Theory of Cyclotron Resonance in Metals*

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Azbel' and Kaner have shown that a cyclotron resonance phenomenon is possible in a metal under extreme anomalous skin-effect conditions when the magnetic field is parallel to the metal surface. Here the correct form of the surface impedance is deduced in a more simple way by using the "ineffectiveness concept" of Pippard. An approximate criterion is also established for how nearly parallel to the surface the magnetic field must be. When the oscillating electric field is not parallel to the constant magnetic field, it produces a polarization of the charge distribution in the metal, and it is shown that this effect does not destroy the resonance in contrast to the situation in semiconductors.

I. INTRODUCTION

THE surface currents in a metal at high frequencies are governed by the usual skin-effect theory, until at low temperatures and high frequencies we come into the anomalous skin effect region where the skin depth δ becomes comparable with or smaller than the mean free path l of the electrons in the metal. In this range the surface impedance has been calculated from the Boltzmann transport equation by various authors,[1] but the calculations are long and complicated. In particular Azbel'[2] and Kaner[3] have considered the case when there is a magnetic field $\mathbf{H}$ (of magnitude H) parallel to the surface, and have shown[3] that this leads to a cyclotron resonance phenomenon although there is no useful resonance when the field is perpendicular to the surface.[4] This resonance is somewhat different from cyclotron resonance in semiconductors in that a resonance is in general expected when the oscillating electric field $\mathbf{E}$ is parallel to $\mathbf{H}$ as well as when it is perpendicular, even if, for instance, the Fermi surface is spherical. In semiconductors, on the other hand, a resonance is obtained with $\mathbf{E}$ and $\mathbf{H}$ parallel only if the Fermi surface is asymmetric and the fields are not along a symmetry axis.[5] This difference is brought about by the fact that in a semiconductor the skin depth is large compared with the specimen thickness, whereas the opposite is true in metals. This fact also has the following consequence. In a semiconductor a particular group of electrons (more precisely a particular closed piece of Fermi surface) gives rise to a single resonance line around a field $\mathbf{H}_c$, say, whereas in a metal additional subsidiary resonances are expected at all submultiples $\mathbf{H}_c/n$ of the field, where n is an integer.

Now Pippard[6] has shown that in the absence of a magnetic field the surface impedance Z_∞ in the extreme anomalous limit when $\delta \ll l$ may be calculated correctly for a Fermi surface of arbitrary shape using a crude but simple physical model known as the "ineffectiveness concept." In this model it is assumed that the electrons only make an effective contribution to the current if they spend most of their time in the skin depth, i.e., if they are moving at an angle less than δ/l with the surface. The purpose of the present paper is to apply a slightly adapted version of the ineffectiveness concept to the case when there is a uniform magnetic field H parallel to the surface. Thus we shall first give, using the ineffectiveness concept, a simple phenomenological derivation of the result of Azbel' and Kaner[3] for the surface impedance near resonance (Sec. II). Actually the result appears to be valid not only near resonance but over a much wider range of H.[2,3] Throughout we shall assume that $\delta/l \ll 1$, $\delta/r \ll 1$, and $\omega\tau\delta/l \ll 1$ so that relaxation effects are negligible,[6] where r is the radius of the orbit in the magnetic field of an electron at the Fermi surface, ω the frequency of the applied fields, and τ the relaxation time of the electrons. In Sec. III we shall consider the reduction in the cyclotron resonance signal when $\mathbf{H}$ is not exactly parallel to the surface.

As already mentioned, the result of Azbel' and Kaner[3] indicates that a resonance should be observed both when $\mathbf{E}$ and $\mathbf{H}$ are parallel and when they are perpendicular to one another, both fields being parallel to the surface. However, these authors do not appear to have taken account of the fact that when $\mathbf{E}$ is perpendicular to $\mathbf{H}$, the individual circular motion of the electrons is strongly coupled to the plasma oscillations of the electron gas as a whole. This effect in semiconductors shifts the resonance considerably,[7] and an analogous theory applied to a metal specimen would lead one to expect that the resonance is shifted to well

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[1] G. E. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1948); R. B. Dingle, Physica **19**, 311 (1953); E. H. Sondheimer, Proc. Roy. Soc. (London) **A224**, 260 (1954); M. I. Kaganov and M. Ia. Azbel', Doklady Akad. Nauk S.S.S.R. **102**, 49 (1955); M. Ia. Azbel' and E. A. Kaner, J. Exptl. Theoret. Phys. (U.S.S.R.) **29**, 876 (1955) [English translation: Soviet Phys. JETP **2**, 749 (1956)].

[2] M. Ia. Azbel', Doklady Akad. Nauk S.S.S.R. **100**, 437 (1955).

[3] M. Ia. Azbel' and E. A. Kaner, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 811 (1956) [English translation: Soviet Phys. JETP **3**, 772 (1956)].

