ever, that the shock velocity and temperature can be substantially increased by merely raising the discharge voltage since the effect of wall cooling becomes increasingly important as the plasma temperature is raised. However, it is not improbable that longitudinal magnetic fields along the expansion chamber can be used to inhibit the heat conduction to the tube walls.

ACKNOWLEDGMENTS

The author would like to thank Dr. W. R. Faust and Dr. C. V. Strain for their constant interest and support during the course of these experiments. In addition, the assistance of Mr. Thomas DeRieux and Mr. Leonard Melhart, who constructed and operated the experimental apparatus, is gratefully acknowledged.

PHYSICAL REVIEW VOLUME 107, NUMBER 2 JULY 15, 1957

Relaxation of a System of Particles with Coulomb Interactions*

W. M. MacDonald,† *Department of Physics, University of Wisconsin, Madison, Wisconsin*
M. N. Rosenbluth,‡ *University of California Radiation Laboratory, Berkeley, California*
AND
Wong Chuck,§ *University of California Radiation Laboratory, Livermore, California*
(Received February 5, 1957)

The relaxation to a Maxwellian distribution of a system of particles interacting through inverse-square-law forces is investigated in the approximation of two-particle interactions resulting in small-angle deflections of particle trajectories. The time required for the relaxation of the distribution in the neighborhood of the average energy is found to agree with the self-collision time defined by Spitzer. The time required for the distribution to become Maxwellian throughout the range from zero energy to several times the average energy is found to be nearly ten times the self-collision time. Filling of the high-energy portion of the Maxwell distribution is also discussed.

I. INTRODUCTION

THE relaxation of the electron or ion component of an ionized gas to a Maxwellian distribution has been of some astrophysical interest. Spitzer[1] has analyzed various aspects of the relaxation phenomenon such as (1) removal of angular anisotropy, (2) energy exchange, and (3) loss of energy of a particle by "dynamical friction." Bohm and Aller[2] have presented a detailed analysis on the relative importance of electron-electron collisions in establishing the velocity distribution of electrons in gaseous nebulae and stellar atmospheres. Although the general conclusions reached by these authors is almost certainly correct, the discussions were based on the rates of change of the distribution function and not on an explicit solution of the time-dependent problem.

In this paper we present an equation for the effect of particle interactions on the one-particle distribution function and the results of a numerical integration of the equation on an electronic digital computer for a distribution initially peaked about a particular energy. The filling of the high-energy portion of the Maxwell distribution is treated approximately.

II. TIME-DEPENDENT EQUATION

In obtaining an equation which describes the effect of interactions between particles of charge e and mass m upon the velocity distribution, we assume that (1) all interactions are a superposition of two-body interactions resulting in (2) small-angle deflections of particle trajectories. Although the validity of these two approximations is not rigorously established, the work of Spitzer, Cohen, and Routly[3] and of Gasiorowicz, Neuman, and Riddell[4] indicate their essential correctness for many phenomena. We shall use the Rutherford scattering law to determine the probability of deflections of a given magnitude. Restricting ourselves to isotropic angular distributions, we can obtain an equation for the time rate of change of the distribution function, either from an expansion of the integrand of the Boltzmann collision integral in powers of the angle of deflection,[5] or from the Fokker-Planck equation,[6]

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

† At University of California Radiation Laboratory, Berkeley, when this work was begun; present address: Department of Physics, University of Maryland, College Park, Maryland.

‡ On leave from Los Alamos Scientific Laboratory at time work was begun; present address: General Atomic, San Diego, California.

§ Present address: 1809 Bonita Avenue, Berkeley, California.

[1] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

[2] D. Bohm and L. Aller, Astrophys. J. **105**, 131 (1947).

[3] Cohen, Spitzer, and Routly, Phys. Rev. **80**, 230 (1950).

[4] Gasiorowicz, Neuman, and Riddell, Phys. Rev. **101**, 922 (1956).

[5] The development by this procedure was considered too lengthy to be given here since the equation given in reference 6 yields the same result much more easily.

