state at 2.34 Mev and the fourth at 4.37 Mev are both required to have high spins and even parity since the ground state of the parent is 4+ or 5+ and both β-ray transitions are allowed. A spin of 4+ for the second excited state is reasonable from the shell model.

The 2.5-Mev γ ray is too intense to be the result of a crossover transition from the second excited state to the ground state and therefore must arise from the third excited level. Hence the order of the 1.74- and 1.47-Mev γ rays given in Fig. 3 is the most likely. This would require a spin of 2 or 3 for the third level. The fact that the third level is not populated by β decay, together with the ratio of the intensities of the 1.74- and 2.03-Mev γ rays, suggest the following level assignments: spin 3+ or 2+ for the third level, spin 4+ for the fifth level, and spin 5+ for the ground state of Co^{62}.

The 1.6-min isomer of Co^{62} found by Parmley[1] was not found here due to the length of time required by the chemical separation. It is quite reasonable to expect, however, that the first excited state of Co^{62} would have a proton configuration of $[(1f_{7/2})^{-1}]_{7/2}$, and a neutron configuration of $[(1f_{5/2})^{-3}]_{5/2}$. These would couple to give a spin of 1+, producing an isomeric state with an energy not much above ground state. This first excited state would have a high probability of decaying directly to the ground state of Ni^{62}. For the sake of completeness this hypothetical excited level has been included in Fig. 3.

ACKNOWLEDGMENTS

The authors wish to express their thanks to Professor W. C. Parkinson, Professor M. L. Wiedenbeck, and the crew of the University of Michigan cyclotron for their help and cooperation.

PHYSICAL REVIEW VOLUME 107, NUMBER 6 SEPTEMBER 15, 1957

Time-Dependent Hartree-Fock Theory of Nuclear Collective Oscillations*†

RICHARD A. FERRELL
University of Maryland, College Park, Maryland
(Received June 7, 1957)

A theory of nuclear collective oscillations is presented which does not involve introducing extra variables and subsidiary conditions. This time-dependent self-consistent field method is applied to the breathing mode of a spherically symmetric nucleus and yields a value for the frequency of oscillation which is more accurate than that from a previous treatment in terms of one-nucleon excitation, but which becomes identical to the latter in the case of weak nucleon-nucleon interaction. In cases where nucleon exchange can be neglected, the new estimate reduces to the frequency derived from the simple classical theory of a compressible fluid. By means of an electric monopole sum rule which is derived for $T=0\rightarrow T=0$ transitions, it is shown that in general the classical formula overestimates the breathing mode frequency. From the sum rule it also follows that the 6.06-Mev 0^+ state in O^{16} is related only indirectly to the breathing mode, which must itself be at a higher excitation energy.

I. INTRODUCTION

IN an earlier publication[1] it was shown how a collective oscillation in a nucleus could be identified as the coherent superposition of one-nucleon excitations. This identification depended on an approximate treatment of the Hill-Wheeler-Griffin[2] wave function. The purpose of the present note is to outline in a very idealized case a time-dependent self-consistent field treatment of collective oscillations which can be formulated within the framework of the shell model, but which does not suffer from the approximation made in reference 1. In order to emphasize the essential features of the approach, we shall limit ourselves to the breathing mode (the simplest type of nuclear oscillation) in a fictitious mass-twelve nucleus composed of a vacant $1s$ shell and a filled $1p$ shell. This has enough nucleons to exhibit collective effects without the complication of coupled shell vibrations. The fact that such a nucleus (actually an excited state of C^{12}) would in fact be unstable need not concern us here, since we can consider that the parts of the nuclear interaction which would give rise to $1p-1s$ transitions have been removed from the Hamiltonian. This procedure does not affect the $1p-2p$ transitions, which are the ones involved in the breathing mode.

II. MONOPOLE SUM RULE

The degree of the inaccuracy of the one-nucleon approximation to a breathing mode excitation can be most easily exhibited by considering a sum rule for electric monopole transitions. Following Sachs and Austern,[3] we consider the double commutator of the

* Research supported by the National Science Foundation.
† A report on this work has been presented at the 1957 New York Meeting of the American Physical Society [Bull. Am. Phys. Soc. Ser. II, 2, 26 (1957)].
[1] R. A. Ferrell and W. M. Visscher, Phys. Rev. **102**, 450 (1956).
[2] References 3 and 4 of reference 1.
[3] R. G. Sachs and N. Austern, Phys. Rev. **81**, 705 (1951).