[4] M. Ia. Azbel' and M. I. Kaganov, Doklady Akad. Nauk S.S.S.R. **95**, 41 (1954); R. G. Chambers, Phil. Mag. **1**, 459 (1956).

[5] W. Shockley, Phys. Rev. **90**, 491 (1953).

[6] A. B. Pippard, Solvay Congress Report **10**, 123 (1954); Proc. Roy. Soc. (London) **A224**, 273 (1954).

[7] Dresselhaus, Kip, and Kittel, Phys. Rev. **100**, 618 (1955).

outside the ordinary range of observation because of the high plasma frequency in a metal. However, such a theory would not be valid, and in Sec. IV we will show by semiquantitative arguments that the cyclotron resonance in a metal is expected to be unaffected by the plasma motion of the electron gas. This difference between metals and semiconductors is again due to the different relationship between the skin depth and the sample thickness.

In applying the ineffectiveness concept, the main part of the calculation consists of obtaining an expression for the current density $\mathbf{J}$ at the surface. Because of the ineffectiveness assumption, this turns out to have the form

$$\mathbf{J}=a\delta\mathbf{E}_0, \tag{1}$$

where $\mathbf{E}_0$ is the electric field at the surface, and where a contains an undetermined numerical factor but otherwise depends only on the shape of the Fermi surface. From (1) we define an effective conductivity

$$\sigma_{\text{eff}}=a\delta, \tag{2}$$

and then calculate Z_∞ using the ordinary skin effect results[6] in electromagnetic units:

$$Z_\infty=(4\pi\omega/\sigma_{\text{eff}})^{\frac{1}{2}}\exp(i\pi/4), \tag{3}$$

$$\delta=(i4\pi\omega\sigma_{\text{eff}})^{-\frac{1}{2}}, \tag{4}$$

where δ is defined by

$$\mathbf{E}(z)=\mathbf{E}_0\exp(-z/\delta). \tag{5}$$

Eliminating δ and σ_{eff} from (2), (3), and (4), we obtain

$$Z_\infty=2(2\pi^2\omega^2/a)^{\frac{1}{3}}\exp(i\pi/3). \tag{6}$$

At first sight it is surprising that such crude arguments correctly give the phase in (6) in agreement with the rigorous theory,[1] but as we shall mention later there are reasons for this. For instance, it is quite unnecessary to appeal to the formulas (3)–(5) from ordinary skin effect theory. The result (6) also follows directly from Maxwell's equations and (1), except that (6) now contains an extra unknown numerical factor depending on the form of $\mathbf{E}(z)$, which incidentally does not have the form (5) in the anomalous skin effect.[8] We shall assume this constant to be absorbed into the constant in a, and thus continue to use (6).

II. THE SURFACE IMPEDANCE

The current density $\mathbf{J}$ at any time and place in the metal may be written in the general form[9]

$$\mathbf{J}(r,t)=-\int\int\int\frac{2d^3\mathbf{p}}{h^3}\frac{df_0}{d\mathcal{E}}e\mathbf{v}\int_c^t e\mathbf{v}(\mathbf{r}',t')\cdot\mathcal{E}(\mathbf{r}',t')\times\exp[-(t-t')/\tau]dt', \tag{7}$$

where $\mathbf{r}'$ and t' refer to the place and time along the electron's trajectory at which the electron that finally has momentum $\mathbf{p}$ at $(\mathbf{r},t)$ finds itself at time t'; f_0 is the Fermi distribution function; $\mathcal{E}$ is the energy of the electronic states as a function of the momentum $\mathbf{p}$; $\mathbf{v}$ is the velocity of an electron, and e is the charge on a proton. The form (7) includes the case of a magnetic field implicitly through the nature of the electron's trajectory. Ordinarily the limit of integration c is $-\infty$, but if the trajectory of an electron cuts the surface of the metal, then c is the latest time prior to t that the trajectory cuts the surface. This takes account of diffuse scattering by the surface.[1] In the Appendix it is shown that (7) follows directly from the Boltzmann transport equation. We shall choose the x axis in the direction of the magnetic field parallel to the surface, the z axis normal to the surface pointing into the metal, and the y axis in the surface making a right-handed triad with z and x (Fig. 1).

We shall now use (7) to calculate the current density at the surface. The time spent by an electron within the skin depth is approximately $2r\phi_m/v_y$ (Fig. 1), where $r-r\cos\phi_m=\frac{1}{2}\delta$, i.e.,

$$\phi_m{}^2=\delta/r\ll 1. \tag{8}$$

This time is sufficiently short for the exponential factor in (7) to remain effectively constant at all fields $H\ll H_1$ for which

$$2r(H_1)\phi_m(H_1)/v_y=\tau. \tag{9}$$

For tin at 40 kMc/sec, $H_1=250/(\omega\tau)^2$, but this estimate may easily be in error (probably underestimated) by a factor as large as γ (see below). The electron travels in a spiral path with a period of rotation $2\pi/\omega_c$, where ω_c is the cyclotron frequency eH/m^*. The effective mass $m^*=(1/2\pi)|\partial S_x/\partial\mathcal{E}|$, where S_x is the cross-sectional area of the Fermi surface in the plane $p_x=$constant, as can easily be shown from the work of Shockley[10] and Onsager.[10] Every revolution the electron returns to the skin depth, and the integrand in (7) changes by a factor

$$e^{-w}=\exp\left[-\frac{2\pi}{\omega_c\tau}-i\frac{2\pi\omega}{\omega_c}\right], \tag{10}$$