[6] Rosenbluth, MacDonald, and Judd, Phys. Rev. (to be published).

$$\frac{\partial f}{\partial t}=\frac{2\pi e^4}{m^2}\ln\Lambda\left\{\frac{2}{3}\frac{\partial^2 f}{\partial v^2}\left[\int_v^\infty du f(u)u+\frac{1}{v^3}\int_0^v du f(u)u^4\right]\right.$$
$$+\frac{4}{3v}\frac{\partial f}{\partial v}\left[\int_0^\infty du f(u)u\right.$$
$$\left.\left.-\int_0^v du f(u)u\left(1-\frac{u}{v}\right)^2\left(1+\frac{u}{2v}\right)\right]+2f^2\right\}, \quad (1)$$

$\Lambda=(3/2e^3)(k^3T^3/\pi n)^{\frac{1}{2}}$.

The quantity $f(v)$ is a distribution function in *magnitude* of velocities and has the normalization

$$n=\int_0^\infty du f(u)u^2, \quad (2)$$

where n is the number density of particles (per unit volume). The Maxwell distribution causes the right side of Eq. (1) to vanish, showing that this distribution is indeed a solution of the static problem.

Equation (1) can be put into dimensionless form by observing that the distribution function can always be written

$$f(v,t)=(A/v_0^3)h(\xi,\tau), \quad \xi=v/v_0, \quad (3)$$

where A is a normalization constant, v_0 is a "characteristic" velocity, and τ is a dimensionless time parameter:

$$\tau=(2\pi e^4/m^2)(A/v_0^3)(\ln\Lambda)t. \quad (4)$$

The equation satisfied by $h(\xi,\tau)$ is then dimensionless:

$$\frac{\partial h}{\partial \tau}=\frac{2}{3}\left\{\frac{\partial^2 h}{\partial \tau^2}\left[\int_\xi^\infty d\eta h(\eta)\eta+\frac{1}{\xi^3}\int_0^\xi d\eta h(\eta)\eta^4\right]\right.$$
$$+\frac{4}{3\xi}\frac{\partial h}{\partial \xi}\left[\int_0^\infty d\eta h(\eta)\eta\right.$$
$$\left.\left.-\int_0^\xi d\eta h(\eta)\eta\left(1-\frac{\eta}{\xi}\right)^2\left(1+\frac{\eta}{2\xi}\right)\right]+2h^2\right\}. \quad (5)$$

We can relate A and v_0 to the number density of particles and the average energy or kinetic temperature (kT) of the system by

$$n=I_2A, \quad E\equiv\tfrac{3}{2}kT=\tfrac{1}{2}mv_0^2(I_4/I_2),$$
$$I_2\equiv\int_0^\infty d\xi h(\xi,\tau)\xi^2, \quad I_4\equiv\int_0^\infty d\xi h(\xi,\tau)\xi^4. \quad (6)$$

The integrals I_2 and I_4 are constants [as one can verify directly from Eq. (5)], determined by the initial distribution. The dimensionless time parameter in terms of n, kT, I_2, and I_4 is

$$\tau=\frac{2\pi e^4 n\ln\Lambda}{m^{\frac{1}{2}}(3kT)^{\frac{3}{2}}}\left(\frac{I_4}{I_2}\right)^{\frac{3}{2}}\frac{1}{I_2}t. \quad (7)$$

The usefulness of this formulation of the time-dependent problem is that all solutions of Eq. (1) which can be obtained by similarity transformations from a given solution are easily constructed by using the solution of Eq. (5). The solutions of the time-dependent Eq. (1) are

$$f(v,t)=\frac{n}{I_2}\left(\frac{m}{3kT}\right)^{\frac{3}{2}}\left(\frac{I_4}{I_2}\right)^{\frac{3}{2}}h(\xi,\tau),$$
$$\xi=v\left(\frac{m}{3kT}\right)^{\frac{1}{2}}\left(\frac{I_4}{I_2}\right). \quad (8)$$

The variable τ is related to the time by Eq. (7). The initial distribution is of course

$$f(v,0)=\frac{n}{I_2}\left(\frac{m}{3kT}\right)^{\frac{3}{2}}\left(\frac{I_4}{I_2}\right)^{\frac{3}{2}}h(\xi,0). \quad (9)$$

The quantities I_2 and I_4 are calculated from $h(\xi,0)$.

