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Energy Dependence of Reactions at Thresholds*

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As a development of Wigner's work on the subject, the behavior of nuclear reactions at thresholds is discussed by employing the $\mathcal{R}$-matrix formalism with special attention to the effect of a nuclear reaction threshold in a particular channel on the energy dependence in other channels. In addition to the characteristic cusp-like behavior for $L=0$, $Z_1Z_2=0$ it is found that for this case it is also possible to have another type of plot of cross section *versus* energy. The second type differs from the first by a relative change of sign of the extra effect on one side of the threshold.

THE energy dependence of nuclear reactions at thresholds has been discussed for the case of Coulomb fields by Ostrofsky, Breit, and Johnson,[1] for photomagnetic capture by Fermi,[2] for slow-neutron capture by various authors[3] and more completely and thoroughly in the general case by Wigner.[4] In the last-mentioned paper an essentially new feature is incidentally brought out, *viz.*, the influence of the threshold of one reaction on the energy dependence of another. In the present note this matter is discussed somewhat more fully with an attempt at a more complete enumeration of the possibilities.

The $\mathcal{R}$-matrix approach to nuclear reaction theory[5–7] will be used in a manner very similar to that used in Wigner's paper[4] on nuclear reaction thresholds. The relations of the $\mathcal{R}$-matrix theory[6] essential for the present purpose are

$$u=-[\mathcal{R}B\omega^*+iA]^{-1}[\mathcal{R}B\omega-iA^*], \tag{1}$$

where u is the scattering matrix corresponding to a definite value of the total angular momentum J,

$$B_{p,L}\omega_{p,L}=k_p^{\frac{1}{2}}i^{1-L}H_L'(k_pb_p),$$
$$A_{p,L}=k_p^{-\frac{1}{2}}i^LH_L(k_pb_p), \tag{1.1}$$

$$H_L(\rho)=G_L(\rho)+iF_L(\rho), \tag{1.2}$$

where $F_L(\rho)$, $G_L(\rho)$ are respectively the regular and irregular Coulomb functions in the notation of Yost, Wheeler, and Breit.[8] The $\mathcal{R}$ matrix occurring in (1) has reference to the same J as u. The rows and columns of all matrices dealt with here are labeled with respect to two indices, the first of which is denoted by p and has reference to the fragment pair of the channel. It specifies the states of both fragments as well. The second index is the orbital angular momentum of the channel expressed in terms of $\hbar$ as a unit. The matrices A, B, ω are diagonal and their elements are defined by (1.1) with the additional requirement that B is real. The simplified boundary condition requiring the vanishing of the derivative of r times the radial function at the channel radius b_p is used. The channels may be either open or closed and all channels are supposed to be included in the rows and columns of u and $\mathcal{R}$. One may rewrite (1) in the form

$$u=-\mathbf{e}^{-1}(\mathcal{R}-q)^{-1}(\mathcal{R}-q^*)\mathbf{e}^*, \tag{2}$$

the prime denoting $d/d\rho$ and the following diagonal matrices having been introduced,

$$q=H/(kH'),\quad \mathbf{e}=i^{L-1}k^{\frac{1}{2}}H'. \tag{2.1}$$

Making use of the Wronskian relation between F and G, one has

$$q-q^*=-2i/(kH'H'^*), \tag{2.2}$$

so that (2) may be rearranged as

$$u=-\mathbf{e}^{-1}\mathbf{e}^*+2i\mathbf{e}^{-1}(\mathcal{R}-q)^{-1}\mathbf{e}^{-1}. \tag{2.3}$$

This formula differs from the corresponding one in Wigner's paper[4] only with respect to superficial matters having to do with notation. The quantity q behaves differently for $L=0$, $Z_1Z_2=0$ and other cases. In the former case

$$q=1/(ik),\quad (L=0,\ Z_1Z_2=0). \tag{3}$$

At threshold $k=0$ and $q=\infty$. On the other hand,

$$\lim_{k\to 0}q=-r/L,\quad (L>0,\ Z_1Z_2=0), \tag{3.1}$$

which is finite.

