Reformulation of the Majorana Theory of the Neutrino

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The Majorana theory for a particle with mass is reformulated in terms of a two-component field. In this form it is seen that the theory goes over continuously to the Weyl-type two-component equation as the mass tends to zero. The asymmetries obtained in β and μ decay experiments are shown to imply only that the neutrino mass is small—not that it is zero. Also the asymmetries are shown to be no more implied by the use of a two-component theory than by the use of a four-component one. Two-component theories do imply certain relations between mass, parity nonconservation, and double β decay. That there are asymmetries is due to the fact that the *interactions* necessary to describe the physical world are not reflection-invariant.

I. INTRODUCTION

IN the following the Majorana theory is reformulated. There are two objects in mind. First we want to investigate the relation of this theory to the "two-component neutrino theory" suggested by Lee and Yang[1] and Landau and Salam.[2] Secondly it would seem worthwhile to have this theory developed completely on its own rather than as an afterthought when treating the Dirac equation. (In the development a number of interesting aspects of quantized field problems are seen.)

In the new form we can at any stage put the mass equal to zero. The result is then just the "two-component neutrino." However, it will be shown that this limit is not necessary to describe present experiments. For a sufficiently small neutrino mass the theory is essentially indistinguishable by experiment from the case of mass zero.

The reformulation also suggests a slightly different view on parity. This view is shown to be in agreement with the treatment of parity in discussing the electromagnetic field. It is also such that when "parity" is conserved there are no left-right asymmetries to be observed. From this viewpoint the mere use of a two-component theory by itself has no more implications as to whether β and μ decays[3,4] are asymmetric than does the use of a four-component theory. The essential point is that the experiments tell us only that the weak *interactions* responsible for these decay processes are *not* reflection-invariant. It is noted that from this point of view parity nonconservation in the decay of K particles into 2π and 3π modes is neither more nor less peculiar than for decays involving neutrinos.

In Sec. II the wave equations for the Majorana neutrino are deduced from the Dirac equation. After obtaining these equations we forget their origin. The theory of free particles described by these equations is developed in Secs. III–VI. Their interactions are considered in Sec. VII. Section VIII discusses the differences between finite and zero neutrino mass.

The Appendix is devoted to an alternative derivation of the Majorana equation.

II. THE MAJORANA ABBREVIATION

Let us write the Dirac equation[5] in the form

$$(\gamma^\mu\partial_\mu+\kappa)\psi=0, \tag{1}$$

where

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu},$$

and

$$g^{00}=-1,\quad g^{ij}=\delta^{ij}.$$

The charge-conjugation matrix C discussed by Pauli[6] has the properties[7]

$$C\gamma^\mu C^{-1}=\gamma^{\mu*},\quad C=\tilde{C},\quad C^*=C^{-1}. \tag{2}$$

Taking the complex conjugate of Eq. (1) and multiplying by C^{-1} gives

$$(\gamma^\mu\partial_\mu+\kappa)C^{-1}\psi^*=0. \tag{3}$$

Introduce the matrix γ^5 by

$$\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{4}$$

This has the properties

$$(\gamma^5)^2=1,\quad \gamma^\mu\gamma^5=-\gamma^5\gamma^\mu,\quad C^{-1}\gamma^{5*}C=-\gamma^5.$$

Let us introduce the projections with respect to the subspaces $\gamma^5=\pm1$ respectively by means of

$$\psi_\pm=\tfrac{1}{2}(1\pm\gamma^5)\psi,\quad \gamma_\pm{}^\mu=\tfrac{1}{2}(1\pm\gamma^5)\gamma^\mu. \tag{5}$$

With this notation Eq. (1) becomes the two equations

$$\gamma_+{}^\mu\partial_\mu\psi_-+\kappa\psi_+=0, \tag{6}$$

and

$$\gamma_-{}^\mu\partial_\mu\psi_++\kappa\psi_-=0. \tag{7}$$

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[1] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1671 (1957).

[2] L. Landau, Nuclear Phys. **3**, 127 (1957); A. Salam, Nuovo cimento **5**, 299 (1957).

[3] E. Ambler, Bull. Am. Phys. Soc. Ser. II, **2**, 65 (1957); Wu, Ambler, Hayward, Hoppes, and Hudson, Phys. Rev. **105**, 1413 (1957).

[4] Garwin, Lederman, and Weinrich, Phys. Rev. **105**, 1415 (1957); J. I. Friedman and V. L. Telegdi, Phys. Rev. **105**, 1681 (1957).

[5] Units such that $\hbar=c=1$ are used. $x^0=t$.

[6] W. Pauli, Revs. Modern Phys. **13**, 203 (1941).

[7] * and $\sim$ denote complex conjugate and transposed respectively.

Similarly Eq. (3) becomes

$$\gamma_+{}^\mu\partial_\mu C^{-1}\psi_+{}^* + \kappa C^{-1}\psi_-{}^* = 0, \tag{6a}$$

and

$$\gamma_-{}^\mu\partial_\mu C^{-1}\psi_-{}^* + \kappa C^{-1}\psi_+{}^* = 0. \tag{7a}$$

The Majorana abbreviation is obtained on putting[8]

$$\psi_- = C^{-1}\psi_+{}^*. \tag{8}$$

Equations (7) and (6) become

$$\gamma_-{}^\mu\partial_\mu\psi_+ + \kappa C^{-1}\psi_+{}^* = 0, \tag{9}$$

and

$$\gamma_+{}^\mu\partial_\mu C^{-1}\psi_+{}^* + \kappa\psi_+ = 0. \tag{10}$$

By using the properties (2) of C it is readily shown that Eq. (10) is just the complex conjugate of Eq. (9). Equations (6a) and (7a) are then identical with Eqs. (9) and (10). Hence the equations of the Majorana neutrino are

$$\eta^\mu\partial_\mu\phi + \kappa\phi^* = 0, \tag{11a}$$

and the complex conjugate

$$\eta^{\mu *}\partial_\mu\phi^* + \kappa\phi = 0. \tag{11b}$$

Here $\eta^\mu = C\gamma_-{}^\mu$ and $\phi = \psi_+$.

