Dispersion Relation for Nonrelativistic Particles*

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It is shown that if the wave function of a nonrelativistic particle satisfies the Schrödinger equation with a velocity-independent potential, then its scattering amplitudes (and the S matrix in general) satisfy the same dispersion formulas as those derived for the scattering of light. In the present derivation, the validity of a perturbation expansion and certain integrability of the potential are assumed, and the requirement of the outgoing wave Green's function replaces the condition of strict causality for the scattering of light. The scattering amplitude for fixed momentum transfer is also shown to satisfy the dispersion relations. Together with the unitarity of the S matrix, the complete S matrix is determined by the Fourier transform of the potential through an iterative procedure using the dispersion relations. If the potential possesses bound states, then all the dispersion formulas are modified to include residue terms corresponding to singularities on the positive imaginary axis of the momentum plane. The necessity of these modifications is related to the divergence of the perturbation series for small wave numbers. Essential singularities of the S matrix due to exponentially damped potentials give no additional contribution to the dispersion formulas.

I. INTRODUCTION

RECENTLY, many authors have derived dispersion relations for the S-matrix of light[1] and of nonzero-mass relativistic particles,[2] based on the causality principle that "no signal can travel faster than the velocity of light *in vacuo*." When one considers the scattering of nonrelativistic particles, such a principle must be modified for two reasons[3]: (1) Maximum velocity does not exist. (2) There are no ingoing or outgoing wave packets that are rigorously zero up to a certain time. In spite of this difference between relativistic and nonrelativistic particles, the corresponding scattering matrices may be shown to satisfy essentially the same dispersion relations. In fact, Van Kampen[3] has modified the causality principle for nonrelativistic particles to the following: "If the ingoing wave packet is so normalized as to represent at $t=-\infty$ one incident particle, the total probability of $t=t_1$ of finding a particle outside of any sphere of radius $r_1\geq a$ cannot be greater than 1." Under this condition, he has shown that $S_l(k)\exp(2ika)$ satisfies the ordinary dispersion relation except for the addition of residue terms corresponding to simple poles on the positive imaginary axis of k, due to the presence of bound states. Here, $S_l(k)$ is the S-matrix corresponding to angular momentum quantum number l and magnitude of momentum $p=\hbar k$. It is the purpose of this note[4] to show that all other dispersion formulas for the scattering of light for various combinations of the S-matrix elements[1] are satisfied by nonrelativistic particles, except again, for residue terms due to bound states. In the present derivation, the causality principle is simply the requirement of the out-going wave Green's function. In addition, we assume the existence of a velocity-independent real potential with certain integrability and the validity of a perturbation expansion. The essential part of the results given in this note has also been obtained recently by Klein and Zemach[5] without recourse to a perturbation expansion. In Sec. II, forward scattering amplitudes for incident plane waves and off-center spherical waves are shown to satisfy the usual dispersion relations. In addition, the integrability of the potential implies that a somewhat stronger dispersion relation is also satisfied. An inversion of the relation between the S matrix and the scattering amplitudes gives the dispersion formula for the S matrix. In Sec. III, the scattering amplitude corresponding to a fixed momentum transfer is also shown to satisfy the dispersion relation. Both Secs. II and III assume uniform convergence of the perturbation series for all real k. Section IV considers the unitarity condition of the S matrix and an extension of the cross-section theorem is given. It is then shown that the complete S matrix is determined by the Fourier transform of the potential through the dispersion relation. An illustrative example is given using the dispersion relations to determine the finite-angle scattering amplitude for the Yukawa well up through the second Born approximation. In Sec. V, bound-state contributions are examined and shown to be related to the divergence of the perturbation series for small k. Essential singularities, for each $S_l(k)$, due to potentials of exponential type, are shown to give no contribution in the scattering amplitudes.

II. DISPERSION RELATIONS FOR THE S-MATRIX

The Schrödinger equation in proper units reads:

$$\nabla^2\psi+k^2\psi-\lambda U(\mathbf{r})\psi=0. \tag{1}$$

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[1] D. Y. Wong and J. S. Toll, Ann. Phys. 1, 91 (1957); hereafter this paper will be referred to as WT.

