$$

where m and $M(\theta,\phi)$ are, respectively, the mass and the M matrix for the deuteron, and $(\mathbf{K}|T|\mathbf{K}')$ is the matrix element of T between the initial and final deuteron momentum eigenstates. The matrix element of $(\mathbf{K}|T|\mathbf{K}')$ (which is a matrix in spin space) between the deuteron states α and α' will be written $(\mathbf{K}\alpha|\mathbf{T}|\mathbf{K}'\alpha')$. In the Born approximation, T becomes the sum of the two interaction Hamiltonians,

$$T=T_1+T_2. \quad (31)$$

The impulse approximation is obtained if Eq. (31) is considered to be valid, not only in the Born approximation, but in general.

The momentum-space matrix elements of T_i must, according to invariance arguments, take the form[9]

$$(\mathbf{k}_i|T_i|\mathbf{k}_i')=n_i[f_i(\Delta k_i)+\mathbf{k}_i\times\mathbf{k}_i'\cdot\boldsymbol{\sigma}_i g_i(\Delta k_i)], \quad (i=1,2) \quad (32)$$

[7] For a more detailed derivation see Henry P. Stapp, thesis, University of California Radiation Laboratory Report UCRL-3098, August, 1955 (unpublished).

[8] W. Lakin, Phys. Rev. 98, 139 (1955). A comparison with Eq. (23) disclosed, however, an incorrect sign in one term of Lakin's result.

[9] See reference 5. The f_i and g_i may, in general, depend upon $\mathbf{k}_i\cdot\mathbf{k}_i$, $\mathbf{k}_i'\cdot\mathbf{k}_i'$, and $\mathbf{k}_i\cdot\mathbf{k}_i'$. However, it will be assumed here that, as in the Born approximation, the f_i and g_i are only functions of the magnitude of the momentum transfer. Also the target is temporarily assumed to be infinitely heavy.

where $\Delta k_i \equiv |\mathbf{k}_i - \mathbf{k}_i'|$, $f_i(\Delta k_i)$ and $g_i(\Delta k_i)$ are scalars, and the $\boldsymbol{\sigma}_i$ are the Pauli spin-matrix vectors for the two particles. The normalization factor n_i is included in order that $f_i(\Delta K)$ be the usual scattering amplitude. Introducing the relative momentum $\mathbf{k}=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$ and the total momentum $\mathbf{N}=(\mathbf{k}_1+\mathbf{k}_2)$ and using Eqs. (31) and (32), one may write the matrix element of T in the relative-total momentum representation as

$$\begin{aligned}(\mathbf{kK}|T|\mathbf{k}'\mathbf{K}')&=n(\mathbf{kK}|f|\mathbf{k}'\mathbf{K}')\\&\quad+n\{\tfrac{1}{2}\mathbf{K}\times\mathbf{K}'\cdot\mathbf{S}+2\mathbf{k}\times\mathbf{k}'\cdot\mathbf{S}\\&\quad+\tfrac{1}{2}(\mathbf{K}\times\mathbf{k}'+\mathbf{k}\times\mathbf{K}')\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\}(\mathbf{kK}|g|\mathbf{k}'\mathbf{K}'),\quad(33)\end{aligned}$$

where $\mathbf{S}\equiv\frac{1}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$, and where, in operator notation,[10]

$$\begin{aligned}(\mathbf{kK}|f|\mathbf{k}'\mathbf{K}')&\equiv\frac{n_1}{n}f_1(\Delta K)(\mathbf{k}|\exp(\tfrac{1}{2}i\Delta\mathbf{K}\cdot\mathbf{x})|\mathbf{k}')\\&\quad+\frac{n_2}{n}f_2(\Delta K)(\mathbf{k}|\exp(-\tfrac{1}{2}i\Delta\mathbf{K}\cdot\mathbf{x})|\mathbf{k}').\quad(34)\end{aligned}$$

Here $\mathbf{x}$ is the relative coordinate $\mathbf{x}_1-\mathbf{x}_2$, $\Delta\mathbf{K}\equiv\mathbf{K}-\mathbf{K}'$, and $\Delta K\equiv|\Delta\mathbf{K}|$. The operator g is defined in the exactly analogous way. If the deuteron state is labeled by the symbol α and $\exp(\frac{1}{2}i\Delta\mathbf{K}\cdot\mathbf{x})$ is abbreviated by e, the matrix element of T may be expressed as

$$\begin{aligned}(\alpha\mathbf{K}|T|\alpha'\mathbf{K}')&=(\alpha|\mathbf{k})(\mathbf{kK}|T|\mathbf{k}'\mathbf{K}')(\mathbf{k}'|\alpha')\\&=n_1f_1(\Delta K)(\alpha|e|\alpha')+n_2f_2(\Delta K)(\alpha|e^{-1}|\alpha')\\&\quad+n_1g_1(\Delta K)[\tfrac{1}{2}\mathbf{K}\times\mathbf{K}'\cdot(\alpha|e\mathbf{S}|\alpha')\\&\quad+2(\alpha|\boldsymbol{\nabla}\times(\boldsymbol{\nabla}e)\cdot\mathbf{S}|\alpha')-\tfrac{1}{2}i(\alpha|e(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{K}\times\boldsymbol{\nabla}|\alpha')\\&\quad-\tfrac{1}{2}i(\alpha|\boldsymbol{\nabla}\times\mathbf{K}'\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)e|\alpha')]+n_2g_2(\Delta K)\\&\quad\times[\tfrac{1}{2}\mathbf{K}\times\mathbf{K}'\cdot(\alpha|e^{-1}\mathbf{S}|\alpha')+2(\alpha|\boldsymbol{\nabla}\times(e\boldsymbol{\nabla}^{-1})\cdot\mathbf{S}|\alpha')\\&\quad-\tfrac{1}{2}i(\alpha|e^{-1}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{K}\times\boldsymbol{\nabla}|\alpha')\\&\quad-\tfrac{1}{2}i(\alpha|\boldsymbol{\nabla}\times\mathbf{K}'\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)e^{-1}|\alpha')].\quad(35)\end{aligned}$$