monopole operator $B\equiv\sum_p r_p^2=\sum_i[1+\tau_z^{(i)}]r_i^2/2$ with the Hamiltonian, H. The sum is over the squares of all the proton coordinates. B can be separated into the two parts, $B^0\equiv\sum_i r_i^2/2$ and $B^1\equiv\sum_i \tau_z^{(i)}r_i^2/2$, scalar and vector in isotopic spin, respectively. Let us now restrict ourselves to $T=0\rightarrow T=0$ transitions, to whose transition probabilities only B^0 contributes. This restriction seems first to have been employed by Gell-Mann and Telegdi[4] (in connection with $E2$ transitions) and has the very great advantage of leading to a sum rule which is independent of the exchange character of the nuclear interaction. In general, for any B^0 of the form $B^0\equiv\sum_i g_i$, where g_i is a function of the spatial coordinates of the ith nucleon alone, we find

$$[B^0,[H,B^0]]=\frac{\hbar^2}{M}\sum_{i=1}^{N}|\mathrm{grad}_i g_i|^2, \tag{1}$$

where N is the number of nucleons and M the nucleon mass. Taking the expectation value with respect to the state 0 and indicating the excitation energy of the various other $T=0$ states by $\hbar\omega_{n0}$, we find the sum rule

$$\sum_n f_{n0}=N, \tag{2}$$

where the oscillator strengths are given by

$$f_{n0}=\frac{2M\hbar\omega_{n0}}{\hbar^2\langle|\mathrm{grad}g|^2\rangle_0}|B_{n0}{}^0|^2. \tag{3}$$

In the present case $g=r^2/2$, so $\langle|\mathrm{grad}g|^2\rangle=\langle r^2\rangle=5/2\gamma$, if one uses oscillator wave functions with Gaussian factors $\exp(-\gamma r^2/2)$. Letting $n=1$ designate the one-nucleon collective excitation of reference 1, it is easily established[5] that $B_{10}{}^0=(2\gamma)^{-1}(5N/2)^{\frac{1}{2}}$. If we further introduce $\hbar\omega'$ to represent the $1p-2p$ excitation energy without collective effects, we can use the oscillator shell model[6] to write $\hbar\omega'=2\hbar^2\gamma/M$. Substituting into Eq. (3) yields $f_{10}=(\omega_{10}/\omega')N$, or the fraction ω_{10}/ω' of the sum rule limit. This result is clearly inconsistent with the fact that B^0 applied to the ground-state wave function yields identically the $n=1$ state, up to a constant. Thus, no other transition can contribute to the sum rule and f_{10} must be just N. The magnitude of this inconsistency is larger the more $\hbar\omega_{10}$ is decreased by collective interaction below the single-nucleon excitation energy of $\hbar\omega'$. As the nucleon-nucleon interaction strength is gradually increased from zero and collective effects first begin to appear, the error is not serious, but for strong collective lowering of $\hbar\omega_{10}$ the theory of reference 1 is clearly inadequate for quantitative purposes.

III. BREATHING MODE FREQUENCY

The most important feature of a collective oscillation is of course its excitation energy, or, speaking more classically, its frequency of vibration. This can be calculated in analogy with a time-dependent self-consistent field derivation[7,8] of the Bohm-Pines dispersion relation for plasma oscillations of a degenerate electron gas. The present work is also related to some earlier work of Inglis[9] and of Araújo.[10] The effect of expansions and contractions on the one-nucleon oscillator orbitals is given by[1]

$$u_{1p}(e^{-\alpha}\mathbf{x})e^{-3\alpha/2}=u_{1p}(\mathbf{x})-\alpha(\tfrac{5}{2})^{\frac{1}{2}}u_{2p}(\mathbf{x})+\cdots, \tag{4}$$

where the higher terms can be neglected for small values of the expansion parameter α. This relationship between differentiation with respect to the scale parameter and excitation in the shell model is illustrated in Fig. 1, where the difference between the $1p$ radial wave function for $\alpha=+0.1$ and $\alpha=-0.1$ is seen to be identical, up to a constant, with the $2p$ radial wave function. Thus, when the nucleus is oscillating we must expect that the one-nucleon orbitals acquire small $2p$ admixtures, with time-dependent coefficients. They can therefore be written in the form

