Algebraic Table of Vector-Addition Coefficients for $j_2=5/2$*

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Vector-addition coefficients are given by a terminating hypergeometric sum of the ${}_3F_2$ type, which is easily converted to a simplified form. This form is equivalent to those found independently by Gaunt, van der Waerden, and Racah. It is used to fill the gap for $j_2=\frac{5}{2}$ in the literature of published algebraic tables.

THE Clebsch-Gordan or vector-addition coefficients,[1] which occur in the irreducible-tensor combinations of products of angular momentum eigenfunctions, are explicitly given by the expression[2]

$$C(j_1j_2j;m_1m_2)=\gamma\ {}_3F_2\begin{pmatrix}-j-m;\ -j+j_1-j_2;\ j_1-m_1+1\\ -j-j_2-m_1;\ j_1-j_2-m+1\end{pmatrix},\tag{1}$$

where

$$\gamma\equiv(-1)^{j_2+m_2}\frac{(j+j_2+m_1)!}{(j_1-j_2-m)!}\left[\frac{(j+j_1-j_2)!(j_1+j_2-j)!(j-m)!(j_1-m_1)!(2j+1)}{(j-j_1+j_2)!(j_1+j_2+j+1)!(j+m)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!}\right]^{\frac{1}{2}}.$$

Here ${}_3F_2\begin{pmatrix}-a,\ \alpha,\ \alpha'\\ \beta,\ \beta'\end{pmatrix}$ represents the familiar[3] generalized hypergeometric series ${}_3F_2$, evaluated at argument unity. Since $-a=-j-m$ is always a negative integer, the series terminates after at most $a+1$ (i.e., $j+m+1$) terms. By means of one of Thomae's identities[3] this series is readily converted[4] to a form with a fewer number of terms, and the vector addition coefficient takes the form

$$C(j_1j_2j;m_1m_2)=(-1)^{j_2+m_2}\bar{\gamma}\sum_{s=0}^{j_2+m_2}\frac{(-1)^s}{s!(j_2+m_2-s)!(j+m-s)!(j-j_1+j_2-s)!(-j+j_1-m_2+s)!(j_1-j_2-m+s)!},\tag{2}$$

where

$$\bar{\gamma}\equiv\left[(j_1-m_1)!(j_1+m_1)!(j_2-m_2)!(j_2+m_2)!(j-m)!(j+m)!\frac{(j-j_1+j_2)!(j+j_1-j_2)!(-j+j_1+j_2)!}{(j+j_1+j_2+1)!}(2j+1)\right]^{\frac{1}{2}}.$$

This expression is essentially equivalent to the expressions arrived at independently by Gaunt,[5] van der Waerden,[6] and Racah.[7] It goes over into the symmetrical expression given by Racah,

$$C(j_1j_2j;m_1m_2)=\bar{\gamma}\sum_{s=0}^{j_2+m_2}\frac{(-1)^z}{(j_2+m_2-z)!z!(j-j_2+m_1+z)!(j-j_1-m_2+z)!(-j+j_1+j_2-z)!(j_1-m_1-z)!},$$

by the substitution j_2+m_2-z for the summation index s.

If we examine the factorials in the denominator of the general term in the summation over s in Eq. (2), we notice that one has to carry out the summation at most only for

$$\max(j-j_1+m_2,0)\leqslant s\leqslant\min(j-j_1+j_2,m_2+j_2).$$

From this it follows that the number of nonvanishing terms in the sum is at most

$$N=\min(j_2+1-|m_2|,\ j_2+1-|j-j_1|).\tag{3}$$

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[1] E. Wigner, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren* (Edwards Brothers, Inc., Ann Arbor, 1944), p. 206; E. U. Condon, and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, New York, 1935), pp. 73–77.
[2] M. E. Rose, *Multipole Fields* (John Wiley and Sons, Inc., New York, 1955), p. 92.
[3] W. N. Bailey, *Generalized Hypergeometric Series* (Cambridge University Press, New York, 1935).
[4] M. A. Melvin and N. V. V. J. Swamy, J. Math. Phys. (to be published).
[5] J. A. Gaunt, Trans. Roy. Soc. (London) **A228**, 151 (1929). Gaunt was not concerned with the vector addition coefficients but with solid-angle integrals of products of three spherical harmonics which of course, except for a factor, come to the same thing.
[6] B. L. van der Waerden, *Die Gruppentheoretische Methode in der Quantenmechanik* (Verlag Julius Springer, Berlin, 1932), p. 69; see also J. M. Keller, Phys. Rev. **55**, 508 (1939).
[7] G. Racah, Phys. Rev. **62**, 438 (1942).