By a straightforward application of the relations of Eq. (2), we have

$$\begin{aligned}\eta^{\mu *}\eta^\nu + \eta^{u *}\eta^\mu &= \tfrac{1}{2}(1+\gamma^5)\{\gamma^\mu\gamma^u + \gamma^\nu\gamma^\mu\}\\ &= \tfrac{1}{2}(1+\gamma^5)\{2g^{\mu\nu}\}.\end{aligned}$$

However, in the two-dimensional space of ψ_+, we have

$$\tfrac{1}{2}(1+\gamma^5)\sim 1.$$

Therefore, the η^μ are two-by-two matrices satisfying[9]

$$\eta^{\mu *}\eta^\nu + \eta^{\nu *}\eta^\mu = 2g^{\mu\nu}. \tag{12}$$

Equations (11) with η^μ satisfying (12) are the analog for the Majorana neutrino of the Dirac equation in the form of Eq. (1). It is a convenient form in which to show that ϕ satisfies the wave equation. Thus on applying $\eta^{\nu *}\partial_\nu$ to Eq. (11a) we obtain

$$\eta^{\nu *}\eta^\mu\partial_\nu\partial_\mu\phi + \kappa\eta^{\nu *}\partial_\nu\phi^* = 0,$$

or

$$\tfrac{1}{2}\{\eta^{\nu *}\eta^\mu + \eta^{\mu *}\eta^\nu\}\partial_\mu\partial_\nu\phi - \kappa^2\phi = 0.$$

Upon using Eq. (12), this is

$$(\Box - \kappa^2)\phi = 0. \tag{13}$$

For many purposes it is more convenient to have the equations in a form analogous to the Dirac equation written as

$$i\frac{\partial\psi}{\partial t} = \frac{1}{i}\boldsymbol{\alpha}\cdot\nabla\psi + \kappa\beta\psi.$$

This is readily obtained on specializing the representation of the η^μ's. Thus let σ^i ($i=1, 2, 3$) by a set of Pauli matrices. There exists a matrix A such that[10]

$$A\sigma^{i *}A^{-1} = -\sigma^i, \quad A = -\tilde{A}, \quad A = A^{-1} = \tilde{A}^*. \tag{14}$$

Let

$$\eta^\mu = -iA\sigma^\mu, \tag{15}$$

where $\sigma^0 = 1$. It is readily verified that these matrices satisfy the relations of Eq. (12). Inserting these matrices in Eq. (11a) and multiplying by A gives

$$i\sigma^\mu\partial_\mu\phi = \kappa A\phi^*,$$

or

$$i\frac{\partial\phi}{\partial t} = \frac{1}{i}\boldsymbol{\sigma}\cdot\nabla\phi + \kappa A\phi^*. \tag{16a}$$

The complex conjugate equation is

$$i\frac{\partial}{\partial t}A\phi^* = -\frac{1}{i}\boldsymbol{\sigma}\cdot\nabla A\phi^* + \kappa\phi. \tag{16b}$$

We shall take Eqs. (16) as the fundamental description of the Majorana neutrino.[11] The method by which they were obtained[12] will be forgotten. It should be noted that when $\kappa=0$ we have identically the "two-component neutrino." From now on this terminology will be dropped, and we shall only speak of the theory with $\kappa=0$ or $\kappa\neq 0$.

III. TRANSFORMATION PROPERTIES

Since this point is somewhat controversial, we would like to state quite explicitly what we mean by invariance. A set of equations is invariant under a group of coordinate transformations if the following conditions are satisfied:

(1) We can find a law by which we can associate the wave functions describing a given state in the new coordinate system with the wave functions describing the same state in the old system.

(2) From the law of association and the equations in the old system we can find the equations for the new wave functions in terms of the new coordinates.

(3) The new equations obtained must have the same form as the old equations.

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[8] It is readily shown that this identification is compatible with the wave equations and the transformation properties. However, since we want to start afresh from the final equations obtained, we omit the proof.

[9] Equations of the form (11) with η^μ satisfying Eq. (12) seem first to have been studied by H. Jehle, Phys. Rev. **75**, 1609 (1949).

[10] These properties are readily verified by noting that, in the conventional representation of the Pauli matrices, $A=\sigma^2$.

[11] Since completing this work I have seen a preprint of an article by James A. McLennan, Jr. [Phys. Rev. **106**, 821 (1957)]. It is pointed out that Eqs. (16) describe the Majorana neutrino. It should be emphasized that essentially all the results in the present paper for the case $\kappa=0$ have been obtained independently and previously by McLennan.

[12] An alternative derivation of Eqs. (16) using a specific representation for the matrices of the Dirac equation is given in Appendix A.

Three possibilities exist:

(a) It is impossible to find a law of association with the required properties. Then we must conclude the equations are not invariant.
(b) The law of association is unique.
(c) There are several satisfactory laws of association.

The possibility (c) is important for the following reason: Suppose we are considering free fields and several laws of association are found. We cannot, *a priori*, decide which is the "correct" law. Let interactions be introduced. If we want to know if the equations are still invariant, we must try all of the possible laws of association found for the free fields. Thus only if *no choice* of association results in the same form of the equations with interaction in the new coordinate system can we say that the system is non-invariant under the given coordinate transformation.

Are Eqs. (16) invariant under the full homogeneous Lorentz group? To prove invariance we must exhibit the law of association.

(a) Proper Transformations

It is readily shown that an appropriate law of association is

$$\phi'(x')=\Lambda\phi(x), \tag{17}$$

where

$$\Lambda=\exp(\tfrac{1}{2}\theta\boldsymbol{\sigma}\cdot\mathbf{q}) \tag{18a}$$

for a rotation of angle θ around the spatial direction $\mathbf{q}$, and

$$\Lambda=\exp(\tfrac{1}{2}v\boldsymbol{\sigma}\cdot\mathbf{q}) \tag{18b}$$

for a Lorentz transformation with velocity v in the direction $\mathbf{q}$. From these two fundamental transformations all proper Lorentz transformations can be constructed.

(b) Spatial Reflections

The coordinate transformation is

$$x^{i'}=-x^i,\quad t'=t. \tag{19}$$

In analogy with Eq. (17) the first attempt is naturally to try

$$\phi'(x')=\Lambda\phi(x),$$

or

$$\phi(x)=\Lambda^{-1}\phi'(x'). \tag{20}$$

Inserting (20) in Eq. (16a) gives

$$i\frac{\partial}{\partial t}\phi'(x')=\frac{1}{i}\Lambda\boldsymbol{\sigma}\Lambda^{-1}\cdot\nabla\phi'(x')+\kappa\Lambda A\Lambda^{-1*}\phi'^*(x'),$$

or, upon using Eq. (19),

$$i\frac{\partial\phi'(x')}{\partial t'}=-\frac{1}{i}\Lambda\boldsymbol{\sigma}\cdot\nabla'\Lambda^{-1}\phi'(x')+\kappa\Lambda A\Lambda^{-1*}\phi'^*(x').$$

However, for this to have the same form as Eq. (16a) we must have

$$\Lambda\boldsymbol{\sigma}\Lambda^{-1}=-\boldsymbol{\sigma}.$$

There is no such Λ since the matrices $\boldsymbol{\sigma}$ and $-\boldsymbol{\sigma}$ are nonequivalent. This is readily seen by remembering that

$$[\sigma^i,\sigma^j]_-=2i\epsilon^{ijk}\sigma^k. \tag{21}$$

The matrices $-\boldsymbol{\sigma}$ then satisfy the equation obtained from this by changing the sign on the right-hand side. Does this mean that Eqs. (16) are not invariant under reflection? It means only that the attempt of Eq. (20) was rather unfortunate.