$$\psi_i(i,t)=[u_i(i)+A_i(t)v_i(i)]e^{-iE_it/\hbar}, \tag{5}$$

where the argument i denotes the space, spin, and

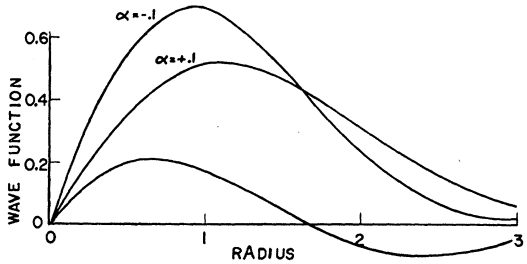

FIG. 1. Radial wave functions (in arbitrary units) *vs* radius (in units of $\gamma^{-\frac{1}{2}}$). The curves labeled $\alpha=\pm0.1$ represent an expanded and a contracted $1p$ wave function, respectively. Their difference is proportional to the $2p$ wave function. In this way a collective oscillation can be expressed in terms of single-nucleon excitation.

[4] M. Gell-Mann and V. L. Telegdi, Phys. Rev. **91**, 169 (1953).
[5] R. A. Ferrell and W. M. Visscher, Phys. Rev. **104**, 475 (1956).
[6] This convenient relation between excitation energy and the scale of length cannot be expected to be strictly valid. We believe that it is probably a satisfactory approximation for the $1p-2p$ excitation but that it may possibly underestimate the $1s-2s$ excitation in O^{16} by a factor of as much as two. Such a correction would raise the estimate in reference 1 of the breathing-mode excitation energy to roughly 15 Mev and make untenable its identification with the 6.06-Mev 0^+ excitation in O^{16}. This conclusion is strengthened by computing from the experimental data the oscillator strength for the transition of this level to the ground state. One finds $f=0.66$, or only about 4% of the sum-rule limit. *Note added in proof.*—Our conjecture concerning the single-nucleon excitation energies finds confirmation in some explicit calculations of F. C. Barker (private communication), who finds 54.4 Mev for s excitation in O^{16}. His p-excitation energy of 36.6 Mev is much closer to $\hbar\omega'=29.7$ Mev.
[7] P. S. Zyrianov and E. M. Eleonski, J. Exptl. Theoret. Phys. U.S.S.R. **30**, 592 (1956) [translation: Soviet Phys. JETP **3**, 620 (1956)].
[8] R. A. Ferrell, Phys. Rev. **107**, 450 (1957).
[9] D. R. Inglis, Phys. Rev. **97**, 701 (1955); see also S. Moszkowski, Phys. Rev. **103**, 1328 (1956).
[10] J. M. Araújo, Nuclear Phys. **1**, 259 (1956).

isotopic spin coordinates of the ith nucleon. u_i and v_i are $1p$ and $2p$ one-nucleon wave functions with the same angular, spin, and isotopic spin factors.

The $A_i(t)$ are to be determined from the time-dependent Hartree-Fock equation,

$$i\hbar\dot{\psi}_i(i,t)=T_i\psi_i(i,t) +\sum_j(\psi_j(j,t),\, V_{ij}(1-P_{ij})\psi_j(j,t))_j\psi_i(i,t). \quad (6)$$

Here time differentiation is represented by a dot, T_i is the kinetic energy operator for the ith nucleon, V_{ij} is the interaction between nucleons i and j, P_{ij} is the exchange operator, and the subscript j on the parenthesis signifies that the scalar product over j is to be evaluated after the exchange operation has been carried out. It is a simple procedure to substitute from Eq. (5) into Eq. (6) and linearize. Exhibiting only the terms of first order in the A_i's, we have

$$i\hbar\dot{A}_iv_i(i)+E_iA_iv_i(i) =[T_i+\sum_j(u_j(j),\, V_{ij}(1-P_{ij})u_j(j))]v_i(i)A_i +\text{perturbation}=(E_i+\hbar\omega')v_i(i)A_i+\text{pert}. \quad (7)$$

The perturbation consists of the first-order terms from the scalar product acting on the zero-order part of $\psi_i(i,t)$. The bracketed terms form just the shell-model operator and yield $E_i+\hbar\omega'$ when operating on $v_i(i)$. (E_i is the $1p$ single nucleon energy.) Taking the scalar product with respect to $v_i(i)$ and writing out the perturbation explicitly gives

$$i\hbar\dot{A}_i=\hbar\omega'A_i +\sum_j[(v_i(i)u_j(j),\, V_{ij}(1-P_{ij})v_j(j)u_i(i))A_j +(v_i(i)v_j(j),\, V_{ij}(1-P_{ij})u_j(j)u_i(i))A_j^*]. \quad (8)$$