Table I. Transformation amplitudes for vector addition, $C(j_1\tfrac{5}{2}j;\, m_1m_2)$, where $m \equiv m_1 + m_2$.

$j \diagdown m_2$	$\frac{5}{2}$	$\frac{3}{2}$
$j_1+\frac{5}{2}$	$\left[\frac{(j_1+m+\frac{5}{2})(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})(j_1+m-\frac{3}{2})}{(2j_1+5)(2j_1+4)(2j_1+3)(2j_1+2)(2j_1+1)}\right]^{\frac{1}{2}}$	$\left[\frac{5(j_1+m+\frac{5}{2})(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})(j_1-m+\frac{5}{2})}{(2j_1+5)(2j_1+4)(2j_1+3)(2j_1+2)(2j_1+1)}\right]^{\frac{1}{2}}$
$j_1+\frac{3}{2}$	$-\left[\frac{5(j_1-m+\frac{5}{2})(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})(j_1+m-\frac{3}{2})}{(2j_1+5)(2j_1+3)(2j_1+2)(2j_1+1)(2j_1)}\right]^{\frac{1}{2}}$	$[-3j_1+5m-(15/2)]\left[\frac{(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})}{(2j_1+5)(2j_1+3)(2j_1+2)(2j_1+1)(2j_1)}\right]^{\frac{1}{2}}$
$j_1+\frac{1}{2}$	$\left[\frac{10(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})(j_1+m-\frac{3}{2})(j_1-m+\frac{5}{2})(j_1-m+\frac{3}{2})}{(2j_1+4)(2j_1+3)(2j_1+1)(2j_1)(2j_1-1)}\right]^{\frac{1}{2}}$	$(2j_1-10m+9)\left[\frac{(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})(j_1-m+\frac{3}{2})}{2(2j_1+4)(2j_1+3)(2j_1+1)(2j_1)(2j_1-1)}\right]^{\frac{1}{2}}$
$j_1-\frac{1}{2}$	$-\left[\frac{10(j_1+m-\frac{1}{2})(j_1+m-\frac{3}{2})(j_1-m+\frac{5}{2})(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})}{(2j_1+3)(2j_1+2)(2j_1+1)(2j_1-1)(2j_1-2)}\right]^{\frac{1}{2}}$	$(2j_1+10m-7)\left[\frac{(j_1+m-\frac{1}{2})(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})}{2(2j_1+3)(2j_1+2)(2j_1+1)(2j_1-1)(2j_1-2)}\right]^{\frac{1}{2}}$
$j_1-\frac{3}{2}$	$\left[\frac{5(j_1-m+\frac{5}{2})(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})(j_1+m-\frac{3}{2})}{(2j_1+2)(2j_1+1)(2j_1)(2j_1-1)(2j_1-3)}\right]^{\frac{1}{2}}$	$-[3j_1+5m-(9/2)]\left[\frac{(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})}{(2j_1+2)(2j_1+1)(2j_1)(2j_1-1)(2j_1-3)}\right]^{\frac{1}{2}}$
$j_1-\frac{5}{2}$	$-\left[\frac{(j_1-m+\frac{5}{2})(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})(j_1-m-\frac{3}{2})}{(2j_1+1)(2j_1)(2j_1-1)(2j_1-2)(2j_1-3)}\right]^{\frac{1}{2}}$	$\left[\frac{5(j_1+m-\frac{3}{2})(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})(j_1-m-\frac{3}{2})}{(2j_1+1)(2j_1)(2j_1-1)(2j_1-2)(2j_1-3)}\right]^{\frac{1}{2}}$