The formula for u will be used first to obtain the asymptotic forms for cases in which the threshold occurs in the initial or in the final channel. The influence of a reaction threshold on another reaction will then be easily understood. Since the right side of (3.1) is independent of k, the dominant term of the off-diagonal elements of (2.3) in an expansion of this quantity in powers of k can be obtained by replacing $(\mathcal{R}-q)^{-1}$ by q^{-1}. In fact, if q is finite the critical dependence arises through other factors than $(\mathcal{R}-q)^{-1}$. If q is infinite as

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[1] Ostrofsky, Breit, and Johnson, Phys. Rev. **49**, 22 (1936).
[2] E. Fermi, Phys. Rev. **48**, 570 (1935).
[3] G. Beck and H. L. Horsley, Phys. Rev. **47**, 510 (1935); H. A. Bethe, Phys. Rev. **47**, 747 (1935); F. Perrin and W. Elsasser, Compt. rend. **200**, 450 (1935); G. Breit and E. P. Wigner, Phys. Rev. **49**, 519 (1936).
[4] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).
[5] E. P. Wigner, Phys. Rev. **70**, 15, 606 (1946); Am. J. Phys. **17**, 99 (1949).
[6] E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947).
[7] T. Teichmann and E. P. Wigner, Phys. Rev. **87**, 123 (1952).
[8] Yost, Wheeler, and Breit, Phys. Rev. **49**, 174 (1936).

in (3), then q^{-1} gives the critical energy dependence of $(\mathcal{R}-q)^{-1}$. To see this more concretely one notes that the matrix u is used with the first subscript referring to the incident channel, the second to the outgoing. Since $\mathbf{e}$ is diagonal the threshold behavior enters only one of the two elements of $\mathbf{e}$ contributing to $\mathbf{e}^{-1}(\mathcal{R}-q)^{-1}\mathbf{e}^{-1}$ in (2.3). If the incident channel 1 is going through a threshold, one is concerned only with $(\mathbf{e}_1)^{-1}[(\mathcal{R}-q)^{-1}]_{1n}$. A consideration of Kramer's rule for solving linear equations shows that if q_1 becomes infinite then all

$$[(\mathcal{R}-q)^{-1}]_{1n}\propto 1/q_1, \quad (q_1\rightarrow\infty). \tag{3.2}$$

Hence $\mathbf{e}^{-1}q^{-1}$ for the threshold channel contains all the critical energy dependent factors in u. The threshold channel thus gives, according to (2.1), the asymptotic form

$$\frac{1}{k^{\frac{1}{2}}}\left(\frac{kdr}{dH}\right)\left(\frac{dH}{Hdr}\right)=\frac{k^{\frac{1}{2}}}{H}, \tag{3.3}$$

where the subscript L is understood to go with H which is supposed to be evaluated at the channel radius. Similarly k refers to the channel in which the threshold occurs. This channel will be referred to as the *critical channel* from now on. If the critical channel is the incident one the square of its wavelength also behaves critically and hence $1/k^2$ enters as an additional critical factor. If, however, particles in the critical channel arise as "new" particles as a result of "old" particles in another channel, the square of the absolute value of the right-hand side of (3.3) can be used directly. The designations of particles as "new" and "old" is the same as in Wigner's paper,[4] the new particles being those in the critical channel. One has thus

$$\sigma(\text{new}\rightarrow\text{old})\propto\frac{1}{k^2}\left|\frac{k}{H_L(kb)}\right|^2, \qquad \sigma(\text{old}\rightarrow\text{new})\propto\left|\frac{k}{H_L(kb)}\right|^2. \tag{3.4}$$