An appropriate law of association is, indeed, readily obtained. Let

$$\phi'(x')=\Lambda\phi^*(x),\quad\text{or}\quad\phi^*(x)=\Lambda^{-1}\phi'(x'). \tag{22}$$

Inserting in Eq. (16b) gives

$$i\frac{\partial\phi'(x')}{\partial t'}=\Lambda A\left(\frac{1}{i}\boldsymbol{\sigma}\cdot\nabla'\right)(\Lambda A)^{-1}\phi'(x')+\kappa\Lambda A\Lambda^{-1*}\phi'^*(x'). \tag{23}$$

In order that this be of the same form as Eq. (16a), we must have

$$\Lambda A\boldsymbol{\sigma}(\Lambda A)^{-1}=\boldsymbol{\sigma}, \tag{24a}$$

and

$$\Lambda A\Lambda^{-1*}=A. \tag{24b}$$

From Eq. (24a) we can conclude that ΛA is a multiple of the unit matrix, i.e.,

$$\Lambda A=\epsilon 1,$$

or

$$\Lambda=\epsilon A. \tag{25}$$

The *only effect of the mass term* is to give the condition of (24b) which restricts the constant ϵ. Thus, using the properties of Eq. (14) gives

$$\epsilon=-\epsilon^*, \tag{26a}$$

or

$$\epsilon=i\rho, \tag{26b}$$

where ρ is a real number.[13]

(c) Time Reflections

The coordinate transformation is

$$x^{i'}=x^i,\quad t'=-t.$$

By similar arguments we obtain

$$\phi'(x')=\Lambda\phi^*(x),$$

where

$$\Lambda=\mu A, \tag{27}$$

and the presence of the mass term requires μ to be real.[13]

The conclusion is then that Eqs. (16) are indeed invariant under the full Lorentz group. The only effect

[13] The involuntary character of the reflection transformations requires μ, ρ both to have absolute value unity.

of the mass term is to slightly decrease the possible choices of the law of association. Alternatively, any interaction which is invariant for $\kappa\neq 0$ is certainly invariant when $\kappa\to 0$, while an interaction which is not invariant for $\kappa=0$ is also not invariant for $\kappa\neq 0$.

Since the relation of Eq. (22) is somewhat different from the customary associations for space reflections, it may be well to show that this is really the same as the association one makes in electrodynamics. The Maxwell equations,

$$\nabla\times\mathbf{H}=\partial\mathbf{E}/\partial t,\quad \nabla\times\mathbf{E}=-\partial\mathbf{H}/\partial t,$$
$$\nabla\cdot\mathbf{H}=0,\qquad \nabla\cdot\mathbf{E}=0,$$

can be rewritten simply[14] in matrix notation. Let

$$\psi_i=H_i-iE_i,\quad i=1,\,2,\,3.$$

Then the two curl equations are

$$i\frac{\partial\psi}{\partial t}=\frac{\mathbf{S}\cdot\nabla}{i}\psi, \tag{28a}$$

where the matrices S^j are given by

$$(S^j)_{ik}=i\epsilon_{ijk} \tag{29}$$

(ϵ_{ijk} is the alternating symbol). It is readily seen that the S^j have all the properties of spin 1 matrices. The divergence equations are simply

$$\mathbf{S}\cdot\nabla\mathbf{S}\cdot\mathbf{q}\psi=\mathbf{q}\cdot\nabla\psi, \tag{30}$$

with $\mathbf{q}$ an arbitrary space vector. The full content of Maxwell's equations are then Eq. (28a), (30), and their complex conjugates. [For the present purposes we can ignore (30).] Therefore the basic equations are (28a) and

$$i\frac{\partial\psi^*}{\partial t}=\mathbf{S}^*\cdot\frac{\nabla}{i}\psi^*. \tag{28b}$$

Are these equations invariant under the reflection $x^{i\prime}=-x^i$, $t'=t$? One immediately sees (by the same argument as for the σ's) that $\mathbf{S}$ and $-\mathbf{S}$ are not equivalent. Hence an association of the form

$$\psi'(x')=\Lambda\psi(x),$$

will not be possible. However, the association

$$\psi'(x')=\Lambda\psi^*(x),$$

is feasible. Indeed, using the representation of Eq. (28) we see at once that $\Lambda=1$ is satisfactory. What is the transformation? Written out in components this is

$$H_i'-iE_i'=H_i+iE_i,$$

or

$$E_i'=-E_i,\quad H_i'=H_i.$$

These are, of course, the conventional transformations that one assumes for $\mathbf{E}$ and $\mathbf{H}$.

[14] This form has been used by J. R. Oppenheimer, Phys. Rev. **38**, 725 (1931).

A somewhat more convincing proof that the association of Eq. (22a) is satisfactory is found below. When we introduce interactions, it will be seen that if parity in this sense is preserved then there are no left-right asymmetries.

Having obtained the transformation properties of ϕ, we can now see what covariants can be formed from bilinear expressions in ϕ and ϕ^*. Under proper Lorentz transformations there are two scalars S_1 and S_2 where

$$S_1=(\phi^*,A\phi^*),\quad S_2=(\phi,A\phi), \tag{31}$$

a vector S^μ where

$$S^\mu=(\phi^*,\sigma^\mu\phi), \tag{32}$$

and a six-vector with components

$$(\phi,A\boldsymbol{\sigma}\phi)\quad\text{and}\quad(\phi^*,\boldsymbol{\sigma}A\phi^*). \tag{33}$$

Under the reflections the two scalars become interchanged as do the two quantities in Eq. (33).

The question of the electromagnetic properties of the particles described by Eq. (16) is at this point somewhat obscure. Suppose we regard the ϕ as c numbers. At first glance Eq. (16a) looks nongauge invariant because of the occurrence of the ϕ^*. However, Jehle[9] has shown that it is possible to construct suitable gauge transformations. Alternatively one can ask whether there is a four-vector which can serve as a charge-current vector. From the above this can only be S^μ. Using the equation of motion, we readily find that

$$i\partial S^\mu/\partial x^\mu=\kappa\{S_1-S_2\}. \tag{34}$$

Now, if ϕ is a c number, both S_1 and S_2 vanish identically since A is antisymmetric. Hence we have a vector which is conserved. However, since we know that we must quantize, a decision as to electromagnetic properties must wait till after the next section.

IV. QUANTIZATION

The difficulty just alluded to prevents us from applying the canonical formalism. Specifically, suppose we try to construct a Lagrangian. The terms involving κ must come from some bilinear scalar. There are only S_1 and S_2—and these vanish identically.