$j \diagdown m_2$	$\frac{1}{2}$	$-\frac{1}{2}$
$j_1+\frac{5}{2}$	$\left[\frac{10(j_1+m+\frac{5}{2})(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1-m+\frac{5}{2})(j_1-m+\frac{3}{2})}{(2j_1+5)(2j_1+4)(2j_1+3)(2j_1+2)(2j_1+1)}\right]^{\frac{1}{2}}$	$\left[\frac{10(j_1+m+\frac{5}{2})(j_1+m+\frac{3}{2})(j_1-m+\frac{5}{2})(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})}{(2j_1+5)(2j_1+4)(2j_1+3)(2j_1+2)(2j_1+1)}\right]^{\frac{1}{2}}$
$j_1+\frac{3}{2}$	$(-j_1+5m-\frac{5}{2})\left[\frac{2(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1-m+\frac{3}{2})}{(2j_1)(2j_1+5)(2j_1+3)(2j_1+2)(2j_1+1)}\right]^{\frac{1}{2}}$	$(j_1+5m+\frac{5}{2})\left[\frac{2(j_1+m+\frac{3}{2})(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})}{(2j_1+5)(2j_1+3)(2j_1+2)(2j_1+1)(2j_1)}\right]^{\frac{1}{2}}$
$j_1+\frac{1}{2}$	$(-2j_1^2-4j_1m+10m^2+\frac{3}{2}-8m-2j_1)\left[\frac{(j_1+m+\frac{1}{2})}{(2j_1+4)(2j_1+3)(2j_1+1)(2j_1)(2j_1-1)}\right]^{\frac{1}{2}}$	$(-2j_1^2+10m^2+4j_1m-2j_1+8m+\frac{3}{2})\left[\frac{(j_1-m+\frac{1}{2})}{(2j_1+4)(2j_1+3)(2j_1+1)2j_1(2j_1-1)}\right]^{\frac{1}{2}}$
$j_1-\frac{1}{2}$	$(2j_1^2-10m^2-4j_1m+2j_1+4m-\frac{3}{2})\left[\frac{(j_1-m+\frac{1}{2})}{(2j_1+3)(2j_1+2)(2j_1+1)(2j_1-1)(2j_1-2)}\right]^{\frac{1}{2}}$	$(-2j_1^2+10m^2-4j_1m-2j_1+4m+\frac{3}{2})\left[\frac{(j_1+m+\frac{1}{2})}{(2j_1+3)(2j_1+2)(2j_1+1)(2j_1-1)(2j_1-2)}\right]^{\frac{1}{2}}$
$j_1-\frac{3}{2}$	$(j_1+5m-\frac{3}{2})\left[\frac{2(j_1+m-\frac{1}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})}{(2j_1+2)(2j_1+1)(2j_1)(2j_1-1)(2j_1-3)}\right]^{\frac{1}{2}}$	$(j_1-5m-\frac{3}{2})\left[\frac{2(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})(j_1-m-\frac{1}{2})}{(2j_1+2)(2j_1+1)(2j_1)(2j_1-1)(2j_1-3)}\right]^{\frac{1}{2}}$
$j_1-\frac{5}{2}$	$-\left[\frac{10(j_1+m-\frac{1}{2})(j_1-m+\frac{1}{2})(j_1+m-\frac{3}{2})(j_1-m-\frac{1}{2})(j_1-m-\frac{3}{2})}{(2j_1+1)(2j_1)(2j_1-1)(2j_1-2)(2j_1-3)}\right]^{\frac{1}{2}}$	$\left[\frac{10(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})(j_1+m-\frac{3}{2})(j_1-m-\frac{1}{2})(j_1-m-\frac{3}{2})}{(2j_1+1)(2j_1)(2j_1-1)(2j_1-2)(2j_1-3)}\right]^{\frac{1}{2}}$