[2] R. H. Capps and G. Takeda, Phys. Rev. 103, 1877 (1956), where many further references are also given.

[3] These are first pointed out by N. G. van Kampen, Phys. Rev. 91, 1267 (1953).

[4] This is a revision of the Physics Department Technical Report No. 62, University of Maryland, January, 1957 (unpublished).

[5] A. Klein and A. C. Zemach (private communication). The author wishes to thank Professor Klein for a preprint of their work.

Solutions of (1) also satisfy the integral equation

$$\psi(\mathbf{r})=\psi_i(\mathbf{r})-\frac{\lambda}{4\pi}\int\frac{e^{ik|\mathbf{r}-\mathbf{r}_1|}}{|\mathbf{r}-\mathbf{r}_1|}U(\mathbf{r}_1)\psi(\mathbf{r}_1)dV_1, \quad (2)$$

where $\psi_i(\mathbf{r})$ is the incident wave and $(1/|\mathbf{r}-\mathbf{r}_1|)\times\exp(ik|\mathbf{r}-\mathbf{r}_1|)$ is the outgoing wave Green's function.

The scattering amplitude is obtained from the asymptotic expression of (2):

$$A(\theta,k)=-\frac{\lambda}{4\pi}\int e^{-i\mathbf{k}_s\cdot\mathbf{r}_1}U(\mathbf{r}_1)\psi(\mathbf{r}_1)dV_1, \quad (3)$$

where $\mathbf{k}_s$ is the momentum vector along the direction of $\mathbf{r}$ making an angle θ with respect to the polar axis. The usual Born series gives

$$A(\theta,k)=\sum_{n=1}^{\infty}\lambda^n A^{(n)}(\theta,k), \quad (4)$$

with

$$A^{(n)}(\theta,k)=\left(\frac{-1}{4\pi}\right)^n\int\left[\prod_{m=1}^{n-1}\frac{U(\mathbf{r}_m)U(\mathbf{r}_n)}{|\mathbf{r}_m-\mathbf{r}_{m+1}|}\right]\psi_i(\mathbf{r}_n)$$
$$\times\exp\left[-i\mathbf{k}_s\cdot\mathbf{r}_1+\sum_{m=1}^{n-1}ik|\mathbf{r}_m-\mathbf{r}_{m+1}|d\right]dV_1\cdots dV_n. \quad (5)$$

We shall assume that (5) exists for every n and (4) converges uniformly with respect to k. The possible divergence of (4) will be related to the existence of bound states which will be considered in Sec. V. We shall now choose some particular incident waves and investigate the analytic properties of $A^{(n)}(\theta,k)$.

First, let

$$\psi_i(\mathbf{r})=\exp(i\mathbf{k}_i\cdot\mathbf{r}), \quad (6)$$

where $|\mathbf{k}_i|=|\mathbf{k}_s|=|k|$ and $\mathbf{k}_i$ is taken along the polar axis. The k-dependent part in (5) becomes

$$\exp\left[-i\mathbf{k}_s\cdot\mathbf{r}_1+i\mathbf{k}_i\cdot\mathbf{r}_n+\sum_{m=1}^{n-1}ik|\mathbf{r}_m-\mathbf{r}_{m+1}|\right]$$
$$=\exp\left[i(\mathbf{k}_i-\mathbf{k}_s)\cdot\mathbf{r}_1-ik|\mathbf{r}_1-\mathbf{r}_n|\cos(\mathbf{k}_i,(\mathbf{r}_1-\mathbf{r}_n))\right.$$
$$\left.+\sum_{m=1}^{n-1}ik|\mathbf{r}_m-\mathbf{r}_{m+1}|\right]. \quad (7)$$