III. NUMERICAL INTEGRATION

The numerical integration was carried out by using the difference equation for Eq. (5):

$$h_j^{n+1}=h_j^n+\Delta\tau\left\{\left[\frac{2}{3}\left(\frac{h_{j+1}^n-2h_j^n+h_{j-1}^n}{(\Delta\xi)^2}\right)S_j^n\right]\right.$$
$$\left.+\frac{4}{3\xi_j}\left(\frac{h_{j+1}^n-h_{j-1}^n}{2\Delta\xi}\right)(S^n-R_j^n)+2(h_j^n)^2\right\}, \quad (10)$$

where

$$S_j^n=\frac{1}{\xi_j^3}\int_0^j d\eta h^n\eta^4+\int_j^J d\eta h^n\eta, \quad S^n=\int_0^J d\eta h^n\eta,$$
$$R_j^n=\int_0^j d\eta h^n\eta-\frac{3}{2\xi_j}\int_0^j d\eta h^n\eta^2+\frac{1}{2\xi_j^3}\int_0^j d\eta h^n\eta^4. \quad (11)$$

Subscripts refer to space points, and superscripts to time intervals. The condition for stability of the numerical integration of Eqs. (10) and (11) is

$$\Delta\tau/(\Delta\xi)^2<(\tfrac{3}{4}S^n). \quad (12)$$

This condition was used to determine the interval in τ for each successive time step.

An initial distribution was chosen which represents the shape assumed by a delta function after a time sufficiently short to be neglected ($\tau\sim 10$), and yet one which has sufficient breadth to be treated simply in a machine calculation. The initial distribution was chosen to be a Gaussian centered at $\xi=0.3$:

$$h(\xi,0)=0.01\exp\{-10[(\xi-0.3)/0.3]^2\}. \quad (13)$$

The distribution function $h(\xi,\tau)$ was computed for 24 values of ξ at intervals of $\Delta\xi=0.03$ from $\tau=0$ to $\tau=484.17$. This initial distribution is a two-parameter

TABLE I. A comparison of the latest distribution $h(\xi, 484.17)$ with the Maxwellian distribution.

ξ	$h_M(\xi)$	$h(\xi, 484.17)$
0	1.76×10^{-2}	1.86×10^{-2}
0.03	1.73	1.86
0.06	1.67	1.79
0.09	1.58	1.68
0.12	1.45	1.54
0.15	1.30	1.37
0.18	1.14	1.20
0.21	9.77×10^{-3}	10.13×10^{-3}
0.24	8.17	8.38
0.27	6.67	6.76
0.30	5.31	5.32
0.33	4.13	4.08
0.36	3.14	3.05
0.39	2.33	2.29
0.42	1.69	1.59
0.45	1.19	1.10
0.48	8.23×10^{-4}	7.47×10^{-4}
0.51	5.52	4.93
0.54	3.66	3.17
0.57	2.34	1.99
0.60	1.47	1.21
0.63	9.98×10^{-5}	7.15×10^{-5}
0.65	5.37	4.09
0.69	3.13	2.26
0.72	1.79	1.21

function since the center and width of such a Gaussian are variable. We did not consider variations of the initial distribution of Eq. (13), with the consequence that all initial velocity distributions obtained from $h(\xi,\tau)$ by Eq. (9) have a half-width in energy at half-maximum of approximately five-eighths the average energy.

IV. RELAXATION TO MAXWELLIAN DISTRIBUTION

A plot of $h(\xi,\tau)$ for a sequence of values of the dimensionless time parameter is given in Fig. 1. For comparison the Maxwellian distribution $h_M(\xi)$ corresponding to the same average energy and number density is also plotted. The function $h_M(\xi)$ is the final steady state which should be approached for sufficiently long times. In Table I the numerical values of $h_M(\xi)$ are given for each of the 24 space points along with the values of $h(\xi,\tau)$ for $\tau=484.17$. Although the latter distribution is rather close to Maxwellian, the low-energy portion of the spectrum is overpopulated and the Maxwell "tail" at high energies is not yet full. An upward diffusion in energy must still occur before the Maxwell distribution is achieved.