TABLE I.—*Continued.*

$j \diagdown m_2$	$-\frac{3}{2}$	$-\frac{5}{2}$
$j_1+\frac{5}{2}$	$\left[\frac{5(j_1+m+\frac{5}{2})(j_1-m+\frac{5}{2})(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})}{(2j_1+5)(2j_1+4)(2j_1+3)(2j_1+2)(2j_1+1)}\right]^{\frac{1}{2}}$	$\left[\frac{(j_1-m+\frac{5}{2})(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})(j_1-m-\frac{3}{2})}{(2j_1+5)(2j_1+4)(2j_1+3)(2j_1+2)(2j_1+1)}\right]^{\frac{1}{2}}$
$j_1+\frac{3}{2}$	$[3j_1+5m+(15/2)]\left[\frac{(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})}{(2j_1+5)(2j_1+3)(2j_1+2)(2j_1+1)(2j_1)}\right]^{\frac{1}{2}}$	$\left[\frac{5(j_1+m+\frac{5}{2})(j_1-m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})(j_1-m-\frac{3}{2})}{(2j_1+5)(2j_1+3)(2j_1+2)(2j_1+1)(2j_1)}\right]^{\frac{1}{2}}$
$j_1+\frac{1}{2}$	$(2j_1+10m+9)\left[\frac{(j_1+m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})}{2(2j_1+4)(2j_1+3)(2j_1+1)(2j_1)(2j_1-1)}\right]^{\frac{1}{2}}$	$\left[\frac{10(j_1+m+\frac{5}{2})(j_1+m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m-\frac{1}{2})(j_1-m-\frac{3}{2})}{(2j_1+4)(2j_1+3)(2j_1+1)(2j_1)(2j_1-1)}\right]^{\frac{1}{2}}$
$j_1-\frac{1}{2}$	$(-2j_1+10m+7)\left[\frac{(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1-m-\frac{1}{2})}{2(2j_1+3)(2j_1+2)(2j_1+1)(2j_1-1)(2j_1-2)}\right]^{\frac{1}{2}}$	$\left[\frac{10(j_1+m+\frac{5}{2})(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1-m-\frac{1}{2})(j_1-m-\frac{3}{2})}{(2j_1+3)(2j_1+2)(2j_1+1)(2j_1-1)(2j_1-2)}\right]^{\frac{1}{2}}$
$j_1-\frac{3}{2}$	$[-3j_1+5m+(9/2)]\left[\frac{(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})}{(2j_1+2)(2j_1+1)(2j_1)(2j_1-1)(2j_1-3)}\right]^{\frac{1}{2}}$	$\left[\frac{5(j_1+m+\frac{5}{2})(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})(j_1-m-\frac{3}{2})}{(2j_1+2)(2j_1+1)2j_1(2j_1-1)(2j_1-3)}\right]^{\frac{1}{2}}$
$j_1-\frac{5}{2}$	$-\left[\frac{5(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})(j_1+m-\frac{3}{2})(j_1-m-\frac{3}{2})}{(2j_1+1)(2j_1)(2j_1-1)(2j_1-2)(2j_1-3)}\right]^{\frac{1}{2}}$	$\left[\frac{(j_1+m+\frac{5}{2})(j_1+m+\frac{3}{2})(j_1+m+\frac{1}{2})(j_1+m-\frac{1}{2})(j_1+m-\frac{3}{2})}{(2j_1+1)2j_1(2j_1-1)(2j_1-2)(2j_1-3)}\right]^{\frac{1}{2}}$

Comparison with the expression for $C(j_1j_2j;m_1m_2)$ given in Condon and Shortley [reference 1, p. 75, Eq. (5)] shows that the number of terms in the summation there is $j_2+1+j-j_1$. This is as small as the number of terms in Eq. (2) only in special cases. This fact, and the greater simplicity of its structure, gives Eq. (2) a considerable advantage for computation over the equation of reference 1.

Table I, which was calculated by Eq. (2), gives explicit algebraic forms of the vector addition coefficients for $j_2=\frac{5}{2}$. In accord with Eq. (3), the maximum number of terms encountered in the summation for any entry in the table was three. Most entries involved only one or two terms. Table I fills the gap between the algebraic tables for $j_2=\frac{1}{2}, 1, \frac{3}{2}, 2$ given by Condon and Shortley[1] and the algebraic table for $j_2=3$ given by Falkoff *et al.*[8] It may be noted that numerical, in contrast to algebraic, tables of some vector addition coefficients for various values of j have been given by Alder[9] and, more completely, for all $j\leqslant 9/2$ by Simon.[10]

We are grateful to Mr. John M. Boardman for checking Table I by independent calculation.