For forward scattering, $(\mathbf{k}_i-\mathbf{k}_s)=0$. From the triangular inequality $\sum_{m=1}^{n-1}|\mathbf{r}_m-\mathbf{r}_{m+1}|\geq|\mathbf{r}_1-\mathbf{r}_n|$, the exponential function (7) is clearly the boundary values of an uniformly bounded function of $\zeta=k+iK$ on the upper half-plane of ζ. It has been shown by Toll[6] that such analytic behavior is sufficient for (7) to satisfy the dispersion relation which involves an integration over k. We now assume that the potential diminishes sufficiently rapidly that interchange of the $\mathbf{r}$ and k integrations is permitted. This then implies that the forward-scattering amplitude satisfies the dispersion relation for each order in λ. The existence of (5) and the validity of interchange of integrations will be loosely referred to as the integrability of the potential. Upon using (7) again, the assumption of the uniform convergence of (4) for real k also implies uniform convergence on the whole upper half-plane. Hence $A(0,k)$ satisfies the dispersion relation

$$\mathrm{Re}A(0,k)=\frac{2k^2}{\pi}P\int_0^{\infty}\frac{\mathrm{Im}A(0,\nu)}{\nu(\nu^2-k^2)}d\nu+\mathrm{Re}A(0,0) \quad (8)$$

where P means the principal part of the integral.

The boundedness of $A^{(n)}(0,k)$ implied by (7) can be made more stringent for our present problem. In fact, for $n>1$, the space integration will introduce a factor $(1/k)$ in the limit of large k, and hence $[A(0,k)-\lambda A^{(1)}(0,k)]$ goes at least as $(1/k)$ for large k. The dispersion relation that $[A(0,k)-A^{(1)}(0,k)]$ satisfies is simply the Hilbert transform formula,[5] which can be rewritten as:

$$\mathrm{Re}A(0,k)=\frac{2}{\pi}P\int_0^{\infty}\frac{\nu\,\mathrm{Im}A(0,\nu)}{\nu^2-k^2}d\nu+\lambda A^{(1)}(0,0), \quad (9)$$

after we have used the symmetry property $A^*(k)=A(-k)$ and the relation

$$A^{(1)}(0,k)=-\frac{1}{4\pi}\int U(\mathbf{r}_1)dV_1$$
$$=A^{(1)}(0,0)=\mathrm{Re}A^{(1)}(0,0). \quad (10)$$

Equation (9) is stronger than (8) in the sense that there is only one constant in (9) as compared with the infinite number of constants in (8); one for each order in λ. In what follows, we shall use the dispersion formula (9) with the understanding that (8) is also satisfied. Incidentally, if the extent of the scatterer is finite $(|\mathbf{r}|\leq a)$, then it is clear from (7) that

$$\exp[2ika\sin(\theta/2)]A(\theta,k)$$

also satisfies the dispersion relations (8) and (9).

Next we consider the incident wave to be a spherical wave originated at a distance r_0 from the center of the scatterer, along the negative polar-axis:

$$\psi_i'=\frac{r_0}{r'}e^{ik(r'-r_0)}=\frac{r_0}{|\mathbf{r}-\mathbf{r}_0|}e^{ik|\mathbf{r}-\mathbf{r}_0|-ikr_0}, \quad (11)$$

where r' is the radial distance from the point source. An argument similar to that for the plane wave case shows that the forward-scattering amplitude for the spherical wave (11) also satisfies the dispersion relations (8) and (9). In what follows, we shall assume a spheri-

[6] J. S. Toll, Phys. Rev. 104, 1760 (1956).

cally symmetric potential. In terms of the S matrix, this forward-scattering amplitude is given by[1]

$$A'(0,k)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)[S_l(k)-1]q_l(kr_0), \quad (12)$$

where

$$S_l(k)=\exp[2i\eta_l(k)] \quad (13)$$

is the S-matrix element with η_l being the lth partial wave phase shift. $q_l(kr_0)$ is an lth degree polynomial in $(1/kr_0)$, given by

$$q_l(kr_0)=\sum_{n=0}^{l}\frac{i^n(l+n)!}{n!(l-n)!}\frac{1}{(2kr_0)^n}. \quad (14)$$

As in WT, an infinite set of functions of k can be constructed from (12), each satisfying the dispersion relation. These functions are

$$A_n(k)=\frac{1}{2ik^{2n+1}}\sum_{l=n}^{\infty}(2l+1)\frac{(l+n)!}{(l-n)!}[S_l(k)-1]. \quad (15)$$