Because of the spherical symmetry of the breathing mode all of the A_i's are equal, so that Eq. (8) reduces to the single equation

$$i\hbar\dot{A}=\hbar\omega'A+CA+C'A^*, \quad (9)$$

involving the quantities C and C' which are the sums over j of the scalar products appearing in the second and third lines, respectively, of Eq. (8). The direct and exchange parts of C and C' are illustrated in Fig. 2. It may be noted that the direct parts of C and C' are equal, since the directions of any of the arrows in the vertical transitions may be reversed. (Only the real radial wave functions are involved in these transitions.) There does not, however, seem to be any simple relation between the exchange parts.

Equation (9) and its complex conjugate comprise a pair of simultaneous linear homogeneous equations for the two unknowns A and A^*. It is easy to show that the only frequency, ω, for which there exists a non-trivial solution is given by

$$\hbar\omega=[(\hbar\omega'+C)^2-C'^2]^{\frac{1}{2}}, \quad (10)$$

which can be compared with the expression

$$\hbar\omega=\hbar\omega'+C, \quad (11)$$

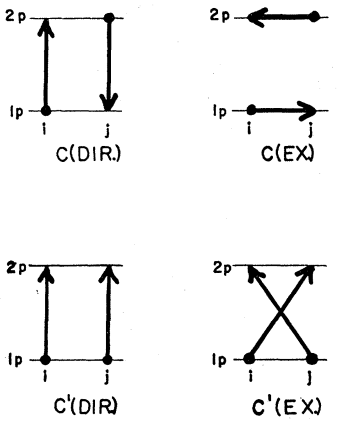

FIG. 2. Nucleon-nucleon interaction matrix elements which determine the frequency of the breathing mode of oscillation. The upper half of the figure illustrates the matrix elements already present in the one-nucleon-excitation theory of the breathing mode, while the lower half shows additional correction terms which are derived by means of the time-dependent self-consistent field. Direct and exchange terms are shown on the left and right, respectively.

obtained in reference 1. Equation (10) clearly reduces to Eq. (11) in the limit of $C'\to 0$, but for not-too-large values of C' the right-hand member of Eq. (11) must be reduced by the second-order fractional correction $-2^{-1}[C'/(\hbar\omega'+C)]^2$. It is of interest to compare Eq. (10) with the classical fluid model for a nucleus vibrating in the breathing mode. The inertial parameter related to α is in the present case $I=60\hbar/\omega'$, while one can also show that the Feenberg compressibility coefficient[11] is given by

$$E_0''=60(\hbar\omega'+C+C'). \quad (12)$$

Designating the classical frequency by ω_{cl}, one finds

$$\hbar\omega_{\text{cl}}=(\hbar^2E_0''/I)^{\frac{1}{2}}=[\hbar\omega'(\hbar\omega'+C+C')]^{\frac{1}{2}}. \quad (13)$$

The ratio of the true frequency to the classical frequency is therefore

$$\frac{\omega}{\omega_{\text{cl}}}=\left(1+\frac{C-C'}{\hbar\omega'}\right)^{\frac{1}{2}}. \quad (14)$$

As mentioned above, C and C' differ only in their exchange parts. Thus, in cases where exchange is negligible, the classical formula gives correctly, as would be expected, the frequency of oscillation.

Note added in proof.—J. Touchard, [Compt. rend. **244**, 2499 (1957)], has derived the classical formula from the Inglis "cranking model,"[9] which seems to indicate that the "cranking model" also does not adequately take into account the effect of exchange.

IV. MONOPOLE MATRIX ELEMENT

The solution of Eq. (9) associated with the frequency of Eq. (10) has the form

$$A(t)=A_0[(1+a)e^{-i\omega t}+(1-a)e^{i\omega t}], \quad (15)$$

where

$$a=\left(\frac{\hbar\omega'+C+C'}{\hbar\omega'+C-C'}\right)^{\frac{1}{2}}.$$

[11] E. Feenberg, Phys. Rev. **59**, 149 (1941).