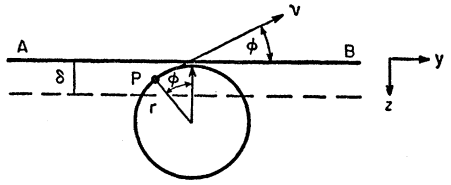

FIG. 1. Orbit of an electron passing through the skin depth at the surface AB. The direction of the magnetic field is along the x direction pointing into the plane of the paper.

[8] Note added since completing the manuscript. These points have previously been made by R. G. Chambers, Can. J. Phys. **34**, 1395 (1956). I am indebted to Dr. Chambers for this and other comments.

[9] R. G. Chambers, Proc. Phys. Soc. (London) **A65**, 458 (1952).

[10] W. Shockley, Phys. Rev. **79**, 191 (1950); L. Onsager, Phil. Mag. **43**, 1006 (1952).

where the second term arises from the change in the phase of the field $\mathbf{E}$. Thus the second integral in (7) becomes

$$e\left(\frac{2r\phi_m v_j}{v_y}\right)E_j(1+e^{-w}+e^{-2w}+\cdots)=\frac{2er\phi_m v_j E_j}{v_y[1-\exp(-w)]}.$$

We use suffices i and j for two directions at right angles in the surface, and also the double suffix summation convention.

Considering a section of the Fermi surface $p_x=$ constant, we have in (7)

$$d^3p\frac{df_0}{dE}=-dp_x\delta(E-E_M)\frac{dE}{v_\perp}\rho d\phi,$$

where $\delta(\mathcal{E}-\mathcal{E}_M)$ is the Dirac delta function, ρ is the radius of curvature in the section, ϕ the polar angle with origin such that $v_z=0$, $v_y>0$ at $\phi=0$, and $v_\perp$ is the component of $\mathbf{v}$ perpendicular to $\mathbf{H}$, i.e., $v_\perp(\phi=0)=v_y$. Since electrons are scattered diffusely at the surface, an electron can only contribute effectively to the current at P (Fig. 1) if it does not hit the surface, i.e., if its perpendicular velocity component $v_\perp$ makes an angle ϕ less than ϕ_m with the surface. This is our form of the basic assumption of the ineffectiveness concept. If one takes only these effective electrons into account, (7) becomes

$$J_i=\gamma\frac{2e^2}{h^3}\int\frac{\phi_m{}^2 n_i n_j\rho_0 2rE_j}{n_y{}^2[1-\exp(-w)]}dp_x,$$

where we have included a real undetermined numerical factor γ to make up for the approximations of the model. The quantity ρ_0 is ρ at $\phi=0$, and n_i, n_j, n_z are the direction cosines of $\mathbf{v}$. Using (8), (1), and (2), we obtain

$$(\sigma_{\text{eff}})_{ij}=a_{ij}\delta, \tag{11}$$

where

$$a_{ij}=\gamma\frac{4e^2}{h^3}\int\frac{n_i n_j}{n_y{}^2}\frac{\rho_0 dp_x}{[1-\exp(-w)]}, \tag{12}$$

whence the surface impedance follows from (6). We shall now define i and j to be the principal directions of (12) because actually the ineffectiveness concept only applies when $\mathbf{E}$ is along one of these directions.[6] Then our result is in agreement with that of Azbel' and Kaner[3] although expressed slightly differently, and Z_∞ for an arbitrary direction[6] in the surface may be obtained from $Z_{\infty i}$ and $Z_{\infty j}$. The denominator in (12) makes the surface impedance go through a resonance near $H=H_c/n$ for $\omega\tau>1$, where H_c is such that $\omega_c(H_c)=\omega$, and n is an integer. The resonance occurs in both $Z_{\infty i}$ and $Z_{\infty j}$ and, thus, for any orientation of the electric field in the surface.

Near resonance, the phase factor in the impedance becomes very important, and to show that it has been calculated correctly we would have to prove (*i*) that γ is real, (*ii*) that the δ in (11) is the complex skin depth δ as in Eqs. (1) to (6) and not for instance its real part, and (*iii*) that the constant which appears in (6) in the general case is real. These results can be proved by making some general assumptions about $\mathbf{E}(z)$, but the assumptions can in turn only be justified by appealing to the rigorous analysis. Instead of bolstering up the argument in this way, we shall take the attitude that the usefulness of the ineffectiveness concept lies in its giving the correct form of the answer quickly rather than in any rigor. Actually the "ineffectiveness" model corresponds closely to the rigorous analysis in many, but not all, important respects.