Upon inverting the relation between $A_n(k)$ and $S_l(k)$, the dispersion formula for the S matrix is obtained:

$$\mathrm{Im}S_l(k)=\sum_{n=l}^{\infty}b_{ln}k^{2n+1}\times\left[\frac{2}{\pi}P\int_0^{\infty}\frac{\sum_{m=n}a_{nm}\nu^{-2n}\,\mathrm{Re}[1-S_m(\nu)]}{\nu^2-k^2}d\nu+\lambda A_n^{(1)}(0)\right], \quad (16)$$

where

$$a_{nm}=\tfrac{1}{2}(2m+1)\frac{(m+n)!}{(m-n)!},$$

$$b_{ln}=\frac{2(-1)^{n-l}}{(n-l)!(n+l+1)!}, \quad (17)$$

$$A_n^{(1)}(0)=\tfrac{1}{2}(2n+1)![\mathrm{Im}S_n^{(1)}(k)/k^{2n+1}]_{k=0},$$

and we have used (9) for each $A_n(k)$. If (8) had been used instead, we would have obtained dispersion formulas identical with those for the scattering of light.[1] The implication of (16) is essentially the same as Eq. (30) of WT except that the low-energy limit here is only a singly infinite set of parameters instead of the doubly infinite set in WT.

III. DISPERSION RELATION FOR SCATTERING AMPLITUDES WITH FIXED MOMENTUM TRANSFER

Returning now to (7) again, if we consider $\mathbf{\Delta}\equiv\mathbf{k}_s-\mathbf{k}_i$ as a fixed vector, then as a function of $\mathbf{\Delta}$, k, and $\{\mathbf{r}_m\}$, (7) becomes

$$\exp\left[-i\mathbf{\Delta}\cdot\mathbf{r}_1+\frac{i}{2}\mathbf{\Delta}\cdot\mathbf{R}_n-ikR_n(1-\Delta^2/4k^2)^{\frac{1}{2}}\times\sin(\mathbf{\Delta},\mathbf{R}_n)\cos\varphi+\sum_{m=1}^{n-1}ik|\mathbf{r}_m-\mathbf{r}_{m+1}|\right], \quad (18)$$

where $\mathbf{R}_n\equiv(\mathbf{r}_n-\mathbf{r}_1)$, $(\mathbf{\Delta},\mathbf{R}_n)$ is the angle between the vectors $\mathbf{\Delta}$ and $\mathbf{R}_n$, φ is the angle between the plane containing $\mathbf{\Delta}$ and $\mathbf{R}_n$ and the plane containing $\mathbf{\Delta}$ and $\mathbf{k}_i$, and

$$\begin{aligned}\mathbf{k}_i\cdot\mathbf{R}_n&=kR_n[\cos(\mathbf{\Delta},\mathbf{R}_n)\cos(\mathbf{k}_i,\mathbf{\Delta})+\sin(\mathbf{\Delta},\mathbf{R}_n)\sin(\mathbf{k}_i,\mathbf{\Delta})\cos\varphi]\\&=-\tfrac{1}{2}\mathbf{\Delta}\cdot\mathbf{R}_n+kR_n(1-\Delta^2/4k^2)^{\frac{1}{2}}\times\sin(\mathbf{\Delta},\mathbf{R}_n)\cos\varphi. \end{aligned}\quad (19)$$

It is easily seen by the triangular inequality that (18) is uniformly bounded on the upper half-plane of $\zeta=k+iK$. Again, if the perturbation series converges uniformly, $A(\Delta,k)$ satisfies the dispersion relation

$$\mathrm{Re}A(\Delta,k)=\frac{2}{\pi}P\int_0^{\infty}\frac{\nu\,\mathrm{Im}A(\Delta,\nu)}{\nu^2-k^2}d\nu+\lambda A^{(1)}(\Delta,0). \quad (20)$$