Since the deuteron wave functions are all states of positive parity, a transformation $x\simeq -x$ may be performed in the $f_2(\Delta K)$ and $g_2(\Delta K)$ terms to eliminate e^{-1}. Then one obtains

$$\begin{aligned}\frac{1}{n}(\alpha\mathbf{K}|T|\alpha'\mathbf{K}')&=f(\Delta K)(\alpha|e|\alpha')\\&\quad+g(\Delta K)[\tfrac{1}{2}\mathbf{K}\times\mathbf{K}'\cdot(\alpha|e\mathbf{S}|\alpha')\\&\quad+2(\alpha|\boldsymbol{\nabla}\times(\boldsymbol{\nabla}e)\cdot\mathbf{S}|\alpha')],\quad(36)\end{aligned}$$

where

$$f(\Delta K)\equiv\frac{n_1}{n}f_1(\Delta K)+\frac{n_2}{n}f_2(\Delta K),$$

and similarly for $g(\Delta K)$. The terms proportional to $(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)$, which appear in Eq. (35), are zero in virtue of their spin-space dependence, and have been dropped from Eq. (36).

The calculation of the matrix elements appearing on the right in Eq. (36) may be carried out in coordinate space. In this representation the deuteron wave function is[11]

$$\begin{aligned}(x|\alpha_i)&=(r^{-1})[u(r)+w(r)S_{12}(8)^{-\frac{1}{2}}]Y_{101}{}^i(\theta,\phi)\\&=(4\pi r^2)^{-\frac{1}{2}}[u(r)+w(r)S_{12}(8)^{-\frac{1}{2}}]X^i\\&\equiv[s(r)+d(r)S_{12}(8)^{-\frac{1}{2}}]X^i,\end{aligned}\quad(37)$$

where X^i are the three triplet-state spin functions, S_{12} is the tensor operator,

$$S_{12}=3(r^{-2})(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,\quad(38)$$

and $s(r)$ and $d(r)$ are the radial S- and D-wave functions. These satisfy the normalization conditions

$$\int d\mathbf{r}\,s^2(r)=\int_0^\infty dr\,u^2(r)=(S\text{-state probability})\cong 96\%,\quad(39)$$

$$\int d\mathbf{r}\,d^2(r)=\int_0^\infty dr\,w^2(r)=(D\text{-state probability})\cong 4\%.\quad(40)$$

When Eq. (37) is substituted into Eq. (36) and the angular integrations and matrix multiplications are carried out, the M matrix reduces to the form

$$M=a_0+b_0S_iN_i+c_0S_{ij}E_iE_j,\quad(41)$$

where the unit vectors $\mathbf{E}$ and $\mathbf{N}$ are defined above Eq. (17), $\mathbf{S}$ is defined above Eq. (34), and a_0, b_0, and c_0 are given by

$$\begin{aligned}a_0&=f(\Delta K)[\langle j_0(\tfrac{1}{2}r\Delta K)\rangle_{ss}+\langle j_0(\tfrac{1}{2}r\Delta K)\rangle_{dd}]\\&\quad+\tfrac{1}{2}g(\Delta K)[-6i\Delta K\langle j_1(\tfrac{1}{2}r\Delta K)/r\rangle_{dd}],\end{aligned}\quad(42)$$

$$\begin{aligned}b_0&=\tfrac{1}{2}g(\Delta K)K^2\sin\theta\Big[\langle j_0(\tfrac{1}{2}r\Delta K)\rangle_{ss}+\frac{1}{\sqrt{2}}\langle j_2(\tfrac{1}{2}r\Delta K)\rangle_{ds}\\&\quad-\tfrac{1}{4}\langle j_2(\tfrac{1}{2}r\Delta K)\rangle_{dd}-\tfrac{1}{2}\langle j_0(\tfrac{1}{2}r\Delta K)\rangle_{dd}\Big],\end{aligned}\quad(43)$$

$$\begin{aligned}c_0&=f(\Delta K)\Big[\frac{-6}{\sqrt{2}}\langle j_2(\tfrac{1}{2}r\Delta K)\rangle_{ds}+\tfrac{3}{2}\langle j_2(\tfrac{1}{2}r\Delta K)\rangle_{dd}\Big]\\&\quad+\tfrac{1}{2}g(\Delta K)\Big[36i\sqrt{2}\Big\langle\frac{j_2(\frac{1}{2}r\Delta K)}{r}\frac{\partial}{\partial r}\Big\rangle_{ds}\\&\quad-72i\Big\langle j_2(\tfrac{1}{2}r\Delta K)\frac{1}{r^2}\Big\rangle_{dd}+9i\Delta K\Big\langle\frac{j_1(\frac{1}{2}r\Delta K)}{r}\Big\rangle_{dd}\Big].\end{aligned}\quad(44)$$

The $j_n(\frac{1}{2}r\Delta K)$ are the usual spherical Bessel functions[12]

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[10] In these expressions x is actually an operator. The quantity $(\mathbf{k}|\exp(\frac{1}{2}i\Delta\mathbf{K}\cdot\mathbf{x})|\mathbf{k}')$ is, according to the normalization conventions which we use, $(2\pi)^3\delta(\mathbf{k}-\mathbf{k}'-\frac{1}{2}\Delta\mathbf{K})$.