In (12) we cannot determine γ directly by comparison with the result of Azbel' and Kaner[3] since this itself contains an undetermined constant. However for an ellipsoidal Fermi surface, and H much greater than H_c and $H_c/\omega\tau$, we can compare our result with that of Azbel'[2] and obtain

$$\gamma=\left(\frac{8}{9}\right)^3\frac{8\pi}{3\sqrt{3}}\quad\text{for large } H.$$

Although our analysis only applies for $H>H_1$, we notice that (12) also has the correct form[6,1] for $H=0$, so that we obtain

$$\gamma=8\pi/3^{\frac{3}{2}}\quad\text{for}\quad H=0.$$

Thus if the relaxation time and the effective mass are constant over the whole Fermi surface, we can express the surface impedance for an arbitrary field and direction in terms of the surface resistance R in zero magnetic field.

$$Z_\infty/R=2C\exp(i\pi/3)\left[1-\exp\left(-\frac{2\pi}{\omega_c\tau}-i\frac{2\pi\omega}{\omega_c}\right)\right]^{\frac{1}{3}}, \tag{13}$$

where $C=1$ at $H=0$, $C=8/9$ at $H\gg H_c$ and $H\gg H_c/\omega\tau$, and where C may be expected to vary rapidly with H near $H=H_1$.

III. NONPARALLEL FIELD

We shall now discuss the question, how nearly parallel to the surface must the magnetic field be for the resonance to occur? Figure 2 shows the trajectory of an electron when the magnetic field in the xz plane makes a small angle ψ with the surface. We shall assume that $\mathbf{E}$ is along the x direction since this avoids the question of polarization effects (Sec. IV). Suffices $\parallel$ and $\perp$ will denote components parallel and perpendicular to the magnetic field and not to the x axis. $E_\perp$ only produces a translation of the trajectory in the y direction and will be neglected, while $E_\parallel$ produces an acceleration along $\mathbf{H}$ every time the electron passes through the skin depth.

For unaccelerated motion in a magnetic field, $p_\parallel$ is a constant of the motion, so that an electron travels in a

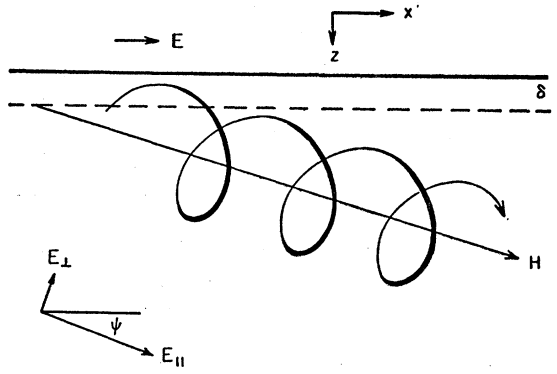

FIG. 2. Trajectory of an electron when **H** is not parallel to the surface.

helical path with its axis parallel to the field. Thus if the electron starts its path at the surface, it will tend to move away from the surface, particularly if it has a large $v_{||}$, so that after a while it ceases to return within the skin depth each revolution and therefore ceases to take part in the conduction process. In particular an electron with $v_{||}=v_\perp$ travels a distance δ downwards in the z direction in each revolution if $\psi=\psi_m$, where

$$\psi_m=\delta/2\pi r. \tag{14}$$

For this angle and a spherical Fermi surface we therefore obtain the following picture of the conduction process. This number of electrons passing through the skin depth is determined as in Sec. II so that there is no change in the impedance on this account. However, half of these electrons ($|v_{||}/v_\perp|>1$) never return to within the skin depth of the surface and do not contribute towards any resonance. A fraction $2\pi/\omega\tau$ ($\omega\tau\gg1$) of the electrons have $|v_{||}/v_\perp|<2\pi/\omega\tau$ and return to the skin depth $\omega\tau/2\pi$ times making their full contribution to the resonance as in (12). The remaining electrons make some contribution to the resonance. Similarly the amplitude of the resonance at other angles is a function of ψ/ψ_m and $\omega\tau$ which begins to fall off considerably at $\psi\approx\psi_m$. This function could easily be calculated by using the ineffectiveness concept except that we do not know at what stage to introduce the constant γ. For instance, the effect of γ in (12) could be interpreted as meaning that the effective conducting surface layer has a thickness $\gamma\delta$ instead of δ, in which case the angle determining the reduction in the resonance would be $\gamma\psi_m$ instead of ψ_m.