Here, $A(\Delta,k)$ is defined, for all k, through the expression (18). For $k\geq\Delta/2$, $A(\Delta,k)$ has the physical meaning of the scattering amplitude for fixed momentum transfer Δ. In the unphysical region $k<\Delta/2$, $A(\Delta,k)$ is just an analytic continuation through (18).‡

IV. UNITARITY CONDITION

For real potentials,[7] the phase shifts are real and we have the simple relation

$$\begin{aligned}&\mathrm{Im}[S_l(k)-1]=\sin 2\eta(k).\\&\mathrm{Re}[S_l(k)-1]=-2\sin^2\eta_l(k)=-\tfrac{1}{2}|S_l(k)-1|^2.\end{aligned} \quad (21)$$

Since $[S_l(k)-1]$ to any order in λ can determine $\mathrm{Re}[S_l(k)-1]$ to the next higher order by (21), one can combine (16) and (21) to form an iterative procedure with $A_n^{(1)}(0)$ as the first approximation. Hence this singly infinite parameter $A_n^{(1)}(0)$ may take the role of the potential when all the assumptions leading to (16) holds. The example of a square-well potential has been given in WT where it is referred to as a dielectric sphere of refractive index $n=[1+(U/k^2)]^{\frac{1}{2}}$. Similarly, we shall show that an iterative procedure can also be set up for $A(\Delta,k)$ by using an extension of the well-known cross-section theorem:

$$\mathrm{Im}A(0,k)=(k/4\pi)\sigma_{\text{total}}. \quad (22)$$

‡ *Note added in proof.*—It has been pointed out by N. N. Khuri (private communication) that, in the unphysical region, (18) is applicable only for $\Delta\leq\alpha/2$ if the potential falls off like $\exp(-\alpha r)$.

[7] If the potential is taken to be complex, then both the symmetry property and the dispersion relations have to be modified. This extension should be straight forward and may be of interest for some problems.

Consider the integral

$$\frac{k}{4\pi}\int A^*(\mathbf{k}_i|\mathbf{k})A(\mathbf{k}|\mathbf{k}_s)d\Omega_{\mathbf{k}}, \tag{23}$$

where

$$|\mathbf{k}_i|=|\mathbf{k}_s|=|\mathbf{k}|, \quad \mathbf{k}_i\cdot\mathbf{k}_s=k^2\cos\theta,$$

and the notation for the argument of the scattering amplitude $A(\theta,k)$ is replaced by $A(\mathbf{k}_i,|\mathbf{k}_s)$, etc.

Upon expanding $A^*(\mathbf{k}_i|\mathbf{k})$ and $A(\mathbf{k}|\mathbf{k}_s)$ in partial waves and using the integral relation[8] for the Legendre polynomials,

$$\int P_l(\mathbf{k}_i|\mathbf{k})P_{l'}(\mathbf{k}|\mathbf{k}_s)d\Omega_{\mathbf{k}}=\frac{4\pi}{2l+1}P_l(\mathbf{k}_i|\mathbf{k}_s)\delta_{ll'}, \tag{24}$$

it is easily verified that[9]

$$\mathrm{Im}A(\theta,k)=\frac{k}{4\pi}\int A^*(\mathbf{k}_i|\mathbf{k})A(\mathbf{k}|\mathbf{k}_s)d\Omega_{\mathbf{k}}. \tag{25}$$

We shall call (25) the extended cross-section theorem.

From (25), one can determine $\mathrm{Im}(\theta,k)$ in terms of lower orders of $A(\theta,k)$. Changing the variable to (Δ,k) and substituting $\mathrm{Im}A(\Delta,k)$ into (20), we see that an iterative procedure is closed. The starting term is $\lambda A^{(1)}(\Delta,0)$ which is nothing more than the Fourier transform of the potential:

$$\lambda A^{(1)}(\Delta,0)=\frac{-\lambda}{4\pi}\int \exp(-i\mathbf{\Delta}\cdot\mathbf{r})U(\mathbf{r}_1)dV_1. \tag{26}$$

