Spin Reversal in Scattering Processes

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The effect on the scattering matrix element of reversing the spins of all particles is discussed. An exact relation is shown to be the consequence of invariance under combined space and time inversion and the unitarity of the S matrix. In lowest order perturbation theory this gives a symmetry property of the matrix element. The implication for particles in arbitrary states of polarization is pointed out.

RECENTLY, with the availability of polarized particles from parity-nonconserving weak-decay sources, there has been a growth of interest in the spin dependence of scattering processes. The purpose of this article is to call attention to an approximate symmetry property to which the spin dependence of the matrix elements is subject.

Let states be specified by the momenta $\mathbf{p}$ of particles in plane-wave states (we take $\hbar=c=1$ throughout), and by $m=-j,\ -(j-1),\ \cdots(j-1),\ j$, where m, the "helicity quantum number," is the eigenvalue of $\mathbf{s}\cdot\mathbf{p}/p$, $\mathbf{s}$ being the spin operator for the particle in question. We wish to discuss the effect on the matrix element of the transformation

$$\mathbf{p}\rightarrow\mathbf{p},\quad m\rightarrow -m, \tag{1}$$

referred to in the following as spin reversal.

As an illustration, let us take the matrix element for bremsstrahlung in first Born approximation in the external Coulomb field.[1] The matrix element is[2]

$$M=(2\pi^2\mathbf{q}^2 i)^{-1}\alpha^{\frac{3}{2}}Zm(\epsilon\epsilon'\omega)^{-\frac{1}{2}}\bar{u}_{r'}(\mathbf{p}')Q_s u_r(\mathbf{p}). \tag{2}$$

Here r and r' are the helicities of the electron in the initial and final states, respectively. The matrix Q_s is given by

$$Q_s(\mathbf{p},\mathbf{p}',\mathbf{k})=e_s\frac{i(p'+k)-m}{2p'\cdot k}n+n\frac{i(p-k)-m}{-2p\cdot k}e_s. \tag{3}$$

The photon helicity s enters into the definition of the complex unit polarization vector $\mathbf{e}_s$ appropriate to left $(s=+1)$ and right $(s=-1)$ circularly polarized photons.[3] Let the vectors $\mathbf{p}$ and $\mathbf{p}'$ lie in the 1–2-plane, the vector $\mathbf{k}$ lie in the 1–3-plane making an angle χ with the 1 axis, and let the real part of $\mathbf{e}_s$ be parallel to the 2 axis. Furthermore, choose γ_1, γ_2, γ_0 as pure imaginary, and γ_3 as real (this is only for convenience; any representation of the Dirac matrices gives the same result). With these conventions one sees easily that all factors are purely imaginary or real, except

$$\begin{aligned} ik&=\omega(i\gamma_1\cos\chi+i\gamma_3\sin\chi+\gamma_0),\\ -ie&=2^{-\frac{1}{2}}(-s\gamma_1\sin\chi-i\gamma_2+s\gamma_3\cos\chi),\end{aligned} \tag{4}$$

and for these factors complex conjugation has the same effect as changing the sign of χ. The latter operation is a reflection of momenta in the 1–2-plane, and leaves the helicities unaffected. A space inversion to which the theory is invariant involves, however, also the reversal of the helicities. Consequently, one concludes that

$$(\mathbf{p}',r',\mathbf{k},s|M|\mathbf{p},r)=-(\mathbf{p}',-r',\mathbf{k},-s|M|\mathbf{p},-r)^*. \tag{5}$$

Under spin reversal the real part of the matrix element changes sign.

To investigate the validity of this result for general scattering processes it is important to note that spin reversal is closely related to combined space-time inversion but is not identical with it. Space-time inversion reverses spins and leaves momenta unchanged, but it also interchanges the role of the initial and final states. Thus, in a theory invariant under this symmetry operation, the S matrix (with a proper choice of arbitrary phases for the states under consideration) satisfies

$$(f|S|i)=(i'|S|f'), \tag{6}$$

where the prime refers to the operation (1). In order to obtain the generalization of (5) it must be shown that the relevant part of the S matrix is anti-Hermitian. This, in turn, is known to be true in lowest order perturbation theory in which the effect occurs, and it is a simple consequence of the unitarity of the S matrix. To formulate this precisely, let $R\equiv S-1$, so that the unitarity condition becomes

$$R^\dagger=-R(1+R)^{-1}. \tag{7}$$

Combining this with (6), one obtains the exact relation

$$-(f'|R|i')^*=(f|R(1+R)^{-1}|i). \tag{8}$$

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[1] The circular polarization in bremsstrahlung from polarized electrons has been detected recently by Goldhaber, Grodzins, and Sunyar, Phys. Rev. **106**, 826 (1957). The theory has been given by K. W. McVoy, Phys. Rev. **106**, 828 (1957). The author is indebted to Dr. Goldhaber and Dr. McVoy for letting him see these papers before publication.