For a nonspherical Fermi surface, the amplitude of the resonance may, however, be greater for $\psi=\psi_m$ than for $\psi=0$ for the following reason. The form of the resonance in the resistive part of the surface impedance is more like a dispersion curve than a resonance one. Thus if m^* varies by more than a factor of 1.5 over the Fermi surface, the different contributions tend to cancel one another resulting in a very much reduced resonance. In this case having $\psi\neq0$ would select a band of the Fermi surface for which $v_{||}\approx0$ and $m^*\approx$ constant, so that the resonance of these electrons by themselves would be observed.[8]

For tin at resonance at 40 kMc/sec we obtain $\psi_m\approx0.4°$ by using the effective mass corresponding to the resistance minima in the resonance curves of Fawcett,[11] but the resonance may be observable at considerably greater angles for one or both of the reasons discussed. The results of the present discussion are at variance with Azbel' and Kaner's assertion[3] that the condition for an observable resonance is $\psi\lesssim(\delta/r)^{\frac{1}{3}}$. This angle is 7° for tin under the same conditions, so that it may be difficult to distinguish between them experimentally in this case.

IV. PLASMA EFFECTS

In Fig. 3, $ABCA$ represents the orbit of an electron which for the sake of convenience we assume has no velocity component along **H**. If the electric field **E** is applied perpendicular to **H**, an electron with speed v is accelerated in the direction of its motion as it passes through the skin depth, so that its speed is increased to $v+\Delta v$. In the absence of further electric fields it would then describe with the same angular velocity ω_c the orbit $ADEA$ which has a larger radius than the unaccelerated orbit. Thus the electron density at E would increase at the expense of that at C, and a space charge would be set up (Fig. 3). This inhomogeneity of charge and the consequent electric fields have not been taken into account in Sec. II, in Azbel''s calculation[2] based on the Boltzmann equation, nor apparently by Azbel' and Kaner,[3] and we shall now do so. In semiconductors these effects, known as plasma or polarization effects, modify and shift the cyclotron resonance considerably, but we shall see that we expect this to be not so under anomalous skin-effect conditions in metals. No polarization arises when **E** is parallel to **H**, and we shall not discuss this case further.

In a metal, a space charge such as shown in Fig. 3 cannot exist and dies away in about 10^{-21} sec.[12] In fact it could never build up, and what actually happens is that a small space charge is produced which sets up a vertical electric field. This field is strong enough to

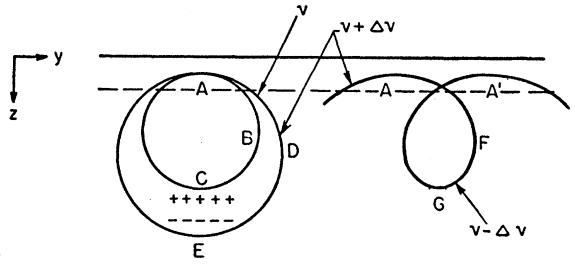

FIG. 3. Production of space charge.

[11] E. Fawcett, Phys. Rev. **103**, 1582 (1956).
[12] J. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 15.

stop further electrons coming down to a depth further than C (Fig. 3). The resulting orbit looks like $AFGA'$. The amount of space charge required to set up these fields is exceedingly minute and in any case tends to zero as $\mathbf{E}$ tends to zero. To obtain the terms in the current linear in $\mathbf{E}$, we may neglect this charge and also consider that A' is within the skin depth if A is.

Thus to calculate the surface impedance in the presence of polarization, we would have to include in the transport equation (7) an unknown vertical electric field E_z which is determined by the condition that the resulting transport maintains the charge density uniform everywhere. This problem has not yet been solved generally. However we assert that an electron when it reaches the skin depth again at A' (Fig. 3) after one revolution, has on the average the same drift velocity as it had when leaving the skin depth after being accelerated at A. It therefore contributes to the current as in Sec. II and is ready to be accelerated further, leading to repeated acceleration in the same direction, and thus, resonance if $\omega=\omega_c$. Although we cannot prove our assertion in general, we shall now show that it is correct in a simple case which would appear to contain all the physical essentials of the situation.

Consider an electron traveling unperturbed in the orbit $ABCA$ (Fig. 3) with speed v. Its position is given by

$$\begin{aligned} y_0&=(v/\omega_c)\sin\omega_c t,\\ z_0&=(v/\omega_c)(1-\cos\omega_c t). \end{aligned} \tag{15}$$

If it receives a single impulse at A which increases its speed to $v+\Delta v$, its motion in the absence of polarization is along $ADEA$ and is given by

$$y_1=\frac{v+\Delta v}{\omega_c}\sin\omega_c t,\quad z_1=\frac{v+\Delta v}{\omega_c}(1-\cos\omega_c t). \tag{16}$$

Its speed at all times is $v+\Delta v$, so that the magnetic field has reversed the drift velocity in half a cycle. In