We shall consider the Yukawa well as an illustrative example:

$$\lambda U=\lambda e^{-\mu r}/(\mu r). \tag{27}$$

The Fourier transform is

$$\lambda A^{(1)}(\Delta,0)=-\lambda/[\mu(\mu^2+\Delta^2)]. \tag{28}$$

By (25), we have

$$\begin{aligned}\mathrm{Im}A^{(2)}(\theta,k) &=\frac{k}{4\pi}\int\frac{1}{\mu^2(\mu^2+|\mathbf{k}_i-\mathbf{k}|)(\mu^2+|\mathbf{k}-\mathbf{k}_s|)}d\Omega_{\mathbf{k}}\\ &=\frac{1}{4k\sin\theta/2}\frac{1}{\mu^2\{\mu^4+4k^2[\mu^2+k^2\sin^2(\theta/2)]\}^{\frac{1}{2}}}\\ &\quad\times\ln\left(\frac{\{\mu^4+4k^2[\mu^2+k^2\sin^2(\theta/2)]\}^{\frac{1}{2}}+2k^2\sin(\theta/2)}{\{\mu^4+4k^2[\mu^2+k^2\sin^2(\theta/2)]\}^{\frac{1}{2}}-2k^2\sin(\theta/2)}\right).\end{aligned} \tag{29}$$

Changing the variable back to (Δ,k) and substituting (29) into (20), one obtains

$$\begin{aligned}\mathrm{Re}A^{(2)}(\Delta,k)&=\frac{2}{\pi}P\int_0^\infty\frac{\nu A^{(2)}(\Delta,\nu)}{\nu^2-k^2}d\nu\\ &=\frac{1}{\Delta}\frac{1}{\mu^2[\mu^4+4k^2(\mu^2+\frac{1}{4}\Delta^2)]^{\frac{1}{2}}}\\ &\quad\times\tan^{-1}\left(\frac{\frac{1}{2}\mu\Delta}{[\mu^4+4k^2(\mu^2+\frac{1}{4}\Delta^2)]^{\frac{1}{2}}}\right).\end{aligned} \tag{30}$$

Equations (29) and (30) agree with the usual perturbation calculation.[10] Furthermore, the present calculation appears to be simpler. Here the usual integration over intermediate states is replaced by two separated integrations, i.e., the angular integration in the extended cross-section theorem and the magnitude of momentum integration in the dispersion relation. More drastic simplifications are seen in examples of forward scattering in quantum electrodynamics.[11]

V. SINGULARITIES OF THE S MATRIX

Thus far, we have derived the dispersion relations under the following assumptions: (1) outgoing wave Green's function; (2) velocity-independent potential; (3) integrability of the potential (this depends on the form but not the size of the potential); (4) uniform convergence of the perturbation series for real k. It is known[3] that, when bound states are present, the S matrix, and hence also the scattering amplitude, has simple poles on the positive imaginary axis. Consequently, the dispersion relation cannot be satisfied. Since the first three assumptions listed above are independent of the strength of the potential, we conclude that the perturbation series must fail to converge uniformly for some k. In particular, this is expected to occur in the limit of small k, since $k=0$ gives an upper bound for each term in the perturbation series when the potential is nonpositive for all r. We shall illustrate the relation between bound states and divergence of the perturbation series by the example of the square well: $\lambda U(r)=-\lambda$ for $r\leq a$ and zero for $r>a$.

The perturbation series for the scattering amplitude at zero energy is

$$\begin{aligned}A(0,0)&=\sum_{n=1}^{\infty}\left(\frac{-\lambda}{4\pi}\right)^n\int\left(\prod_{m=1}^{n-1}\frac{U(r_m)}{|\mathbf{r}_m-\mathbf{r}_{m+1}|}\right)\\ &\qquad\times U(r_n)dV_1\cdots dV_n\\ &=a\left[\frac{1}{3}(a^2\lambda)+\frac{2}{15}(a^2\lambda)^2+\frac{17}{315}(a^2\lambda)^3+\cdots\right].\end{aligned} \tag{31}$$

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[8] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), p. 1327.

[9] This result has been obtained by R. Glauber and V. Schomaker [Phys. Rev. **89**, 667 (1953)].

[10] P. M. Morse and H. Feshbach, reference 8, p. 1084.