These results are more easily interpreted after τ is related to the time. First we shall give the distributions derived from $h(\xi,0)$ and $h_M(\xi)$:

$$f(v,0)=62.93(n/v_0^3)\times\exp\{-10[v/v_0-0.3)/0.3]^2\},$$
$$f_M(v)=110.51(n/v_0^3)\exp\{-13.285(v/v_0)^2\}. \quad (14)$$

From $f_M(v)$ we find that v_0 is related to the rms velocity by

$$v_0=2.976\langle v^2\rangle^{\frac{1}{2}}=2.976(3kT/m)^{\frac{1}{2}}. \quad (15)$$

The time is then given by

$$t=0.00212[m^{\frac{1}{2}}(3kT)^{\frac{3}{2}}/\pi ne^4\ln\Lambda]\tau. \quad (16)$$

We can also express t in terms of τ and the relaxation time t_c called by Spitzer[1] "the self-collision time":

$$t_c=m^{\frac{1}{2}}(3kT)^{\frac{3}{2}}/8\times0.714\pi ne^4\ln\Lambda. \quad (17)$$

We find $t=(0.0121\tau)t_c$. The relaxation time t_c is really the average time required for a particle having an energy equal to the average energy to suffer a 90° deflection of trajectory and a 100% change in energy. We define a corresponding relaxation time from our calculation as the time required for the distribution function to achieve the Maxwell value in the neighborhood of the maximum at $\xi=0.3$. This occurs at $\tau\approx60$ or $t=0.73t_c$. The relaxation time defined in this way agrees quite well with the "self-collision time." We see from the distribution obtained at $\tau=484.17$ or $t=5.9t_c$, however, that the concept of a relaxation time can be misleading. While the distribution in the neighborhood of the average energy is Maxwellian within a few percent, at an energy six times the average energy only 75% of the Maxwell amplitude has been achieved. Certainly more time is required for the higher energy parts of the Maxwell distribution to be filled. This aspect of the problem was not treated in detail by the machine calculations although Eq. (5) is approximated by a much simpler equation for large values of ξ.

V. DIFFUSION OF PARTICLES INTO THE MAXWELL TAIL

For values of ξ above the main portion of the distribution, an approximate equation can be obtained from Eq. (5) by neglecting all integrals from ξ to infinity and

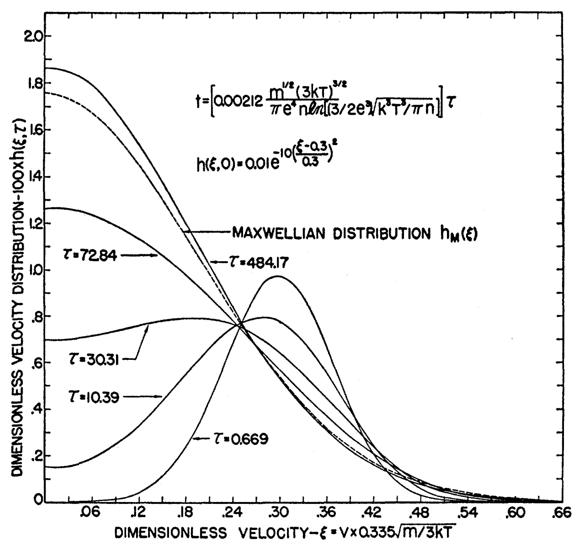

FIG. 1. Relaxation to Maxwellian distribution in a gas with inverse-square-law forces.

extending the finite integrals to infinity.

$$\partial h/\partial\tau=(2I_4/3\xi^2)(\partial/\partial\xi)(\xi^{-1}\partial h/\partial\xi+2\alpha h),$$
$$\alpha=3I_2/2I_4. \tag{18}$$