These formulas are equivalent to those stated in Wigner's paper. They may also be written in the form

$$\sigma(\text{new}\rightarrow\text{old})\propto\frac{1}{kG_L^2(kb)}, \qquad \sigma(\text{old}\rightarrow\text{new})\propto\frac{k}{G_L^2(kb)}, \tag{3.5}$$

since at threshold $F_L/G_L=0$. In these formulas there enters the channel radius b. It does not affect the law of energy dependence because at threshold $kb\ll 1$ and the function $G_L\propto k^{-L}$ if there is no Coulomb field, while in the presence of a Coulomb interaction it also approaches a limit with a definite energy dependence. One may write therefore

$$\begin{aligned}\sigma(\text{new}\rightarrow\text{old})&\propto k^{2L-1}\propto E^{L-\frac{1}{2}},\\ \sigma(\text{old}\rightarrow\text{new})&\propto k^{2L+1}\propto E^{L+\frac{1}{2}},\ (Z_1Z_2=0).\end{aligned} \tag{3.6}$$

In the presence of a repulsive Coulomb field there is available[8] an asymptotic form of G_L which was rederived by Beckerley[9] and for which more systematic asymptotic expansions have been correctly surmised in reference 8 and established by Breit and Hull.[10] The dominant term in the asymptotic form is given by

$$\begin{aligned}G_L&=D_L\rho^{-L}\Theta_L;\quad D_L=1\cdot3\cdot5\cdots(2L-1)\\ &\times\left[\left(1+\frac{\eta^2}{L^2}\right)\left(1+\frac{\eta^2}{(L-1)^2}\right)\cdots\left(1+\frac{\eta^2}{1^2}\right)\right]^{-\frac{1}{2}}\times\left(\frac{e^{2\pi\eta}-1}{2\pi\eta}\right)^{\frac{1}{2}},\\ \Theta_L&\sim\frac{2}{(2L)!}\left(\frac{x}{2}\right)^{2L+1}K_{2L+1}(x),\quad x=(8\rho\eta)^{\frac{1}{2}},\end{aligned} \tag{3.7}$$

and K_n is the Bessel function of imaginary argument of the second kind in notation of Whittaker and Watson.[11] It is apparent from this form that for repulsive fields

$$G_L\propto e^{\pi\eta}\eta^{-\frac{1}{2}}, \tag{3.8}$$

and hence

$$\begin{aligned}\sigma(\text{new}\rightarrow\text{old})&\propto k^{-2}e^{-2\pi\eta},\\ \sigma(\text{old}\rightarrow\text{new})&\propto e^{-2\pi\eta},\end{aligned} \tag{3.9}$$

again in agreement with older work. The case of attractive Coulomb fields can be obtained by noting that in (3.7) the $e^{2\pi\eta}-1$ is dominated by the -1 in place of $e^{2\pi\eta}$ so that one may simply omit the factors $e^{-2\pi\eta}$ in (3.9). This results in there being no energy dependence in the dominant term of the (old $\rightarrow$ new) cross section for an attractive Coulomb field. This is a well-known fact in the theory of the absorption of light by atoms, the absorption in the discrete part of the spectrum merging practically continuously with the continuum.

If the critical channel 1 is neither the incident nor the final one and is $L=0$, $Z_1Z_2=0$, the infinite value of q_1 results in first approximation in the matrix element of $(\mathcal{R}-q)^{-1}$ between the incident and emergent channels being such as though the row and column corresponding to the threshold channel did not exist in $\mathcal{R}-q$. This may be seen for example from Kramer's rule. The next approximation involves, however,

$$1/q_1=ik_1 \tag{4}$$

as a factor in a correction term. This may be seen again from the familiar Kramer's rule used as a way of forming the reciprocal of a matrix. The matrix u and the cross section contain therefore the factor $(E-E_1)^{\frac{1}{2}}$. Above the threshold the plot of cross-section *vs* energy

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[9] J. G. Beckerley, Phys. Rev. **67**, 11 (1945). The remark in the paper by Wigner[4] regarding the unavailability of numerical coefficients of the Bessel function must have been caused by failure to notice the direct reduction in the paper by Yost, Wheeler, and Breit.[8]

[10] G. Breit and M. H. Hull, Jr., Phys. Rev. **80**, 392, 561 (1950).