An invariant quantization procedure is fortunately straightforward. Since our ϕ transforms as a two-valued representation of the proper Lorentz group, we know[15] that anticommutators must be employed. Further, since we shall demand causality to hold, the anticommutators can involve only Pauli's[6] invariant D function. We shall express the relations in terms of ϕ and $A\phi^*$ since then the transformation properties are slightly more perspicuous. The quantization is

$$[\phi_\alpha(\mathbf{x},t),(A\phi^*)_\beta(\mathbf{x}',t')]_+=(\theta_1)_{\alpha\beta}D(\mathbf{x}-\mathbf{x}',\,t-t'), \tag{35a}$$

$$[\phi_\alpha(\mathbf{x},t),\phi_\beta(\mathbf{x}',t')]_+=(\theta_2)_{\alpha\beta}D, \tag{35b}$$

$$[(A\phi^*)_\alpha(\mathbf{x},t),(A\phi^*)_\beta(\mathbf{x}',t')]_+=(\theta_3)_{\alpha\beta}D. \tag{35c}$$

[15] See W. Pauli, Phys. Rev. **58**, 716 (1940).

The matrix operators θ_1, θ_2, θ_3 are still to be determined. Clearly, the relations for ϕ^* (35c) can be found from those for ϕ (35b) by taking the Hermitian conjugate. This gives the relation

$$\theta_3 = -A\theta_2{}^*A^{-1}. \tag{36}$$

Further, the equations of motion imply a relation between θ_1 and θ_2. This is

$$\kappa\theta_2 D = -\theta_1 A\left(i\frac{\partial}{\partial t}-\frac{1}{i}\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)A^{-1}D. \tag{37}$$

A set of operators which satisfy Eqs. (36) and (37) (and hence are compatible with the equations of the field) is found on remembering that D satisfies the wave equation for mass κ. The final result for the anticommutation relations is

$$[\phi_\alpha(\mathbf{x},t),(A\phi^*)_\beta(\mathbf{x}',t')]_+ = \left\{\left(-\frac{\partial}{\partial t}+\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)A\right\}_{\alpha\beta} D(\mathbf{x}-\mathbf{x}',\,t-t'), \tag{38a}$$

and

$$\begin{aligned}[\phi_\alpha(\mathbf{x},t),\phi_\beta(\mathbf{x}',t')]_+ &= -[(A\phi^*)_\alpha(\mathbf{x},t),(A\phi^*)_\beta(\mathbf{x}',t')]_+ \\ &= -i\kappa A_{\alpha\beta}D.\end{aligned} \tag{38b}$$

It is readily verified that these relations are compatible with the transformation properties previously found.

An interesting feature is that, as a consequence of the peculiar form of the Eqs. (16), the ϕ's at two different times *do not* anticommute precisely.

Specializing to equal times and using the results

$$D(\mathbf{x},0)=0,\ \left.\frac{\partial D}{\partial t}(\mathbf{x},t)\right|_{t=0}=\delta(\mathbf{x}), \tag{39}$$

we obtain the more familiar relations

$$[\phi_\alpha(\mathbf{x},t),\phi_\beta{}^*(\mathbf{x}',t)]_+ = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'), \tag{40a}$$

$$[\phi_\alpha(\mathbf{x},t),\phi_\beta(\mathbf{x}',t)]_+ = [\phi_\alpha{}^*(\mathbf{x},t),\phi_\beta{}^*(\mathbf{x}',t)]_+ = 0. \tag{40b}$$

V. INTERPRETATION

For interpretation purposes we clearly need quantities which can be considered as energy and momentum. These can be constructed in the canonical manner from a Lagrangian. A purely quantum-mechanical Lagrangian is indeed now possible. The point is that because of the anticommutivity of the ϕ's and ϕ^*'s the scalars S_1 and S_2 no longer vanish. For example, using the representation

$$A=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},$$

we have

$$S_1=\phi^*A\phi^*=i(\phi_2{}^*\phi_1{}^*-\phi_1{}^*\phi_2{}^*)\neq 0.$$

An appropriate Lagrangian is

$$\mathfrak{L}=\frac{1}{2i}\{\phi^*\sigma^\mu\partial_\mu\phi-\partial_\mu\phi^*\sigma^\mu\phi\}+\tfrac{1}{2}\kappa\{\phi^*A\phi^*+\phi A\phi\}. \tag{41}$$

Varying ϕ^*, we obtain

$$\delta\mathfrak{L}=\delta\phi^*\left\{\frac{1}{i}\sigma^\mu\partial_\mu\phi\right\}+\tfrac{1}{2}\kappa\delta\phi^*A\phi^*+\tfrac{1}{2}\kappa\phi^*A\delta\phi^*. \tag{42}$$

For consistency with our commutation relations, we must require the variation $\delta\phi^*$ to anticommute with ϕ^*. Hence

$$\phi^*A\delta\phi^*=-\delta\phi^*(\phi^*A)=-\delta\phi^*\tilde{A}\phi^*=\delta\phi^*A\phi^*.$$

Thus Eq. (42) is

$$\delta\mathfrak{L}=\delta\phi^*\left\{\frac{1}{i}\sigma^\mu\partial_\mu\phi+\kappa A\phi^*\right\}. \tag{43}$$

For $\delta\mathfrak{L}$ to vanish, we obtain

$$i\sigma^\mu\partial_\mu\phi=\kappa A\phi^*, \tag{44}$$

which is just the requisite equation. Similarly, varying ϕ yields the conjugate equation. We define a stress tensor by

$$T_\mu{}^\nu=\frac{\partial\mathfrak{L}}{\partial(\partial\phi/\partial x^\nu)}\frac{\partial\phi}{\partial x^\mu}+\frac{\partial\phi^*}{\partial x^\mu}\frac{\partial\mathfrak{L}}{\partial(\partial\phi^*/\partial x^\nu)}-\mathfrak{L}\delta_\mu{}^\nu. \tag{45}$$

Since $\mathfrak{L}$ vanishes in virtue of the equations of motion, we obtain

$$T_\mu{}^\nu=\frac{1}{2i}\{\phi^*\sigma^\nu\partial_\mu\phi+\partial_\mu\phi^*\sigma^\nu\phi\}. \tag{46}$$

In view of the considerable ambiguity of the ordering of the factors in Eq. (45), it is important to note that the use of $T_\mu{}^\nu$ given by Eq. (46) as a stress tensor can be justified in its own right. Thus it obviously has the transformation properties indicated by the tensor indices. In virtue of the equations of motion, the conservation laws hold, i.e.,

$$\partial T_\mu{}^\nu/\partial x^\nu=0. \tag{47}$$

For the energy-momentum four-vector, we take

$$P^\mu=\int T^{\mu 0}d^3\mathbf{x},$$

or

$$E=\frac{-1}{2i}\int\left\{\phi^*\frac{\partial\phi}{\partial t}-\frac{\partial\phi^*}{\partial t}\phi\right\}d^3x, \tag{48}$$

and

$$\mathbf{P}=\frac{1}{2i}\int\{\phi^*\boldsymbol{\nabla}\phi-(\boldsymbol{\nabla}\phi^*),\,\phi\}d^3x. \tag{49}$$

The displacement relations,

$$\partial\phi/\partial t = i[E,\phi]_-,$$

and

$$\nabla\phi = -i[\mathbf{P},\phi]_-$$

do indeed follow from these definitions and the commutation relations for the field operators.