[11] The notation of Blatt and Weisskopf is followed here. J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 100.

[12] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 77.

and, for any A,

$$\langle A\rangle_{ds}\equiv\int d\mathbf{r}\; d(\mathbf{r})As(\mathbf{r}). \qquad (45)$$

The $\langle A\rangle_{ss}$ and $\langle A\rangle_{dd}$ are defined analogously. When the D-state contributions are neglected, only the terms proportional to $\langle j_0(\frac{1}{2}r\Delta K)\rangle_{ss}$ remain. This factor is the square root of the usual sticking factor.

The expression for the M matrix given in Eq. (41) may be put in the form given in Eq. (17) by using the identity

$$N_iN_j+D_iD_j+E_iE_j=\delta_{ij}.$$

The coefficients a, b, c, and d appearing in Sec. I are then expressed in terms of the coefficients defined in Eqs. (42) through (44) by the relations

$$a=a_0,\quad b=b_0,\quad c=-\tfrac{1}{2}c_0,\quad d=-\tfrac{1}{2}c_0. \qquad (46)$$

These relations, when used in Eqs. (23) through (28), give the differential cross section and polarization effects in terms of the S and D radial deuteron wave functions and $f(\Delta K)$ and $g(\Delta K)$, the two parameters which describe the scattering of the individual particles of the deuteron.

In order to obtain estimates for the various expectation values appearing in Eqs. (42) through (44), some assumption regarding the forms of the radial wave functions must be made. The problem of determining S- and D-state wave functions that are consistent with the known proporties of the deuteron—in particular its binding energy, quadrupole moment, and effective range—has been studied by Sugawara.[13] He uses the forms

$$u(r)=N(e^{-\alpha r}-e^{-\beta r}), \qquad (47)$$

$$w(r)=N'(1-e^{-\gamma r})^2e^{-\alpha}\times\left[1+\frac{3(1-e^{-\gamma r})}{\alpha r}+\frac{3(1-e^{-\gamma r})^2}{(\alpha r)^2}\right]. \qquad (48)$$

If the percentage D state is taken as 4% and the deuteron effective range is approximated by the triplet n-p effective range, then Sugawara finds for the parameters in Eqs. (47) and (48) the values

$$\alpha=0.23171\times10^{13}\ \text{cm}^{-1},\quad \beta=5.751\alpha,\quad \gamma=2.922\alpha. \qquad (49)$$

Using these values, one obtains for the various expectation values appearing in Eqs. (41) through (44) the values given in Fig. 1. A second apparently reasonable form for the deuteron wave function was also investigated, and it gave similar results.

If the values given in Fig. 1 are used, one finds that the contributions to the coefficients a_0 and b_0 in Eqs.

[13] Masao Sugawara, *Handbuch der Physik* [Springer-Verlag, Berlin (to be published)], Vol. 39. I wish to thank Dr. Sugawara for advanced information concerning his results.

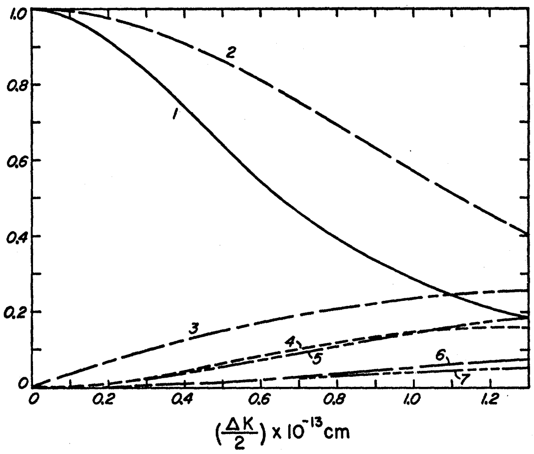

FIG. 1. Curve (1) is $\langle j_0(\frac{1}{2}r\Delta K)\rangle_{ss}\times P_s^{-1}$; Curve (2) is $\langle j_0(\frac{1}{2}r\Delta K)\rangle_{dd}\times P_d^{-1}$; Curve (3) is $\langle r^{-1}j_1(\frac{1}{2}r\Delta K)\rangle_{dd}\times P_d^{-1}$; Curve (4) is $\langle j_2(\frac{1}{2}r\Delta K)\rangle_{ds}\times (P_dP_s)^{-2}$; Curve (5) is $\langle j_2(\frac{1}{2}r\Delta K)\rangle_{dd}\times P_d^{-1}$; Curve (6) is $\langle r^{-2}j_2(\frac{1}{2}r\Delta K)\rangle_{dd}\times P_d^{-1}$; Curve (7) is $\langle r^{-1}j_2(\frac{1}{2}r\Delta K)\partial/\partial r\rangle_{ds}\times(P_dP_s)^{-2}$. The wave functions used are those of Sugawara that are given in Eqs. (47), (48), and (49) of the test. The ordinate for Curve (3) is in units of (10^{13} cm^{-1}) and the ordinate for Curves (6) and (7) is in units of (10^{13} cm^{-1})^2. P_s and P_d are the deuteron S-state and D-state probabilities, respectively.