[11] For example, J. S. Toll, thesis, Princeton, 1952 (unpublished); F. Rohrlich and R. L. Gluckstern, Phys. Rev. **86**, 1 (1952).

This series converges to $(\lambda)^{-\frac{1}{2}}[\tan(a\lambda^{\frac{1}{2}})]$ for $-\infty<\lambda<(\pi/2a)^2$. The latter is exactly the condition for bound states to occur. Incidentally, since $A(0,0)$ converges for all square wells possessing no bound state and each term in $A(0,0)$ is greater or equal to the corresponding term in $A(0,k)$, we conclude that $\sum\lambda^n A^{(n)}(0,k)$ converges uniformly for $\lambda<(\pi/2a)^2$. This result may be of greater generality; however, we shall not consider it any further in the present investigation.

Next we come to the question of what modifications are necessary in the dispersion relations for scatterers that possess bound states. Since we can no longer use perturbation series for all k, the existence and the form of the dispersion relation must be derived from some other arguments. It is reasonable to assume that, for any potential satisfying the integrability previously considered, there exists a positive number k_m, such that the perturbation series converges for $|\zeta|\geq k_m$ on the upper half-plane, and hence $A(0,\zeta)$ is analytic and properly bounded on the exterior of the half-circle. Since the Born series is usually rapidly convergent for extremely large k and (7) also gives a convergent factor for the upper half-plane, the above assumption should cover most physical cases. Now, excluding a neighborhood containing the positive imaginary axis, Van Kampen[3] has shown that $S_l(\zeta)$ is an analytic function on the interior of the half-circle with finite radius k_m. In summing $S_l(\zeta)$ to form the scattering amplitude, we see that the sum is uniformly convergent with respect to ζ since ζ is bounded and $S_l(\zeta)$ becomes very small for l much greater than k_m times the mean width of the potential. We thus conclude that $A(0,k)$ is analytic also in the interior of k_m except, perhaps, in a neighborhood containing the positive imaginary axis. Hence $A(0,k)$, and similarly $A_n(k)$ and $A(\Delta,k)$, satisfies the usual dispersion relations (8) and (9) except for contributions from the positive imaginary axis. Van Kampen[3] has also shown that for scatterer of finite extent, the only singularity of $S_l(\zeta)$ appears as simple poles located at $\zeta=iK_{ln}$, where $-(K_{ln})^2$ is the energy of the nth bound state for the lth partial wave. For infinite potentials that are exponentially damped, it is well known that essential singularities will also occur in each $S_l(\zeta)$ on the positive and negative imaginary axes. This is, however, independent of the value of λ; hence it should give no contribution to the scattering amplitudes. Otherwise, even for λ small enough that the perturbation series converges, the scattering amplitudes would still have singularities on the upper half-plane, and this would be a contradiction. We shall again illustrate this point by the example of the Yukawa well (27).

The S-matrix in the first order is

$$S_l^{(1)}(\zeta)-1=-2i\zeta\int_0^\infty j_l(\zeta r)\frac{\lambda e^{-\mu r}}{\mu r}j_l(\zeta r)r^2dr$$
$$=-\frac{i\lambda}{\mu\zeta}Q_l\left(\frac{\mu^2+2\zeta^2}{2\zeta^2}\right), \tag{32}$$

where Q_l is the Legendre function of the second kind which has essential singularities at $\zeta=\pm i\mu/2$. However,[12]

$$\frac{1}{2i\zeta}\sum_{l=0}^{\infty}(2l+1)[S_l^{(1)}(\zeta)-1]$$
$$=-\frac{\lambda}{2\mu\zeta^2}\sum_{l=0}^{\infty}(2l+1)Q_l\left(\frac{\mu^2+2\zeta^2}{2\zeta^2}\right)=-\frac{\lambda}{\mu^3} \tag{33}$$

has no singularity and is equal to the first-order forward-scattering amplitude.

The final conclusion is: When bound states occur, residue terms due to the corresponding poles on the positive imaginary axis of ζ should be added to the dispersion relations in Secs. II and III.

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[12] P. M. Morse and H. Feshbach, reference 8, p. 1328.