The expression of Eq. (48) contains a certain amount of zero-point energy. This can be eliminated by the Heisenberg hole-theory symmetrization. However, we shall for compactness keep the expressions of Eqs. (48) and (49) in their simple form and merely drop the zero-point terms when they are encountered.

Now we can consider the electromagnetic properties of our field. From Eq. (34), we see that the divergence of the *only* four-vector (S^μ) that can be formed is

$$i\partial S^\mu/\partial x^\mu = \kappa\{S_1 - S_2\}.$$

Since now the scalars S_1 and S_2 do not vanish, this four-vector is conserved only when $\kappa=0$. Thus, only when the mass is zero can this theory describe a charge-bearing field. The existence of the six-vector previously mentioned suggests that the particles of this field might have a magnetic moment. However, the six-vector now vanishes identically. Thus consider

$$(\phi, A\boldsymbol{\sigma}\phi). \tag{50}$$

We note that A, $A\boldsymbol{\sigma}$ are four linearly independent two-by-two matrices. A is antisymmetric. Since there must be three symmetric and one antisymmetric matrix in a complete set, we conclude that $A\boldsymbol{\sigma}$ are all symmetric. (This may be verified by using a specific representation.) But

$$\phi_\alpha(\mathbf{x},t)\phi_\beta(\mathbf{x},t)+\phi_\beta(\mathbf{x},t)\phi_\alpha(\mathbf{x},t)=0.$$

Hence, the expression of (50) vanishes. A similar conclusion holds for $(\phi^*, \boldsymbol{\sigma}A\phi^*)$. Therefore, for finite κ there can be *no* static electromagnetic interactions.

VI. NORMAL-MODE DECOMPOSITION

This is clearly necessary for an interpretation in terms of particles. It can be obtained as follows: For each vector $\mathbf{k}$ we consider the operator

$$\phi(\mathbf{x},t;\mathbf{k}) = v(\mathbf{k})a(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-k_0t)} + v^*(\mathbf{k})b(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-k_0t)}. \tag{51}$$

Here $v(\mathbf{k})$ and its Hermitean conjugate $v^*(\mathbf{k})$ are operators; $a(\mathbf{k})$ and $b(\mathbf{k})$ are spinor c numbers. Since ϕ is to satisfy the Klein-Gordon equation, we require $k_0 = (k^2+\kappa^2)^{\frac{1}{2}}$. Inserting this in Eq. (16a), we see that (51) describes a solution provided

$$b(\mathbf{k}) = \frac{-\kappa}{k_0 - \boldsymbol{\sigma}\cdot\mathbf{k}} A a^*(\mathbf{k}), \tag{52a}$$

or alternatively

$$a(\mathbf{k}) = \frac{\kappa}{k_0 - \boldsymbol{\sigma}\cdot\mathbf{k}} A b^*(\mathbf{k}). \tag{52b}$$

Two linearly independent solutions are necessary in order to form a complete set. Let $\alpha(\mathbf{k})$, $\beta(\mathbf{k})$ be two spin functions such that

$$\boldsymbol{\sigma}\cdot\mathbf{k}\alpha(\mathbf{k}) = |\mathbf{k}|\alpha(\mathbf{k}), \quad \boldsymbol{\sigma}\cdot\mathbf{k}\beta(\mathbf{k}) = -|\mathbf{k}|\beta(\mathbf{k}). \tag{53b}$$

These can be chosen so that $(\alpha^*,\alpha) = (\beta^*,\beta) = 1$. They are orthogonal, so that $(\alpha^*,\beta) = (\beta^*,\alpha) = 0$. The relative phases can be fixed so that

$$A\alpha^*(\mathbf{k}) = i\beta(\mathbf{k}) \quad \text{and} \quad A\beta^*(\mathbf{k}) = -i\alpha(\mathbf{k}). \tag{54}$$

Two independent solutions are obtained with

$$a^{(1)}(\mathbf{k}) = \lambda(\mathbf{k})\alpha(\mathbf{k}), \quad b^{(2)}(\mathbf{k}) = \lambda(\mathbf{k})\alpha(\mathbf{k}). \tag{55}$$

Equations (52) give

$$b^{(1)}(\mathbf{k}) = \frac{-i\kappa\lambda(\mathbf{k})\beta(\mathbf{k})}{k_0+k}, \quad a^{(2)}(\mathbf{k}) = \frac{i\kappa\lambda(\mathbf{k})\beta(\mathbf{k})}{k_0+k}. \tag{56}$$

For convenience in the future development, we choose the normalization constant $\lambda(\mathbf{k})$ so that

$$\lambda(\mathbf{k}) = 1/[1+\kappa^2/(k_0+k)^2]^{\frac{1}{2}}. \tag{57}$$

[It may be noted that in the limit $\kappa\rightarrow 0$ we have

$$\begin{aligned} a^{(1)}(\mathbf{k}) &= \alpha(\mathbf{k}), \quad & b^{(1)}(\mathbf{k}) &= 0, \\ a^{(2)}(\mathbf{k}) &= 0, & b^{(2)}(\mathbf{k}) &= \alpha(\mathbf{k}). \end{aligned} \tag{58}$$

This is the same result as occurs when $|\mathbf{k}| \equiv k \gg \kappa$.]

Now expand the field operator ϕ in terms of these solutions; i.e.,

$$\phi(\mathbf{x},t) = \frac{1}{\sqrt{V}} \sum_{kr} \{v^r(\mathbf{k})a^r(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-k_0t)} + v^{r*}(\mathbf{k})b^r(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-k_0t)}\}. \tag{59}$$