[11] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1920).

has therefore an infinite slope. The value of the coefficient multiplying $(E-E_1)^{\frac{1}{2}}$ depends on the magnitude of the cross-product term arising from $1/q_1$ in u.

Just below threshold the calculations still apply formally even though the threshold channel is closed. It is necessary, however, to take into account the difference in the requirements on the function in the nuclear exterior. This has to be taken as a constant multiple of

$$(H)_{L=0}=\exp(ik_1r)=\exp(-\beta_1r), \tag{4.1}$$

where the quantity ik_1r is meant to be taken in the convention

$$k_1=i\beta_1, \quad \beta_1=|k_1|. \tag{4.2}$$

One may still use

$$(F)_{L=0}=\sin k_1r=i\sinh\beta_1r, \tag{4.3}$$

all other relations working out similarly to the case above threshold. Accordingly there appears in u below threshold the quantity $-\beta_1$ where ik_1 occurs above threshold. Since

$$\beta_1=|(E_1-E)^{\frac{1}{2}}|, \quad (E<E_1) \tag{4.4}$$

the plot of σ against E below threshold should also have an infinite slope at $E=E_1$. In σ the coefficient of β_1 below E_1 is in general different from that of k_1 above E_1 because of the factor i in (4.2). On account of this factor the cross-product terms of k_1 above threshold and β_1 below threshold arise from different parts in the remainder of u differing in phase by 90°. There is therefore no general relationship between the coefficients of these quantities above and below threshold. It will be remembered that u contains in general phase factors which can be made to vary by arbitrary amounts as, for instance, through the introduction of central potentials. There can be therefore no general relationship between the signs of the coefficients of $(E-E_1)^{\frac{1}{2}}$ above threshold and of $(E_1-E)^{\frac{1}{2}}$ below threshold. If the two coefficients have the same sign, the curve of cross section *vs* energy should show a cusp, a phenomenon pointed out by Wigner. If the coefficients have opposite signs, the plot has approximately the appearance of the central portion of the letter S turned on its side.

For $L>0$, $Z_1Z_2=0$ the q_1 is not infinite and the row and column corresponding to the critical channel do not disappear from $(\mathfrak{R}-q)^{-1}$ as they do for $L=0$. This formal difference between $L=0$ and $L>0$ is a consequence of the choice of boundary condition and will not be discussed here further. Its practical significance is however that q_1 enters the formulas in the first approximation. Its dependence on k_1 and $L>0$ can be inferred from

$$q_1^{-1}=k_1(G_L'G_L+F_L'F_L+i)/(G_L^2+F_L^2), \tag{5}$$

it being understood that the argument of F_L and G_L is k_1b_1. The denominator on the right side of (5) contains only even powers of k_1. The combination $G_L'G_L+F_L'F_L$ is therefore odd in k_1. The real part of $1/q_1$ is accordingly even in k_1 while the imaginary part is odd. It is also readily verified that the highest power of $1/k_1$ in H_L is $(1/k_1)^{2L}$. It follows therefore from (5) that the lowest odd power of k_1 in $1/q_1$ is

$$k_1^{2L+1}\propto \text{term of lowest odd power in } k_1. \tag{5.1}$$

This term gives the first nonvanishing effect of a discontinuity in a derivative of the reaction cross section, the even powers of k_1 giving a smooth transition across the threshold. According to (5.1) the Lth energy derivative of the reaction cross section should show therefore the same type of dependence on the energy as the cross section shows for $L=0$, $Z_1Z_2=0$.

The increase in the centrifugal barrier caused by increasing L makes the threshold channel assume some of the characteristics of a stationary state even when $E>E_1$. If the state were truly stationary, there would be no distinction between $E>E_1$ and $E<E_1$. Qualitatively this is the explanation of the difference between $L=0$ and $L>0$. One expects an even smoother transition between the two energy regions in the case of a repulsive Coulomb field. This expectation is in fact confirmed on closer inspection as follows.