[2] J. M. Jauch and F. Rohrlich, *The Theory of Electrons and Photons* (Addison-Wesley Publishing Company, Cambridge, 1955), Sec. 15–6. We follow the notation of this work.

[3] This seemingly illogical convention originates in optics where a right circularly polarized wave has clockwise-rotating electromagnetic vectors when viewed facing the light source. Positive (negative) helicity, on the other hand, means clockwise (counterclockwise) rotation when looking along the propagation vector. The latter convention has the advantage that a photon with positive (negative) helicity has its spin parallel (antiparallel) to its direction of propagation.

In perturbation theory one retains only the lowest order term of the geometrical series on the right side, and then the analog of (5) results for an arbitrary process.

A few remarks should be made about the case when $|i)$ and $|f)$ cannot be taken as free-particle plane wave states. This occurs, for instance, in the description of radiation phenomena in the presence of a strong external potential. It is convenient to discuss this in a picture where the scattering operator S represents only the effects of the radiation interaction, and the external field influences only the state vectors with which matrix elements are formed. Scattering is described by a matrix element of the form $(f_{\text{in}}|S|i_{\text{out}})$. Here $|i_{\text{out}})$ represents a state which consists asymptotically of plane waves characterized by the label i, and scattered waves due to the external potential satisfying the "outgoing" boundary condition at infinity; analogously for $|f_{\text{in}})$ with the "ingoing" boundary condition.[4] Space-time inversion invariance implies in this case

$$(f_{\text{in}}|S|i_{\text{out}})=(i_{\text{in}}'|S|f_{\text{out}}'), \tag{9}$$

because time inversion also converts the "ingoing" into the "outgoing" boundary condition, and vice versa. If now only the lowest order in the radiation interaction $S_{(1)}$ is considered, we still have $S_{(1)}=-S_{(1)}^\dagger$, but now this implies that

$$(f_{\text{in}}|S_{(1)}|i_{\text{out}})=-(f_{\text{out}}'|S_{(1)}|i_{\text{in}}')^*, \tag{10}$$

whereas the spin-reversed matrix element is $(f_{\text{in}}'|S_{(1)}|i_{\text{out}}')$. It is not difficult to see, however, that as long as the external potential is weak so that it can be treated in first Born approximation, the matrix element changes under spin reversal as shown by the example (5).

There remains the question as to what the spin-reversal symmetry just discussed implies with regard to scattering from and into states of arbitrary polarization. For the sake of simplicity, let us restrict the discussion to systems of spin-$\frac{1}{2}$ particles and photons. Also, consider only one particle in both the initial and the final states. The relation

$$(m'|M|m)=-(-m'|M|-m)^* \tag{11}$$

can be written in matrix form as[5]

$$M=-\sigma_1 M^*\sigma_1. \tag{12}$$

The transition probability is in general proportional to the quantity

$$\Gamma\equiv\text{Tr}(\epsilon M\rho M^\dagger). \tag{13}$$

Here ρ is a statistical density matrix with respect to the helicity indices of the initial state, and ϵ is an "efficiency matrix"[6] with respect to the indices of the final state. The former characterizes the state of polarization of the initial beam of particles, and the latter the polarization sensitivity of the detector response. A convenient parametrization of these 2×2 matrices is given by

$$\rho=\tfrac{1}{2}(1+\boldsymbol{\xi}_i\cdot\boldsymbol{\sigma}),\quad \epsilon=\tfrac{1}{2}(1+\boldsymbol{\xi}_f\cdot\boldsymbol{\sigma}), \tag{14}$$

where the $\boldsymbol{\sigma}$ are the Pauli matrices in the standard representation. The real "polarization vectors" $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_f$ characterize the polarization properties of the experimental arrangement, and Γ appears as a function of them.[7] The question to be decided is what symmetry property of this function is implied by (12). Substituting that relation into (13), one gets after some simple algebra

$$\Gamma(\boldsymbol{\xi}_i,\boldsymbol{\xi}_f)=\Gamma(\boldsymbol{\xi}_i',\boldsymbol{\xi}_f'). \tag{15}$$

Here the prime denotes the reversal in sign of the 3 component. We conclude then that the transition probability remains invariant with respect to "reflection in the 1–2-plane" of all polarization vectors. Phrased this way, the statement is true when an arbitrary number of spin-$\frac{1}{2}$ particles and photons are involved.

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[4] For a discussion that these are the correct boundary conditions see H. A. Bethe and L. C. Maximon, Phys. Rev. **93**, 768 (1954), Sec. IV.

[5] The asterisk denotes complex conjugation, as contrasted to Hermitian conjugation which is denoted by a dagger.

[6] F. Coester and J. M. Jauch, Helv. Phys. Acta **26**, 3 (1953).

[7] This formalism has been exploited recently by H. A. Tolhoek, Revs. Modern Phys. **28**, 277 (1956).