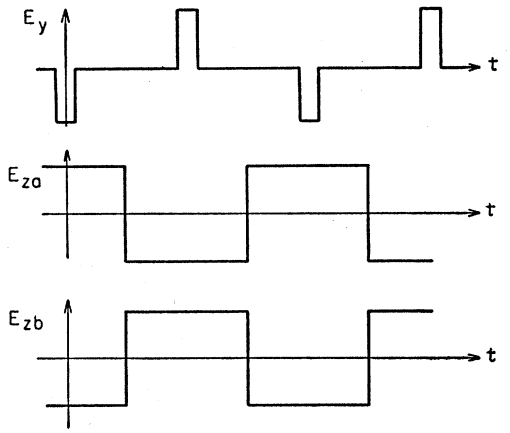

FIG. 4. Electric fields at resonance with polarization. E_y is the applied field at the surface; $E_z=E_{za}$ for $0\leq z<v/\omega_c$, $E_z=E_{zb}$ for $v/\omega_c<z\leq 2v/\omega_c$.

semiconductors like germanium with a low carrier concentration, where the radius v/ω_c is small compared with the skin depth, and where plasma effects are absent, this leads to a resonance since $\mathbf{E}$ also reverses in half a cycle. However in the presence of polarization the equations of motion are

$$dv_y/dt=-\omega_c v_z,\quad dv_z/dt=-(e/m^*)E_z+\omega_c v_y, \tag{17}$$

where E_z is determined by the condition that the vertical charge density is the same as if the electron had not been accelerated, i.e., in our case by the condition

$$z(t)=z_0(t). \tag{18}$$

The solution of (17) and (18) is

$$\begin{aligned} z_2&=(v/\omega_c)(1-\cos\omega_c t),\\ y_2&=(v/\omega_c)\sin\omega_c t+t\Delta v,\\ v_{y2}&=v\cos\omega_c t+\Delta v,\\ E_z&=e\omega\Delta v/m^*, \end{aligned} \tag{19}$$

giving the orbit $AFGA'$. Thus the drift velocity remains constant as if no magnetic field were present instead of being reversed after half a cycle. Thus, in a semiconductor like InSb with a high carrier concentration where plasma effects are strong but where the orbit radius is still less than the skin depth, we would expect no resonance in our approximation because the field accelerates an electron at A (Fig. 3) and decelerates it half a cycle later at G. However from (19) after a full cycle at A' the velocity is again $v+\Delta v$. In a metal, therefore, under anomalous skin effect conditions, an electron contributes to the surface current just as in Sec. II and it is immaterial whether or not the drift velocity is reversed at G because the electric field does not penetrate as deep as this. Thus if we apply the periodic field E_y (Fig. 4), bunches of electrons are accelerated or retarded every half cycle, the field E_{za}, E_{zb} (Fig. 4) is set up so that each bunch sees a constant E_z on its path, and we obtain at the surface repeated resonance acceleration or retardation in phase, which is only limited by the relaxation time, τ.

Another way of stating the situation makes it clearer why only the drift velocity at AA' matters and not that at G (Fig. 3). Neglecting polarization, the last integral in (7) can be written in the form

$$\int_0^\infty K_0(z,\zeta;\mathbf{p})E_y(\zeta)d\zeta, \tag{20}$$

where K_0 is the memory that an electron with momentum $\mathbf{p}$ at z has of the drift velocity it picked up from the field $E_y(\zeta)$ at a depth ζ below the surface. K_0 has a large cusp at $z-\zeta=0$ due to the electrons moving parallel to the surface at $z(v_z=0)$, and the small wavelength Fourier components of K_0 due to this cusp determine the surface impedance in the extreme anomalous limit.[1] Thus Z_∞ is determined only by

$K_0(z,\zeta;v_z\approx 0)$ for $z-\zeta\approx 0$. In the presence of polarization effects, (20) should be replaced by

$$\int_0^\infty [K_0(z,\zeta;\mathbf{p})E_y(\zeta)+K_1(z,\zeta;\mathbf{p})E_z(\zeta)]d\zeta,$$

but since E_z depends linearly on E_y this can be written as

$$\int_0^\infty K_2(z,\zeta;\mathbf{p})E_y(\zeta)d\zeta.$$

In the example with a pulsed field E_y (Fig. 4), we saw that

$$K_0(z,\zeta;\, v_z=0)=K_2(z,\zeta;\, v_z=0) \quad \text{for} \quad z-\zeta=0$$

so that the surface impedance is the same. The fact that

$$K_0(z,\zeta;\, v_z=0)=-K_2(z,\zeta;\, v_z=0) \quad \text{for} \quad z-\zeta=2v/\omega_c$$

is immaterial in the extreme anomalous limit.

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I am indebted to Professor C. Kittel for suggesting this problem, and to him as well as other members of the Department for stimulating discussions.