(42) and (43) are changed by less than 35% at angles less than 30° unless $|\frac{1}{2}g(\Delta K)|$ (measured in units of 10^{-26} cm^2) is larger than $f(\Delta K)$. In the same angular range, c_0—which was zero when D-state contributions were neglected—becomes roughly $a^0[-\frac{1}{3}+\frac{1}{2}ig(\Delta K)f^{-1}(\Delta K)]$, a^0 being the original a_0. Equations (46) and (24) then show that the D-state contributions may significantly affect the differential cross section, but not to the large extent required to fit the experimental values. For instance, if one assumes $\frac{1}{2}g(\Delta K)=if(\Delta K)$, then the effect of the D-state contributions at $\Delta K=1.8\times10^{13}$ cm^{-1} (i.e., $\theta\simeq31°$) is to increase a by $\sim$32%, to leave b virtually unchanged, and to change both c and d from zero to approximately $-\frac{1}{2}a^0\{-0.35+1.15i\times[\frac{1}{2}ig(\Delta K)f^{-1}(\Delta K)]\}=0.75a^0$. The cross section is then increased by about 75%. However, the increase in the differential cross section needed to fit the experimental values is about ten times as large as this representative increase due to the D-state contributions.

Although the D-state effects are evidently not primarily responsible for the large differential cross sections at large angles, they must evidently be considered in any quantitative treatment of the large-angle scattering for which the contributions corresponding to the S-state terms considered in this section are important. In particular, since the interference between the contributions considered in this section and the contributions of processes in which both particles of the deuteron are scattered is very important at both large and small scattering angles, the D-state contributions must be considered in any quantitative treatment of large-angle scattering. Detailed numerical considerations are given in the forthcoming paper.[4]

III. IMPULSE APPROXIMATION FOR THE SIMULTANEOUS SCATTERING OF BOTH PARTICLES OF THE DEUTERON

The contributions to the scattering calculated in the preceding section are linear in T_1 and T_2 and contain no term proportional to T_1T_2. Terms of the latter type would correspond to processes in which both particles of the deuteron were scattered. In order to treat processes of this nature, it is convenient to use a time-dependent formulation of scattering theory. When one uses this formulation, the transition matrix element $\mathfrak{M}$ is[14]

$$\mathfrak{M}=\langle K\alpha|\tilde{T}_1\tilde{T}_2|K'\alpha'\rangle(i\hbar)^{-2}, \tag{50}$$

where the tilde above the operators distinguishes these time-dependent operators from the time-independent operators used in Sec. II. The α designates the relative coordinate part of the deuteron state and K' and K denote the initial and final energy-momentum corresponding to the center-of-mass coordinate. The operators $\tilde{T}_1$ and $\tilde{T}_2$ depend upon the free-particle energy operators E_1 and E_2. If in the spirit of the impulse approximation these operators are replaced by E_1^0 and E_2^0, the appropriate free-particle energies, then $\tilde{T}_1$ and $\tilde{T}_2$ become the time-independent operators T_1 and T_2. The matrix element then reduces to the form

$$\mathfrak{M}=(i\hbar)^{-2}\int_{-\infty}^{\infty}e^{i(\Omega-\Omega')\tau}d\tau\int_{-\infty}^{\infty}dt(K\alpha(\tau,t)|T_1T_2|K'\alpha'(\tau,t))$$
$$=(i\hbar)^{-2}2\pi\delta(\Omega-\Omega')\int_{-\infty}^{\infty}dt(K\alpha(0,t)|T_1T_2|K'\alpha'(0,t)). \tag{51}$$

Before the integrand in the above equation for the general case is evaluated, the special case $t=0$ will be treated. When the relative time t is zero, $|K\alpha(0,t))$ and $|K'\alpha'(0,t))$ are just the usual time-independent eigenvectors.[15] These will be denoted by $|\mathbf{K}\alpha)$ and $|\mathbf{K}'\alpha')$ in accordance with the notation of earlier sections. The calculation of $(\mathbf{K}\alpha|T_1T_2|\mathbf{K}'\alpha')$ give

$$(\alpha\mathbf{K}|T_1T_2|\alpha'\mathbf{K}')$$
$$=\int\frac{d^3K''}{(2\pi)^3}(\alpha|\exp[i(\mathbf{K}+\mathbf{K}'-2\mathbf{K}'')\cdot\tfrac{1}{2}\mathbf{x}]|\alpha')\times T_1(\mathbf{K}-\mathbf{K}'')T_2(\mathbf{K}''-\mathbf{K}'), \tag{52}$$

where the expression $(\alpha|\exp[i(\mathbf{K}+\mathbf{K}'-2\mathbf{K}'')\cdot\frac{1}{2}\mathbf{x}]|\alpha')$ is the Fourier component of the square of the deuteron wave function. If only the S-state part of the deuteron wave function is considered, then $(\alpha|\exp[i(\mathbf{K}+\mathbf{K}'-2\mathbf{K}'')\cdot\frac{1}{2}\mathbf{x}]|\alpha')$ is a function of $|\mathbf{K}+\mathbf{K}'-2\mathbf{K}''|$ with a maximum at $\mathbf{K}''=\frac{1}{2}(\mathbf{K}'+\mathbf{K})$ and with the sharpness of the peak varying inversely as the size of the deuteron. In the limit in which the impulse approximation becomes exact, the deuteron is very loosely bound and very large in extent. The wave function of the deuteron in momentum space is then sharply peaked. For this limit $T_1(\mathbf{K}-\mathbf{K}'')$ and $T_2(\mathbf{K}''-\mathbf{K}')$ can be considered to be slowly varying functions of $\mathbf{K}''$, and may be evaluated at $\mathbf{K}''=\frac{1}{2}(\mathbf{K}'+\mathbf{K})$ and taken out of the integral. This gives