(A box for quantization purposes has been assumed.) To obtain the commutation relations, we must solve for the v's in terms of ϕ and ϕ^*. For this purpose it is useful to relate $\alpha(\mathbf{k})$, $\beta(\mathbf{k})$ to $\beta(-\mathbf{k})$, $\alpha(-\mathbf{k})$. Obviously

$$\beta(-\mathbf{k}) = \eta(\mathbf{k})\alpha(\mathbf{k}),$$

and

$$\alpha(-\mathbf{k}) = \eta'(\mathbf{k})\beta(\mathbf{k}),$$

where η, η' are certain phase factors. For consistency with Eq. (54), we must have

$$\eta'(\mathbf{k}) = -\eta^*(\mathbf{k}). \tag{60}$$

The requirements

$$\alpha(-(-\mathbf{k})) = \alpha(\mathbf{k}), \quad \beta(-(-\mathbf{k})) = \beta(\mathbf{k}),$$

yield the condition

$$\eta(\mathbf{k}) = -\eta(-\mathbf{k}). \tag{61}$$

By using the properties outlined above, it is possible to solve for the v^r, v^{r*} in terms of ϕ and ϕ^*. The result is

$$v^{(1)}(\mathbf{k})e^{-ik_0t}=\frac{1}{\sqrt{V}}\int e^{-i\mathbf{k}\cdot\mathbf{x}}\left\{(a^{(1)*}(\mathbf{k}),\phi(\mathbf{x},t))+\frac{i\kappa\eta(\mathbf{k})}{k_0+k}(\phi^*(\mathbf{x},t),a^{(1)}(-\mathbf{k}))\right\}d^3\mathbf{x}, \quad (62)$$

$$v^{(2)}(\mathbf{k})e^{-ik_0t}=\frac{1}{\sqrt{V}}\int e^{-i\mathbf{k}\cdot\mathbf{x}}\left\{(\phi^*(\mathbf{x},t),b^{(2)}(\mathbf{k}))+\frac{i\kappa\eta^*(\mathbf{k})}{k_0+k}(b^{(2)*}(-\mathbf{k}),\phi(\mathbf{x},t))\right\}d^3\mathbf{x}. \quad (63)$$

Using Eqs. (62), (63), their Hermitean conjugates, and the commutation relations of Eqs. (40) gives

$$[v^{(r)}(\mathbf{k}),v^{r'*}(\mathbf{k}')]_+=\delta_{rr'}\delta(\mathbf{k},\mathbf{k}'), \quad (64)$$

and

$$[v^r(\mathbf{k}),v^{r'}(\mathbf{k}')]_+=[v^{r*}(\mathbf{k}),v^{r'*}(\mathbf{k}')]_+=0. \quad (65)$$

We are therefore justified in regarding $v^r(\mathbf{k})$, $v^{r*}(\mathbf{k})$ as absorption and emission operators for particles of type "*r*." Inserting the expansion of Eq. (59) into the expressions of Eqs. (48) and (49) gives (on omitting the zero-point energy)

$$E=\sum_k k_0\{v^{(1)*}(\mathbf{k})v^{(1)}(\mathbf{k})+v^{(2)*}(\mathbf{k})v^{(2)}(\mathbf{k})\}, \quad (66)$$

and

$$\mathbf{P}=\sum_k \mathbf{k}\{v^{(1)*}(\mathbf{k})v^{(1)}(\mathbf{k})+v^{(2)k}(\mathbf{k})v^{(2)}(\mathbf{k})\}. \quad (67)$$

VII. INTERACTIONS

(1) β Decay

These interactions can be described in exactly the same manner as has been done by Lee and Yang.[1] Thus, using the representation of the appendix:

$$\boldsymbol{\alpha}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix},\quad \beta=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},$$

we can write

$$\psi_\nu=\begin{pmatrix}\phi\\ 0\end{pmatrix}.$$

A possible interaction describing β decays is

$$H=\sum_i 2C_i(\psi_p^*,O_i\psi_n)(\psi_e^*,O_i\psi_\nu). \quad (68)$$

Corresponding to this term, we can also consider the parity conjugate. This is obtained by replacing all operators by those into which they transform under the parity operation. For ψ_p, ψ_n, ψ_e we must, of course, take the conventional association

$$\psi'_{p,\,\eta,\,e}(x')=\eta_{p,\,\eta,\,e}\beta\psi_{p,\,\eta,\,e}(x). \quad (69)$$

In agreement with what has been found in Sec. III, we take

$$\phi'=\epsilon A\phi^*,$$

and therefore

$$\psi_\nu'=\begin{pmatrix}\epsilon A\phi^*\\ 0\end{pmatrix}.$$

With these rules, we obtain

$$H'=\sum_i 2C_i(\psi_p^*,O_i\psi_n)(\psi_e^*,O_i\beta\psi_\nu')\eta_p^*\eta_n\eta_e^*. \quad (70)$$

We note that

$$\beta\psi_\nu'=\begin{pmatrix}0\\ i\epsilon A\phi^*\end{pmatrix}. \quad (71)$$

Now what relative amounts of H and H' should we use? *A priori there is absolutely no method of deciding.* We can, of course, try to require that parity (with the present meaning) be conserved. This means we must use H and H' with equal weights. Consider the combination $H+H'$. Under reflection, $H\to H'$ while

$$H'\to\sum_i 2C_i(\psi_p^*,O_i\psi_n)\left(\psi_e^*,O_i\begin{pmatrix}-\phi\\ 0\end{pmatrix}\right)(\eta_p^*\eta_n\eta_e^*)^2. \quad (72)$$

(Here we have used result $\epsilon\epsilon^*=1$.) Thus if the phases η are chosen so that $\eta_p^*\eta_n\eta_e^*=i$, we have a reflection-invariant interaction. With this choice and choosing $\epsilon=-i$, our invariant interaction becomes the same as in Eq. (68) with the replacement

$$\psi_\nu\to\Psi=\begin{pmatrix}\phi\\ iA\phi^*\end{pmatrix}. \quad (73)$$

In this case there are no asymmetries in β decay. This can readily be seen from the normal-mode decomposition given above. The point is that, for each process in which an electron and a neutrino are emitted in given directions with given spin components, there is a process with *exactly* the same probability in which an electron and a neutrino are emitted in just the opposite directions.

Alternatively, we can note that it follows from Eq. (A10) of the appendix that Ψ of Eq. (73) is a completely normal Dirac wave function. The interaction is then of the usual β-decay form and hence conserves parity.

It is clear then that the maximum degree of parity nonconservation (i.e. maximum asymmetries) occurs when we use only H or H'.

How are the interactions related to double β decay? Suppose we use only the interaction term H and for simplicity consider the case $\kappa=0$. The normal-mode decomposition becomes

$$\phi=\frac{1}{\sqrt{V}}\sum_k\{v^{(1)}(\mathbf{k})\alpha(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-k_0t)}+v^{(2)*}(\mathbf{k})\alpha(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-k_0t)}\}. \quad (74)$$

This, and therefore H, contains only the emission operators for neutrinos of type "2" and the absorption operators for neutrinos of type "1." Hence, a neutrino of type "2," if emitted in one β process, cannot be absorbed in a subsequent decay. The "double β decay" does not occur.[16] Conversely, it is readily seen that the maximum probability of double β decay results when we use the parity-conserving interaction term.