APPENDIX

Chambers[9] has derived the expression

$$f=f_0+\phi, \tag{A.1a}$$

where

$$\phi=-\frac{df_0}{d\mathcal{E}}\int_c^t dt'(-e)\mathbf{E}(\mathbf{r}',t')\cdot\mathbf{v}(\mathbf{r}',t') \times\exp\left(-\int_{t'}^t\frac{dt''}{\tau(t'')}\right) \tag{A.1b}$$

for the electron distribution function $f(\mathbf{r},\mathbf{p},t)$ in a transport process. The symbols have the meaning defined in connection with Eq. (7), and the integrations with respect to t', t'' are carried out along the electron's trajectory. This expression was derived by Chambers from "kinetic" arguments similar to Shockley's method[10] of "tube integrals." However in view of the generality and usefulness of the result (A.1) as a starting point for the study of many transport phenomena, it seems worthwhile confirming its validity further by showing that it is also an exact solution of the Boltzmann transport equation[13]

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial\mathbf{r}}+(-e\mathbf{E}-e\mathbf{v}\times\mathbf{H})\cdot\frac{\partial f}{\partial\mathbf{p}}=-\frac{f-f_0}{\tau(\mathbf{p})}, \tag{A.2}$$

where $\partial f/\partial\mathbf{r}$ means grad f. This we shall now do. Incidentally the expression (7) for the current density follows directly from (A.1) and $\mathbf{J}=-(2e/h^3)\int f\mathbf{v}d^3\mathbf{p}$ if we assume a constant relaxation time.

We first observe that (A.1) satisfies the following boundary conditions which are usually required by physical considerations. Thus $f\to f_0$ in regions where $\mathbf{E}=0$. At a surface scattering electrons diffusely,[1,14] (A.1) gives $f=f_0$ for electrons leaving the surface. Similarly at a surface reflecting electrons specularly, f is the same for electrons arriving at and reflected from the surface.

We next note that the independent variables in (A.1b) are t, $\mathbf{r}$, $\mathbf{p}$, t', t'' and that $\mathbf{r}'$ has the form

$$\mathbf{r}'=\mathbf{r}+\mathbf{F}(T,\mathbf{p}),$$

where $\mathbf{F}$ is a function only of $\mathbf{p}$ and $T=t-t'$. Also

$$\partial\mathbf{r}'/\partial\mathbf{r}=\mathbf{1} \quad \text{and} \quad \partial\mathbf{r}'/\partial t=\partial\mathbf{F}/\partial T, \tag{A.3}$$

where $\mathbf{1}$ is the unit tensor. Now since $\mathbf{r}'$ represents the trajectory of an electron, the function $\mathbf{F}$ is determined by the law of force acting on the electron. Consider an electron traveling along its trajectory and arriving at the point $\mathbf{r}$ at time t with momentum $\mathbf{p}$, and suppose it then travels an extra distance $\delta\mathbf{r}$ in a time δt. We have

$$\begin{aligned}\mathbf{r}'&=\mathbf{r}+\mathbf{F}(t-t',\,\mathbf{p})\\ &=\mathbf{r}+\delta\mathbf{r}+\mathbf{F}(t+\delta t-t',\,\mathbf{p}+\delta\mathbf{p}),\end{aligned}$$

where

$$\delta\mathbf{r}=\mathbf{v}(\mathbf{r},t)\delta t, \quad \delta\mathbf{p}=[-e\mathbf{E}(r,t)-e\mathbf{v}\times\mathbf{H}]\delta t.$$

Thus we obtain

$$\mathbf{v}(\mathbf{r},t)\delta t+\frac{\partial\mathbf{F}}{\partial T}\delta t+[-e\mathbf{E}(r,t)-e\mathbf{v}\times H]\cdot\frac{\partial\mathbf{F}}{\partial\mathbf{p}}\delta t=0,$$

and by using (A.3)

$$\frac{\partial\mathbf{r}'}{\partial t}=-\mathbf{v}(\mathbf{r},t)\cdot\frac{\partial\mathbf{r}'}{\partial\mathbf{r}}-[-e\mathbf{E}(r,t)-e\mathbf{v}\times\mathbf{H}]\cdot\frac{\partial\mathbf{r}'}{\partial\mathbf{p}}. \tag{A.4}$$

We can now proceed to differentiate (A.1) and hence verify that it satisfies (A.2). In the following manipulation, we assume for simplicity that $\tau(\mathbf{p})=$ constant and replace $\int dt''/\tau$ by $(t-t')/\tau$, since the extension of the proof to the more general case is straightforward. Thus

$$\begin{aligned}\frac{\partial\phi}{\partial t}=&-\frac{df_0}{d\mathcal{E}}(-e)\mathbf{E}(\mathbf{r},t)\cdot\mathbf{v}(\mathbf{r},t)-\frac{\phi}{\tau}\\ &-\frac{df_0}{d\mathcal{E}}\int dt'\left\{\frac{\partial\mathbf{r}'}{\partial t}\cdot\frac{\partial}{\partial\mathbf{r}'}[-e\mathbf{E}(\mathbf{r}',t')\cdot\mathbf{v}(\mathbf{r}',t')]\right.\\ &\left.\times\exp[-(t-t')/\tau]\right\}.\end{aligned} \tag{A.5}$$

If we make the substitution (A.4) in the last term of

[13] In fact it would be rather surprising if (A.1) did not satisfy the Boltzmann equation since (A.1b) and (A.2) have been derived from the same physical principles.