$$(\alpha\mathbf{K}|T_1T_2|\alpha\mathbf{K}')\simeq T_1(\tfrac{1}{2}\Delta K)T_2(\tfrac{1}{2}\Delta K)\times\int\frac{d^3K''}{(2\pi)^3}(\alpha|\exp[i(\mathbf{K}+\mathbf{K}'-2\mathbf{K}'')\cdot\tfrac{1}{2}\mathbf{x}]|\alpha')$$
$$=T_1(\tfrac{1}{2}\Delta K)T_2(\tfrac{1}{2}\Delta K)|\phi(0)|^2, \tag{53}$$

where $\Delta K=|\mathbf{K}-\mathbf{K}'|$ and $\phi(0)$ is the deuteron wave function at $x=0$.

In order to extend this result to the case $t\neq 0$, some assumption regarding the relative time dependence of the deuteron wave function must be made. This question of the relative time dependence is a familiar one in the history of the attempts to use multitime wave functions in bound-state problems. Lévy and Klein[16] assume that during the time interval between the two times t_1 and t_2 the particle whose time is later moves as a free particle. The slightly different assumption made here is that the second particle remains in the deuteron state. More precisely, it will be assumed that for a deuteron at rest the wave function is $\phi(x)\exp[(Et_1+Et_2)(2i\hbar)^{-1}]$, where E is the deuteron total energy. Over the short period of the collision the difference between wave functions obtained by using the two different assumptions is small for high-energy deuterons.

The generalization of this expression for the wave function to the case of a moving deuteron is obtained by making the Galilean transformation $\mathbf{x}_1\rightarrow\mathbf{x}_1-\mathbf{v}t$, $\mathbf{x}_2\rightarrow\mathbf{x}_2-\mathbf{v}t$, where $\mathbf{v}$ is the velocity of the deuteron. The relative coordinate part of the deuteron wave function is therefore

$$\phi(\mathbf{x},t)=\phi(\mathbf{x}-\mathbf{v}t), \tag{54}$$

where t is the relative time t_1-t_2. The integral in Eq. (51) may now be obtained from Eq. (53) by replacing $|\phi(0)|^2$ by $\phi^*(-\mathbf{v}t)\phi(-\mathbf{v}'t)$, where $\mathbf{v}'$ and $\mathbf{v}$ are the initial and final velocities of the deuteron. This gives

$$\mathfrak{M}=(i\hbar)^{-2}2\pi\delta(\Omega-\Omega')\times T_1(\tfrac{1}{2}\Delta K)T_2(\tfrac{1}{2}\Delta K)\int_{-\infty}^{\infty}dt|\phi(vt)|^2. \tag{55}$$

This equation expresses the fact that for the simultaneous-scattering process the effective transition matrix is the product of the individual transition matrices

[14] The operators $\tilde{T}_1$ and $\tilde{T}_2$ operate in the full space-time coordinate space and the angular bracket symbol denotes a vector in the corresponding generalized Hilbert space. For a detailed discussion of the formalism that is used here see reference 6.

[15] F. J. Dyson, Phys. Rev. **91**, 1543 (1954).

[16] M. Lévy, Phys. Rev. **88**, 72, 972 (1952); and A. Klein, Phys. Rev. **90**, 1101 (1953).

times $(i\hbar)^{-1}$ times an average time for the collision. Another form of the equation is

$$\mathfrak{M}=(i\hbar)^{-2}2\pi\delta(\Omega-\Omega')T_1(\tfrac{1}{2}\Delta K)T_2(\tfrac{1}{2}\Delta K)\times 2\int_0^\infty \frac{dr}{v}|\phi(r)|^2$$
$$=(i\hbar)^{-2}2\pi\delta(\Omega-\Omega')T_1(\tfrac{1}{2}\Delta K)T_2(\tfrac{1}{2}\Delta K)\times\frac{2}{4\pi v}\langle r^{-2}\rangle, \quad (56)$$

where $\langle r^{-2}\rangle$ is the expectation value of r^{-2} for the deuteron. This formula for the scattering matrix element is similar in form to, and consistent with, an expression for the forward-scattering amplitude derived by Glauber.[17] It gives a contribution to the differential cross section of

$$\Delta\sigma(\theta)=\left|\frac{-2m}{4\pi\hbar^2}\left(\frac{T_1(\tfrac{1}{2}\Delta K)T_2(\tfrac{1}{2}\Delta K)}{(i\hbar)}\right)\times\frac{2}{4\pi v}\langle r^{-2}\rangle\right|^2, \quad (57)$$

where m is the mass of the deuteron. The quantity inside the absolute-value signs is the scattering amplitude. In terms of $\sigma_1(\theta)$ and $\sigma_2(\theta)$, the individual particle cross sections, $\Delta\sigma(\theta)$ may be expressed as

$$\Delta\sigma(\theta)=\sigma_1(\theta)\sigma_2(\theta)/\sigma_0, \quad (58)$$

where

$$\sigma_0^{-1}=\left|\frac{i}{|K|}\frac{m^2}{m_1m_2}\langle r^{-2}\rangle\right|^2. \quad (59)$$

Here K is the incident momentum of the deuteron. There is, of course, also a contribution to the cross section from the cross terms between the contributions to the scattering amplitudes due to the $G_1T_1+G_2T_2$ parts of the scattering matrix and the contributions considered in this section.