With a finite mass the situation is similar. The maximum double β decay occurs when there is reflection symmetry, the minimum when there is not. The only difference is that in the case of maximum degree of parity nonconservation the double β-decay probability does not vanish. It is, however, reduced from the maximum probability by a factor $(\kappa/\langle k_0+k\rangle_{\mathrm{Av}})^2$, where $\langle k\rangle_{\mathrm{Av}}$ is an average momentum of the intermediate-state neutrino. Clearly this can be made arbitrarily small by taking κ sufficiently close to zero.

In summary, we can say that the assumption of a two-component neutrino with or without mass has no implications as to whether parity is or is not conserved in β decay. The only experimental conclusion that follows from the theory is that a maximum degree of nonconservation of parity implies a minimum probability of double β decay while conservation of parity implies a maximum probability of double β decay. Loosely speaking, this says that "parity nonconservation" and "double β decay" are conjugate observables.

(2) μ Decay

Again we follow Lee and Yang[1] and consider only interactions with the vector S^μ. With our law of association and the commutation rules given previously, S^μ is actually a pseudovector; i.e., under the transformations $x^{i\prime}=-x^i$, $x^{0\prime}=x^0$ we have $S^{i\prime}=+S^i$, $S^{0\prime}=-S^0$. The only parity-conserving interaction is

$$-if_A\psi_e^*\gamma^0\gamma^\nu\gamma^5\psi_\mu S_\nu.$$

This gives rise to no decay asymmetries. However, there is nothing to stop one from writing the nonreflection-invariant interaction

$$f_V\psi_e^*\gamma^0\gamma^\nu\psi_\mu S_\nu+f_A\psi_e^*(-i\gamma^0\gamma^\nu\gamma^5)\psi_\mu S_\nu.$$

This certainly does exhibit asymmetries, as has been shown by Lee and Yang. It is important to note,[17] however, that these arise only from cross terms of the form $f_Vf_A^*$. The two-component theory has no prediction as to whether one of the f's is zero or not.

VIII. DIFFERENCE BETWEEN $\kappa=0$ AND $\kappa\neq0$

As shown above, these differences are quite small. The most significant point is that for $\kappa\neq0$ the charge must be zero. Since the neutrino has zero charge anyway, this distinction is rather trivial. It does, though, give a point of view from which it seems reasonable to expect the charge to be zero even if the mass is zero.

The other differences all seem to depend continuously on the mass in the sense that all observable effects can be made as small as desired by choosing the mass sufficiently small. Thus while the maximum nonconservation of parity in β decay is indeed obtained by using H or H' and $\kappa=0$, we can obtain nonconservation of parity arbitrarily close to this by using a sufficiently small but nonzero mass. Similarly, while "double β decay" does not occur if we use H or H' and $\kappa=0$, it has only an infinitesimal probability if κ is infinitesimal.

IX. THE $\tau-\theta$ PROBLEM

The view presented here is that there is absolutely no "explanation" of the observed β- and μ-decay asymmetries. (By an "explanation" we mean some compelling theoretical reason which would require us to expect the asymmetries.) These asymmetries have nothing to do with the fact that there are neutrinos present—either two- or four-component ones, massless or massive. The situation may well be compared to that in the conventional β-decay theory. There the "theory" admits five arbitrary coupling constants. Experiments tell us which ones really are present and what their magnitude are. *A priori* we have no idea as to whether the couplings are parity-conserving or not. Experiment tell us how much parity nonconservation is present.

Let us now consider the $\tau-\theta$ problem from this viewpoint. For some time the experiments have pointed quite strongly to the fact that there is only one K-particle which has several different decay modes. These decays markedly suggest that parity is not conserved by the weak interactions responsible for them. Naturally physicists were reluctant to give up a law as useful as parity conservation on the basis of one phenomenon. Then Lee and Yang[17] suggested that there were other weak interactions which could be investigated and which might not conserve parity. The β- and μ-decay experiments[4] do indeed show quite conclusively that parity is not conserved in some weak interactions. The general law would seem to be that weak *interactions* do not conserve parity. As has been seen, the nonconservation of parity is not an intrinsic kinematical property of the neutrino but a dynamical property of the coupling. The zero mass seems to have no fundamental relevance. A simple description of the $\tau-\theta$ situation is then the following: There is one K-particle which interacts with the π field by means of a term

$$\Phi_\kappa(a\phi_\pi^2+b\phi_\pi^3).$$

The question as to "why" parity is not conserved by this pion interaction would seem neither more nor less deep than the corresponding question for neutrino interactions.

Indeed, a possible problem of interest might be the

[16] This has also been noted by McLennan (reference 11).
[17] T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956).

inverse one. Why do we have the "accidental degeneracy" of parity conservation in strong interactions?

In the present context, if there is a problem[18] of weak interactions that requires a dynamical explanation, it is the question as to why the degree of parity nonconservation is apparently maximal. The two-component neutrino, while not implying this, is a peculiarly convenient tool for describing the observed facts. In this connection, Salam's[2] condition that the neutrino mass and self-mass be zero should be mentioned. It is readily seen that this is equivalent to the condition that the double β-decay process should not occur—and hence that the β-decay interaction should be of the maximal nonconservation of parity violation type, and $\kappa=0$. If κ were not zero, there is a finite double β-decay probability and a mass renormalization is necessary. However, this renormalization is "small" in the following sense: If $\delta\kappa$ were finite it would tend to zero with κ. Salam's condition has the appeal of simplicity. There does not, however, seem to be a compelling reason to adopt it. From the point of view of two-component theories, we can only say that there are various relations implied between different conditions that can be imposed.

X. CONCLUSION

It should be emphasized that the claim is not being made that the mass of the neutrino is different from zero. The statement is only that there is no necessity at present, either theoretical or experimental, for requiring the mass to be *exactly* zero. (This would, though, certainly be the most appealing possibility.)

Of course, the use of a two-component neutrino theory may be a particularly elegant means of describing the parity-nonconserving weak β-decay interactions. The point is only that there is no logical connection between the two: the left-right asymmetries neither imply nor are necessarily implied by the use of a two-component theory. The theory does imply certain relations between the neutrino mass, double β decay, and parity nonconservation. Thus, for example, if the maximum degree of nonconservation of parity for $\kappa=0$ be taken as 1, the maximum degree of nonconservation for finite κ is $1-\kappa^2/\langle(k_0+k)^2\rangle_{\mathrm{Av}}$. Similarly if the maximum double β-decay probability (which occurs when parity is conserved) be 1, the decay probability with maximum degree of nonconservation of parity is $\kappa^2/\langle(k_0+k)^2\rangle_{\mathrm{Av}}$.

It is a pleasure to thank the various members of the Institute for Advanced Study with whom I have had stimulating discussions on many of the points described above.

APPENDIX A

Since the derivation of the abbreviated Eqs. (16a) and (16b) in the text does not lead directly to the representation used, we give here an alternative derivation which does.

[18] I am indebted to Dr. A. Pais for pointing this out to me.