[8] Falkoff, Colladay, and Sells, Can. J. Phys. **30**, 253 (1952).
[9] K. Alder, Helv. Phys. Acta **25**, 235 (1952).
[10] A. Simon, Oak Ridge National Laboratory Report ORNL-1718, 1954 (unpublished).

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Spins of Thallium-197, -198m, -199, and -204,* and the Hyperfine-Structure Splitting of Thallium-204†

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The nuclear angular momenta of four isotopes of thallium and the hyperfine-structure splitting of Tl^{204} in the $^2P_{\frac{1}{2}}$ state have been measured by the atomic-beam magnetic resonance method. The results are: for 2.7-hr Tl^{197}, $I=\frac{1}{2}$; for 1.8-hr Tl^{198m}, $I=7$; for 7.2-hr Tl^{199}, $I=\frac{1}{2}$; for 4.1-yr Tl^{204}, $I=2$ and $\Delta\nu(^2P_{\frac{1}{2}})=732\pm5$ Mc/sec. The neutron-deficient isotopes are produced by alpha-particle bombardment of gold in the Berkeley 60-inch cyclotron; Tl^{204} is produced by neutron activation of metallic thallium samples.

INTRODUCTION

THE atomic-beam method, as adapted to measurements on radioisotopes, has been used to determine some of the presently available unmeasured spins and moments in the thallium series. This research, dealing with an element near magic numbers 82 and 126, is part of a general program of the Berkeley atomic-beam group to determine as far as possible the unknown spins and nuclear moments of one or more elements in regions of special significance from the point of view of nuclear theory. In this frame of

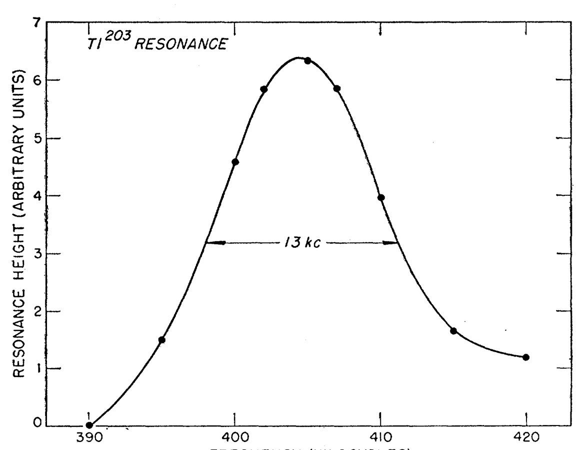

FIG. 1. Typical resonance in natural thallium.

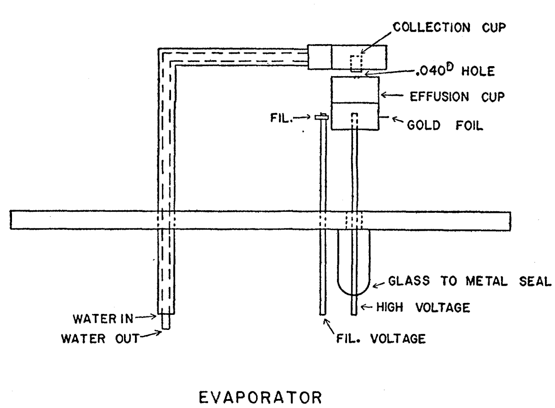

FIG. 2. Evaporation unit used to separate radiothallium from gold targets.

* These measurements, previously reported by G. O. Brink *et al.*, Bull. Am. Phys. Soc. Ser. II, **1**, 343 (1956), were announced at the Brookhaven Conference on Molecular Beams, October 5 and 6, 1956, and are reported in the proceedings of that meeting. At this conference others reported negative results on attempts to measure the spin of Tl^{204}.

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