To get an idea of the order of magnitudes, some typical values may be inserted in the above formulas. The measurement by Strauch[18] of the proton-carbon center-of-mass cross section at ~84 Mev and at 27.2° is about 50 mb. With the parameters given in Sec. II, the value of $\langle r^{-2}\rangle$ for the deuteron S-state is $\sim 0.68\times 10^{26}$ cm^{-2}. The value of K for a 157-Mev deuteron on carbon is $\sim 3.3\times 10^{13}$ cm^{-1}. This gives $\sigma_0\simeq 1.47\times 10^{-26}$ cm^2. The contribution to the cross section is therefore

$$\Delta\sigma(\theta)\cong(2500/14.7)\text{ mb}\approx 170\text{ mb}.$$

Because of the difference in energies the corresponding scattering angle for the deuteron scattering is about 31.5° (c.m.). There the center-of-mass cross section is about 10 mb.

This large discrepancy between experiment and theory is due, in part, to the assumption that the deuteron is very loosely bound and hence large. Although the impulse approximation becomes valid when this condition is satisfied, the condition is in fact not satisfied here. In particular, the deuteron is not a large object in comparison with the scattering nucleus, and the transition between Eq. (52) and Eq. (53) is not legitimate. It is necessary, therefore, to obtain a more exact treatment of Eq. (52).

In order to carry out explicitly the integration over K'' in Eq. (52), the form of $T_1(K-K'')$ and $T_2(K''-K')$ must be prescribed. The cross-section data of Strauch for 96-Mev protons on carbon may be represented to an accuracy of 10% in the range between 15° and 40°, and qualitatively at all angles, by a scattering amplitude of the form[19]

$$f(y)=c_1\exp(-\tfrac{1}{2}\alpha y^2)+c_2\exp(-\tfrac{1}{2}\beta y^2)+c_3\exp(-\tfrac{1}{2}\gamma y^2), \quad (60)$$

where $y=\Delta K$ and the parameters are

$$c_1=10.83\times 10^{-13}\text{ cm},\quad \alpha=4.68\times 10^{-26}\text{ cm}^2,$$
$$c_2=0.42\times 10^{-13}\text{ cm},\quad \beta=0.72\times 10^{-26}\text{ cm}^2, \quad (61)$$
$$c_3=0.27\times 10^{-13}\text{ cm},\quad \gamma=0.34\times 10^{-26}\text{ cm}^2.$$

This form will be assumed to represent the scattering amplitude of the neutron as well as the proton. In order to simplify the calculation, the S-state deuteron wave function is represented by a Gaussian:

$$\phi(\mathbf{x})=N\exp(-x^2/2R^2), \quad (62)$$

with[20] $R=2.64\times 10^{-13}$ cm and $N^2=0.00973\ (10^{+13}\text{ cm}^{-1})^3$. The integrations may then easily be performed to give

$$\int_{-\infty}^{\infty}dt(K\alpha(0,t)|T_1T_2|K'\alpha'(0,t))\equiv X$$
$$=\frac{N^2}{v}R\pi^{\frac{3}{2}}\left(\frac{4\pi\hbar^2}{-2m_1}\right)\left(\frac{4\pi\hbar^2}{-2m_2}\right)[c_1^2\Gamma(\alpha)+c_2^2\Gamma(\beta)+c_3^2\Gamma(\gamma)+2c_1c_2\Gamma(\alpha,\beta)+2c_1c_3\Gamma(\alpha,\gamma)+2c_2c_3\Gamma(\beta,\gamma)], \quad (63)$$

where

$$\Gamma(x)=\exp(-xa^2)\left(\frac{R^2}{R^2+4x}\right)\left(\frac{R^2}{R^2+4x\sin^2(\theta/2)}\right)^{\frac{1}{2}}, \quad (64)$$

$$\Gamma(x,y)=\Gamma\left(\frac{x+y}{2}\right)\exp\left(\frac{[(x-y)a]^2}{R^2+2(x+y)}\right). \quad (65)$$

Here $a=|\tfrac{1}{2}(\mathbf{K}-\mathbf{K}')|$ and θ is the angle between $\mathbf{K}$ and $\mathbf{K}'$. Equation (32) has been used to convert the expressions for $f_1(\Delta K)$ and $f_2(\Delta K)$ to those for $T_1(\Delta K)$

[17] R. J. Glauber, Phys. Rev. **100**, 242 (1955).
[18] K. Strauch and F. Titus, Phys. Rev. **103**, 200 (1956).
[19] This fit to the data was obtained by Kenneth Greider for use in another problem.
[20] T. Y. Wu and J. Ashkin, Phys. Rev. **73**, 986 (1948).