The only solution of the static equation which vanishes at infinity is $h_M(\xi)=A\exp(-\alpha\xi^2)$. [This is also the static solution of Eq. (5) if $A=4\alpha I_2(\alpha/\pi)^{\frac{1}{2}}$.] Introduce the function $g(\xi,\tau)$, defined from

$$f(\xi,\tau)=g(\xi,\tau)\exp(-\alpha\xi^2), \tag{19}$$

into Eq. (18):

$$\partial g/\partial\tau=(2I_4/3\xi^3)[\partial^2 g/\partial\xi^2-(2\alpha\xi+\xi^{-1})\partial g/\partial\xi]. \tag{20}$$

The function $g(\xi,\tau)$ becomes a constant as $h(\xi,\tau)$ approaches a Maxwell distribution. For a detailed analysis of the time-dependent behavior of $h(\xi,\tau)$ at high energies, one would use for an initial $g(\xi,\tau)$ the function reached after the lower energy portion of the distribution has reached nearly a Gaussian dependence. The approach of $g(\xi,\tau)$ to the final constant value at any energy then can be found from numerical integration of Eq. (20).

We should expect g to have the behavior of a diffusion wave; i.e., g is constant for small values of ξ and is zero for large values of ξ, with a transition region connecting the two asymptotic values of g. To examine the change in $g(\xi,\tau)$ with time, we can look at the motion of the points on the curve $g(\xi,\tau)$.

$$(\partial\xi/\partial t)_g=(2I_4/3\xi^3)\times\{2\alpha\xi+\xi^{-1}+(\partial^2\xi/\partial g^2)_\tau/(\partial\xi/\partial g)_\tau{}^2\}. \tag{21}$$

Now we let $\eta=\xi^3$ and find

$$(\partial\eta/\partial\tau)_g=2I_4\{2\alpha-\eta^{-\frac{2}{3}}+3\eta^{\frac{1}{3}}(\partial^2\eta/\partial g^2)_\tau/(\partial\eta/\partial g)_\tau{}^2\}. \tag{22}$$

The first two terms in Eq. (22) clearly represent an upward diffusion of particles to higher η. For large η we can neglect the second term. The last term in Eq. (22) will tend to increase the width W of the transition region in η. If $\bar{\eta}$ is the value of η which corresponds to the midpoint of the transition region where $(\partial^2\eta/\partial g^2)_\tau\approx 0$, the last term is clearly of order $\bar{\eta}^{\frac{1}{3}}/W$ and is positive on the left and negative on the right. We can write

$$\bar{\eta}=4I_4\alpha\tau, \tag{23}$$

and by comparing $\partial\eta/\partial\tau$ at the left and right of $\bar{\eta}$, we also have

$$\partial W/\partial\tau\sim 10I_4(\eta^{\frac{1}{3}}/W),$$

or

$$W\sim(2/\alpha^{\frac{1}{2}})\bar{\eta}^{\frac{2}{3}}. \tag{24}$$

Although Eq. (24) is only qualitative, it does indicate that the width of the transition region in w increases more slowly than $\bar{\eta}$, the upper edge of the Maxwell region in $g(\xi,\tau)$. Consequently $g(\xi,\tau)$ maintains its diffusion character. Equation (23) can be used to estimate the time required to fill the Maxwell tail to velocity v.

$$t=m^2v^3/12\pi e^4 n\ln\Lambda. \tag{25}$$

This time is independent of the average temperature of the gas and is approximately equal to the self-collision time for particles of velocity v given by Eq. (17).

In discussing the filling of the high-energy portion of the Maxwell distribution, we have not considered the effect of collisions which result in large-angle deflections of particle trajectories. The quantity $\ln\Lambda$ defined in Eq. (1) gives approximately

$$\ln\Lambda\sim\left(\frac{\text{frequency of small energy exchanges}}{\text{frequency of large energy exchanges}}\right).$$

For an electron gas $\ln\Lambda\sim 10-30$[1], and therefore according to Eq. (25) the large-angle collisions resulting in large-energy exchanges are also unimportant for filling the Maxwell distribution at high energies.

ACKNOWLEDGMENT

The authors wish to thank Dr. David L. Judd of the University of California Radiation Laboratory at Berkeley for his interest in this problem and for the computing facilities which he made available.