[14] K. Fuchs, Proc. Cambridge Phil. Soc. **34**, 100 (1938).

(A.5) it becomes

$$-\mathbf{v}\cdot\frac{\partial\phi}{\partial\mathbf{r}}-(-e\mathbf{E}-e\mathbf{v}\times\mathbf{H})\cdot\frac{\partial\phi}{\partial\mathbf{p}}, \quad (A.6)$$

since ϕ depends on $\mathbf{r}$ and $\mathbf{p}$ only through $\mathbf{r}'$. We also have

$$\frac{\partial f_0}{\partial t}=\frac{\partial f_0}{\partial\mathbf{r}}=0, \quad \frac{\partial f_0}{\partial\mathbf{p}}=\frac{df_0}{d\mathcal{E}}\mathbf{v}, \quad \frac{\partial f_0}{\partial\mathbf{p}}\cdot\mathbf{v}\times\mathbf{H}=0. \quad (A.7)$$

It is now easy to verify that we really have a solution of the Boltzmann equation by substituting (A.1a), (A.5), (A.6), (A.7) into (A.2). Incidentally, if in (A.1b) we neglect the effect of the electric field on the trajectory of an electron, and thus drop the term in $\mathbf{E}$ in (A.4), then we obtain those terms in the Boltzmann equation which depends only linearly on the electric field.

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Selection Rules for the Absorption of Polarized Electromagnetic Radiation by Mobile Electrons in Crystals

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Group-theoretical methods are employed to obtain the selection rules for the absorption of polarized $E1$, $M1$, and $E2$ electromagnetic radiation by mobile electrons which are near energy-band extrema. Both the single-valued and double-valued groups appropriate to several symmetry points in the reduced zones of simple-cubic, face-centered-cubic, body-centered-cubic, and hexagonal-close-packed lattice symmetries are considered. The symmetry properties at various points in the reduced zone are also used to derive expressions for the electron energy as a function of the wave vector $\mathbf{k}$ in the neighborhood of nondegenerate extremum states.

The results obtained are applied to a consideration of the change in optical selection rules when single crystals of indium antimonide are subjected to a shearing stress.

I. INTRODUCTION

IT has been shown by Bloch[1] that the behavior of mobile electrons[2] in the periodic potential field of a crystal can be described in terms of eigenfunctions of the form

$$\psi_{\mathbf{k}}=e^{i\mathbf{k}\cdot\mathbf{r}}u_{\mathbf{k}}(\mathbf{r}), \quad (1)$$

where the modulating function $u_{\mathbf{k}}(\mathbf{r})$ has the periodicity of the lattice, and where $\mathbf{k}$ is the wave vector for an electron with a given energy $E(\mathbf{k})$. The corresponding energy eigenvalues fall into allowed and forbidden regions in $\mathbf{k}$-space, with energy discontinuities occurring at the Bragg reflection planes

$$\mathbf{k}\cdot\mathbf{n}=(m\pi)/a, \quad (2)$$

where $\mathbf{n}$ is a unit vector normal to a lattice plane. The lattices in $\mathbf{k}$-space bounded by these planes are the Brillouin zones. It can be shown that, owing to the translational symmetry of the lattice, the $\mathbf{k}$-space under consideration can be reduced to a single unit cell in the reciprocal lattice, with the electron wave functions being multivalued functions of $\mathbf{k}$. The unit cell is called the first Brillouin zone or the reduced zone.[3]

The set of symmetry properties associated with a given point in the reduced zone determines the selection rules for the absorption of polarized electromagnetic radiation and the functional dependence of the electron energy on the wave vector $\mathbf{k}$. The selection rules referred to in this paper are those which are in addition to the selection rule obtained from the law of conservation of linear momentum,

$$\mathbf{k}_i+\mathbf{q}=\mathbf{k}_f, \quad (3)$$

where $\mathbf{k}_i$ and $\mathbf{k}_f$ are the initial and final wave vectors of the electron, and $\mathbf{q}$ is the wave vector of the electromagnetic field. Only the vertical transitions ($\Delta\mathbf{k}=0$) will be treated in this paper.[4]

The contention is made that a knowledge of the selection rules for the absorption of polarized radiation by electrons near energy-band extrema could lead to further information about the finer details of the band structure. Although there are, in general, a large number of possible initial or final states, the selection rules could help to eliminate some residual ambiguities in the description of the band structure as obtained from other types of experiment. This point has been illustrated by

[1] F. Bloch, Z. Physik 52, 555 (1929).
[2] The term "electron" will be used to denote electrons or holes.
[3] See for example J. R. Reitz, *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press, Inc., New York, 1955), Vol. 1, Chap. I.
[4] Since the de Broglie wavelength of an electron near a zone boundary is of the order of magnitude of the lattice constant, the assumption of vertical transitions is equivalent to the assumption that the lattice constant is small compared to the photon wavelength.