and $T_2(\Delta K)$. The contribution to the cross section $\Delta\sigma$ is

$$\Delta\sigma=\left|\left(\frac{-2m}{4\pi\hbar^2}\right)\frac{1}{i\hbar}\int_{-\infty}^{\infty}dt(K\alpha(0,t)|T_1T_2|K'\alpha'(0,t))\right|^2, \quad (66)$$

which, with the aid of Eq. (63), becomes

$$\Delta\sigma=\left|iN^2R(\pi)^{\frac{1}{2}}\frac{2\pi}{|\mathbf{K}|}\left(\frac{m^2}{m_1m_2}\right)Y\right|^2, \quad (67)$$

where Y represents the term inside the square brackets on the right-hand side of Eq. (63). The quantity inside the absolute-value sign is the scattering amplitude. The correspondence between this expression for $\Delta\sigma$ and the earlier expression in Eq. (58) is seen if one notes that when the wave function is given by Eq. (62) the value of $\langle r^{-2}\rangle$ is $2\pi N^2R\sqrt{\pi}$, and that in the limit $R\rightarrow\infty$ the quantity $|Y|^2$ approaches $\sigma_1\sigma_2$.

The above computations can be generalized to include also the spin-dependent contributions. In order to include these, the spin-dependent part of the individual-particle transition matrices must be considered. The general form of the single-particle transiton matrices given in Eq. (32) is

$$(\mathbf{k}_i|T_i|\mathbf{k}_i')=n_i[f_i(\Delta k_i)+\mathbf{k}_i\times\mathbf{k}_i'\cdot\boldsymbol{\sigma}_ig_i(\Delta k_i)]. \quad (68)$$

These forms may be substituted into Eqs. (52) and (53) and the calculations carried out if a sufficiently simple form is used for $g_i(\Delta k_i)$. A not unreasonable assumption is that $g_i(\Delta k_i)$ and $f_i(\Delta k_i)$ may be represented by the forms

$$f_i(\Delta k_i)=xf_a(\Delta k_i), \quad (69)$$

and

$$g_i(\Delta k_i)=yf_a(\Delta k_i), \quad (70)$$

where $f_a(\Delta k_i)$ is a real function of the form given in Eq. (60), x is a phase factor, and y is a complex constant which determines the phase and strength of the spin-dependent term. The general form specified in Eqs. (69) and (70), but with arbitrary f_a, is what is obtained in the Born approximation if the real and imaginary potentials have the same form factor and if there is a spin-orbit potential proportional to the gradient of this form factor.[21] Using the forms given by Eqs. (69) and (72), one obtains for the quantity X, defined in Eq. (63),

$$\begin{aligned}X=&\frac{N^2R(\pi)^{\frac{1}{2}}}{v}\left(\frac{4\pi\hbar^2}{-2m_1}\right)^2\Bigg[x^2Y+\tfrac{1}{2}xy\left(\frac{\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2}{2}\right)\cdot\mathbf{K}\times\mathbf{K}'\left\{Y+\frac{2Z'}{|\mathbf{K}|^2}\right\}+y^2(\boldsymbol{\sigma}_1\cdot\mathbf{N})(\boldsymbol{\sigma}_2\cdot\mathbf{N})\\&\times\Bigg\{\tfrac{1}{16}\sin^2\theta|\mathbf{K}|^4Y-\tfrac{1}{2}|\mathbf{K}|^2\cos^2(\theta/2)Z-|\mathbf{K}|^2\cos^2(\theta/2)Z''-\frac{|\mathbf{K}|^2\sin^2(\theta/2)}{2R^2}Y+\tfrac{1}{4}|\mathbf{K}|^2\sin^2\theta Z'\\&+\frac{2Z''}{R^2}+\frac{Z}{R^2}+3\cos^2(\theta/2)\sin^2(\theta/2)\tilde{Z}\Bigg\}+y^2(\boldsymbol{\sigma}_1\cdot\mathbf{D})(\boldsymbol{\sigma}_2\cdot\mathbf{D})\left\{\frac{-|\mathbf{K}|^2\sin^2(\theta/2)}{2R^2}Y+\frac{Z}{R^2}+\frac{2Z''}{R^2}+\cos^2(\theta/2)\tilde{Z}'\right\}\\&+y^2(\boldsymbol{\sigma}_1\cdot\mathbf{E})(\boldsymbol{\sigma}_2\cdot\mathbf{E})\left\{-\tfrac{1}{2}|\mathbf{K}|^2\cos^2(\theta/2)Z+\frac{Z}{R^2}+\sin^2(\theta/2)\frac{Z'}{R^2}\right\}\Bigg]. \quad (71)\end{aligned}$$

The unit vectors $\mathbf{N}$, $\mathbf{D}$, and $\mathbf{E}$ are defined by Eqs. (14)–(16), and the Y and Z's are defined by

$$Y=[c_1^2\Gamma(\alpha)+c_2^2\Gamma(\beta)+c_3^2\Gamma(\gamma)+2c_1c_2\Gamma(\alpha,\beta)+2c_1c_3\Gamma(\alpha,\gamma)+2c_2c_3\Gamma(\beta,\gamma)],$$

$$Z=\left[\frac{c_1^2\Gamma(\alpha)}{R^2+4\alpha}+\frac{c_2^2\Gamma(\beta)}{R^2+4\beta}+\frac{c_3^2\Gamma(\gamma)}{R^2+4\gamma}+\frac{2c_1c_2\Gamma(\alpha,\beta)}{R^2+2\alpha+2\beta}+\frac{2c_1c_3\Gamma(\alpha,\gamma)}{R^2+2\alpha+2\gamma}+\frac{2c_1c_3\Gamma(\beta,\gamma)}{R^2+2\beta+2\gamma}\right],$$