Write the Dirac equation in the form

$$i\frac{\partial\psi}{\partial t}=\boldsymbol{\alpha}\cdot\frac{1}{i}\nabla\psi+\kappa\beta\psi. \qquad \text{(A1)}$$

Using the charge conjugation matrix C, we can write the complex conjugate equation as

$$iC^{-1}\frac{\partial\psi^*}{\partial t}=\boldsymbol{\alpha}\cdot\frac{\nabla}{i}C^{-1}\psi^*+\kappa\beta C^{-1}\psi^*. \qquad \text{(A2)}$$

Specifically, we use the representation

$$\boldsymbol{\alpha}=\rho_1\boldsymbol{\sigma},\ \beta=\rho_3, \qquad \text{(A3)}$$

where

$$\rho_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \rho_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \rho_3=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}.$$

Then

$$C=\rho_3 A. \qquad \text{(A4)}$$

Now decompose into the subspaces $\rho_1=\pm1$. That is

$$\psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix},$$

where ψ_+, ψ_- are each two-component functions on which the matrices A, $\boldsymbol{\sigma}$ act. The two equations (A1) and (A2) become the four equations

$$i\frac{\partial\psi_+}{\partial t}=\frac{1}{i}\boldsymbol{\sigma}\cdot\nabla\psi_+-i\kappa\psi_-, \qquad \text{(A5)}$$

$$i\frac{\partial\psi_-}{\partial t}=-\frac{1}{i}\boldsymbol{\sigma}\cdot\nabla\psi_-+i\kappa\psi_+, \qquad \text{(A6)}$$

$$i\frac{\partial}{\partial t}A\psi_-^*=\frac{1}{i}\boldsymbol{\sigma}\cdot\nabla A\psi_-^*+i\kappa A\psi_+^*, \qquad \text{(A7)}$$

$$i\frac{\partial}{\partial t}A\psi_+^*=-\frac{1}{i}\boldsymbol{\sigma}\cdot\nabla A\psi_+^*-i\kappa A\psi_-^*. \qquad \text{(A8)}$$

The Majorana abbreviation consists in the identification

$$\psi_-=iA\psi_+^*. \qquad \text{(A9)}$$

Equation (A5) becomes

$$i\frac{\partial\psi_+}{\partial t}=\frac{1}{i}\boldsymbol{\sigma}\cdot\nabla\psi_++\kappa A\psi_+^*, \qquad \text{(A10)}$$

while Eq. (A6) becomes the complex conjugate of this. Equations (A7) and (A8) then merely repeat these two equations.

APPENDIX B

Charge Conjugation

Let us ask whether this Majorana theory of the neutrino is invariant under charge conjugation. By this we mean the following: Can we find a law of association giving operators ϕ^C in terms of ϕ, ϕ^* so that as a consequence of the original equations of motion we have equations of the same form for the charge-conjugate quantities except that e occurs with the opposite sign?

First consider purely electromagnetic interactions. In the case $\kappa=0$ we have seen it is possible to have a charge. The equations of motion are

$$i\sigma^\mu(\partial_\mu - ieA_\mu)\phi = 0, \tag{B1}$$

and

$$i\sigma^{\mu*}(\partial_\mu + ieA^\mu)\phi^* = 0. \tag{B2}$$

The question here is then whether we can find a law of association giving a ϕ^C in terms of ϕ and ϕ^* such that as a consequence of (B1) and (B2) we have

$$i\sigma^\mu(\partial_\mu + ieA_\mu)\phi^C = 0, \tag{B3}$$

$$i\sigma^{\mu*}(\partial_\mu - ieA_\mu)\phi^{C*} = 0. \tag{B4}$$

Now by trying laws of association of the form $\phi^C=\Lambda\phi$ and $\phi^C=\Lambda\phi^*$, we can readily convince ourselves that no law of association is possible. Hence, *if the charge of the neutrino were not zero and if it were to be described by a two-component theory*, we could conclude at this point that the theory is not invariant under charge conjugation. However, the charge *is* zero. In this case a trivial law of association is indeed possible. This is

$$\phi^C = \delta_\nu \phi, \tag{B5}$$

where δ_ν is an arbitrary phase factor.

The argument is similar for $\kappa\neq 0$. Then the charge must be zero. A law of association of the form (B5) is indeed satisfactory. The only modification is that the occurrence of the ϕ^* term then requires δ_ν to be real, i.e., $\delta_\nu=\pm 1$. Since an appropriate law of association has been found, we conclude that the free Majorana field is indeed invariant under charge conjugation.

Are the weak interactions such that charge-conjugation invariance is maintained? Consider β decay. Suppose we have an interaction

$$H=\sum_i 2C_i(\psi_p^*,O_i\psi_n)(\psi_e^*,O_i\psi_\nu). \tag{B6}$$

Corresponding to this term, we can consider a term H_C obtained from H by replacing all operators by their charge conjugates. We use the usual relations

$$\psi_{p,n,e}{}^C = \delta_{p,n,e}C^{-1}\psi_{p,n,e}{}^*. \tag{B7}$$

Then

$$\begin{aligned} H_c &= \sum_i 2C_i(\psi_p{}^{c*},O_i\psi_n{}^c)(\psi_e{}^{c*},O_i\psi_\nu{}^c) \\ &= \sum_i 2C_i(\psi_p,O_i^*\psi_n^*)(\psi_e,O_i^*C\psi_\nu)\delta_p^*\delta_n\delta_e^*\delta_\nu. \end{aligned} \tag{B8}$$

Under charge conjugation, we have

$$H\rightarrow H_C, \quad H_C\rightarrow H. \tag{B9}$$

Hence, if the interaction is $H+H_C$ we have charge-conjugation invariance *with the present definition.* It should be noted that this invariance corresponds to the intuitive picture. H_C just has the particle and antiparticle operators of H *for the charged particles* interchanged. Hence, in the antiworld all processes would look just like those in our world.

Now *a priori*, we can only suppose that the interaction term is

$$\alpha H + \alpha_C H_C. \tag{B10}$$

Experiment must tell us the relative weights of α and α_C. If they turn out to be equal, we have charge-conjugation invariance. If they are not equal, this law is violated.

APPENDIX C

The transformation properties discussed in Sec. III pertain to the c-number theory. In the q-number theory we must use exactly the same transformations for operators for proper transformations and space reflections. For time reflection we must adjoin the operation of taking the complex conjugate of all c numbers (Wigner time reversal). Thus, while space reflection is related to an antiunitary transformation in the c-number theory, it is related to a unitary transformation in the q-number theory. Time reflection is, on the contrary, antiunitary in both theories.[19]

[19] *Note added in proof.*—I am indebted to Professors M. Fierz and W. Pauli for calling my attention to an article by J. Serpe, Physica **18**, 295 (1952). In this article the relation between the Weyl equation and the Majorana equation for mass zero is clearly and correctly shown.