A_0 depends on the degree of excitation of the nucleus,[12] or the number of quanta, ν. By taking the expectation value of the Hamiltonian, one can show that $\nu\hbar\omega = 4aA_0^2N\hbar\omega$, or

$$A_0 = (\nu/4aN)^{\frac{1}{2}}. \tag{16}$$

Calculating the expectation value of the monopole operator B^0 to first order in A_0, we find

$$\begin{aligned}&\tfrac{1}{2}\textstyle\sum_i (\psi_i(i,t), r_i^2\psi_i(i,t))_i\\ &\quad = \tfrac{1}{2}N\langle r^2\rangle_0 + \tfrac{1}{2}N(v,r^2u)(A^*+A)\\ &\quad = \tfrac{1}{2}N\langle r^2\rangle_0 + (\nu N/4a)^{\frac{1}{2}}(v,r^2u)(e^{-i\omega t}+e^{i\omega t}).\end{aligned} \tag{17}$$

The static term is of no interest here but the fluctuating term proportional to $e^{-i\omega t}$ can produce an electron-positron pair and de-excite the breathing mode. Taking $\nu=1$, we find $B_{10}^0 = (2\gamma)^{-1}(5N/2a)^{\frac{1}{2}}$, or just $a^{-\frac{1}{2}}$ times the matrix element for the one-nucleon collective excitation discussed in Sec. II. The fraction of the sum rule limit accounted for by the breathing mode is consequently a^{-1} times that found there, or

$$a^{-1}\frac{\omega}{\omega'} = 1 + \frac{C-C'}{\hbar\omega'}. \tag{18}$$

But according to Eq. (14) this is just the square of ω/ω_{01}. Since the sum rule limit may never be exceeded, we seem to have arrived at the general result

$$\omega/\omega_{01} \leq 1; \tag{19}$$

i.e., the actual frequency is always less than, or at most equal to, the classical estimate. Equation (19) would also seem to require $C \leq C'$. Whether or not this is a genuine restriction on the nucleon-nucleon interaction is not clear at the present time.

V. SUMMARY

In conclusion, we would like to emphasize that the simple shell model description of collective excitation given in reference 1 remains correct for the case of weak nucleon-nucleon interaction. For stronger interaction, where the excitation energy undergoes a relatively large shift downward from the one-nucleon excitation energy, it still provides a useful qualitative picture but can no longer be applied quantitatively. The qualitative picture is equivalent to the nuclear version of Hund's rule: For a given excited configuration the state having the lowest multiplicity (greatest spatial symmetry) lies energetically lowest. By expanding the Slater determinant composed of one-nucleon wave functions of the form of Eq. (5), one sees that the improvement brought about by the time-dependent Hartree-Fock calculation of the vibrational frequency and monopole matrix element amounts to including three-nucleon, five-nucleon, etc., admixtures in the one-nucleon collective excitation of reference 1. Although not exhibited explicitly in the above work, the two-nucleon, four-nucleon, etc., admixtures in the ground state have also been taken into account. By working in the classical limit of large quantum numbers[13] we have been able to avoid explicitly writing down the ground-state wave function for the nucleus. It is nevertheless clear that, for many problems, explicitly exhibiting the wave function will be inescapable, and it will be essential to cope with the correlation in the ground state.

As noted in reference 6, we no longer believe the 6.06-Mev 0^+ excitation in O^{16} to be essentially the breathing mode. It provides only 4% of the limit of the sum rule derived in Sec. II. Since the remaining 96% must still be accounted for, the true breathing-mode excitation must lie at a higher energy. The breathing mode must nevertheless play an important role in determining the properties of the 6.06-Mev level. By mixing to some extent with the two-nucleon excitation the breathing mode pushes the energy level down, and also provides the nonzero monopole matrix element. Before the present theory can be applied numerically to the O^{16} breathing mode, it will be necessary to extend it to coupled oscillations of the $1s$ and $1p$ shells. Also it will be necessary to have better estimates of the one-nucleon excitation energies. Besides these improvements it should also be possible to extend the above time-dependent Hartree-Fock treatment to a study of damping. In case the breathing-mode excitation energy turns out to be above 12 Mev, it will probably be broadened considerably by proton emission, and if above 16 Mev, also by neutron emission.

[12] For a more complete discussion of this type of calculation see Sec. III of reference 8.

[13] This is, of course, an idealization, since the quantum number of excitation cannot be allowed to become comparable to the number of nucleons, which in the present case is only twelve, without considerably altering the excitation properties of the nucleus. We are confident, however, that the applicability of Eqs. (10) and (18) extends on down to the first-excited-state–ground-state transition, and that the restriction to large quantum numbers is only a convenience, enabling the time-dependent Hartree-Fock method to be applied.