$$\begin{aligned}Z'=\Bigg[&\frac{c_1^2\Gamma(\alpha)}{R^2+4\alpha\sin^2(\theta/2)}\frac{c_2^2\Gamma(\beta)}{R^2+4\beta\sin^2(\theta/2)}\frac{c_3^2\Gamma(\gamma)}{R^2+4\gamma\sin^2(\theta/2)}+\frac{2c_1c_2\Gamma(\alpha,\beta)}{R^2+(2\alpha+2\beta)\sin^2(\theta/2)}\\&+\frac{2c_1c_3\Gamma(\alpha,\gamma)}{R^2+(2\alpha+2\beta)\sin^2(\theta/2)}+\frac{2c_2c_3\Gamma(\beta,\gamma)}{R^2+(2\beta+2\gamma)\sin^2(\theta/2)}\Bigg],\end{aligned}$$

$$\begin{aligned}Z''=&2c_1c_2\Gamma(\alpha,\beta)\left[\frac{(\alpha-\beta)|\mathbf{K}|\sin(\theta/2)}{R^2+2\alpha+2\beta}\right]^2+2c_1c_3\Gamma(\alpha,\gamma)\left[\frac{(\alpha-\gamma)|\mathbf{K}|\sin(\theta/2)}{R^2+2\alpha+2\gamma}\right]^2\\&+2c_2c_3\Gamma(\beta,\gamma)\left[\frac{(\beta-\gamma)|\mathbf{K}|\sin(\theta/2)}{R^2+2\beta+2\gamma}\right]^2,\end{aligned}$$

[21] E. Fermi, Nuovo cimento 11, 407 (1954).

$$\tilde{Z}=\frac{c_1{}^2\Gamma(\alpha)}{[R^2+4\alpha\sin^2(\theta/2)]^2}+\frac{c_2{}^2\Gamma(\beta)}{[(R^2+4\beta\sin^2(\theta/2)]^2}+\frac{c_3{}^2\Gamma(\gamma)}{[R^2+4\gamma\sin^2(\theta/2)]^2}+\frac{2c_1c_2\Gamma(\alpha,\beta)}{[R^2+(2\alpha+2\beta)\sin^2(\theta/2)]^2}$$
$$+\frac{2c_1c_3\Gamma(\alpha,\gamma)}{[R^2+(2\alpha+2\gamma)\sin^2(\theta/2)]^2}+\frac{2c_2c_3\Gamma(\beta,\gamma)}{[R^2+(2\beta+2\gamma)\sin^2(\theta/2)]^2},$$

$$\tilde{Z}'=\frac{c_1{}^2\Gamma(\alpha)}{[R^2+4\alpha\sin^2(\theta/2)](R^2+4\alpha)}+\frac{c_2{}^2\Gamma(\beta)}{[R^2+4\beta\sin^2(\theta/2)](R^2+4\beta)}+\frac{c_3{}^2\Gamma(\gamma)}{[R^2+4\gamma\sin^2(\theta/2)](R^2+4\gamma)}$$
$$+\frac{2c_1c_3\Gamma(\alpha,\beta)}{[R^2+(2\alpha+2\beta)\sin^2(\theta/2)](R^2+2\alpha+2\beta)}+\frac{2c_1c_3\Gamma(\alpha,\gamma)}{[R^2+(2\alpha+2\gamma)\sin^2(\theta/2)](R^2+2\alpha+2\gamma)}$$
$$+\frac{2c_2c_3\Gamma(\beta,\gamma)}{[R^2+(2\beta+2\gamma)\sin^2(\theta/2)](R^2+2\beta+2\gamma)}. \quad (72)$$

The formulas given above were calculated with the assumption that the phase of the scattering amplitudes f_i and g_i were angle-independent. Although this is true in the Born approximation, it is certainly not completely correct. In the limit of a large deuteron the calculations may be carried out for arbitrary f_i and g_i. For this limit the deuteron scattering amplitude at a given scattering angle depends on the nucleon-scattering amplitudes only in the immediate neighborhood of this same scattering angle—as may be seen, for example, in Eq. (57). When the deuteron is not assumed infinitely large, a first approximation for the phase may be obtained by assuming the phase of the nucleon-scattering amplitudes to be constant at that value which the phase assumes at the angle for which the deuteron-scattering amplitude is being calculated. The same approximation can be made for the ratio of f_i and g_i as a function of angle. With these approximations Eqs. (71), (63), and (51), together with Eqs. (41) through (46) and (23) through (28), give the differential cross section and polarization effects in the scattering of deuterons explicitly in terms of x and y, the parameters that give the phase and relative magnitudes of the scattering amplitudes f_i and g_i; and the function $f_a(x)$ that determines the magnitude of the scattering amplitudes for the scattering of the nucleons. If the spin-dependent effects are omitted by setting $y=0$, then the value of $f_a(x)$ given in Eq. (60) may be used, and the only variable is the phase factor x. This phase factor may be determined at small angles by use of the optical theorem. At other angles it is necessary to use some detailed model of the nucleon-nucleon interaction. The results obtained by use of the model of Fernbach, Serber, and Taylor[22] were described in the introductory section. The numerical details of these results, together with considerations of more realistic models that include polarization effects, will be discussed in a subsequent paper.

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[22] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).