For $E>E_n$ the large values of $G_L^2+F_L^2$ make the imaginary part of $1/q_1$ disappear as is clear from (5). Since furthermore F_L^2/G_L^2 contains the square of the barrier penetration factor, it may be neglected. Hence (5) may be replaced by

$$1/q_1\cong k_1G_L'(k_1b_1)/G_L(k_1b_1), \quad (Z_1Z_2>0) \tag{6}$$

which is equivalent to

$$1/q_1\cong -L/r+d\Theta_L/(\Theta_Ldr) \tag{6.1}$$

in the same notation as (3.7). The quantity Θ_L is known[10] to have an asymptotic expansion in powers of $E-E_1$. This series does not represent a part of Θ_L containing $\exp(-2\pi\eta)$ but this quantity is negligible just above threshold.

For $E<E_n$ it may be shown that the Whittaker function satisfying the differential equation for r times the radial function may be expressed as

$$W_{-1/(a\beta),\,L+(\frac{1}{2})}(2\beta r)=\frac{(2L)!\,``\Theta_L"}{\Gamma(L+1+(1/a\beta))(2\beta r)^L}, \tag{6.2}$$

where "Θ_L" is an extension of the usual Θ_L to negative energies and may be expressed as

$$``\Theta_L"=[\Theta_L]_{PS}-\frac{(x/2)^{2L+1}}{(2L)!}\times\int_0^{\beta xa/4}\left[u^2-\beta^2\left(\frac{xa}{4}\right)^2\right]^L\exp\left\{-\frac{x}{2}\left(u+\frac{1}{u}\right)-\frac{x}{2u}\left[\frac{\beta^2}{3}\left(\frac{xa}{4u}\right)^2+\frac{\beta^2}{5}\left(\frac{xa}{4u}\right)^4+\cdots\right]\right\}du. \tag{6.3}$$

Here $[\Theta_L]_{PS}$ is the power series in the energy already referred to. The coefficients of the series are expressible in terms of Bessel functions of imaginary argument of the second kind K_ν. The argument of the K_ν is $(8r/a)^{\frac{1}{2}}$, with $a=\hbar^2/(\mu Z_1Z_2e^2)$ standing for the Bohr length of the channel and μ for the reduced mass. The second part of the right hand side of (6.3) contains an integral which is very small for small β, the factor $\exp[-x/(2u)]$ in the integrand being very small in the range of integration. The part of "Θ_L" which is not expressible by the power series is therefore negligible at threshold. There is therefore no discontinuity in any energy derivative at threshold.

For attractive Coulomb fields the continuous merging of effects of the infinite level density just below threshold with effects of the continuum above threshold leads one to expect the absence of discontinuities in the derivatives of a reaction cross section as a result of the occurrence of a threshold of another reaction. The threshold phenomenon merges in this case with the establishment of an infinite number of new open channels each of these corresponding to the excitation to the discrete level in the attractive Coulomb field. Each of the newly opened channels can have effects of the type already discussed but it is simpler to consider these not as threshold effects of the Coulomb field channel but of other channels.

The difference between effects just below and just above threshold which leads one to expect the possibility of other than cusp-like shapes of the σ *versus* E plots in the discussion immediately after (4.4) is analogous to effects in electromagnetic oscillations of cavities. The critical channel introduces a resistive component into the effective mutual impedance of the cavity if the propagation made in the critical channel is oscillatory. If the mode is attenuated the component is reactive. The two effects are 90 degrees out of phase and combine with different parts of the mutual impedance in the calculation of the current. Hence one has the possibility of different signs of the effect in the nuclear case.

The cusp phenomenon has been looked for by Hemmendinger, Jarvis, and Taschek[12] and by Ennis and Hemmendinger[13] in $p-\mathrm{T}$ scattering and some evidence of it appears to have been found. Employing the data of the latter reference as the more complete and plotting the values of the differential scattering cross section, one sees at least a strong suggestion of the effect for the scattering angle $\theta_{\text{c.m.}}$ of 150° in the center-of-mass system. If, on the other hand, one plots their values for $\theta_{\text{c.m.}}=46°$ or 64.8° against energy, the shape of the plot at $E=1.1(5)$ Mev is suggestive of the S-shaped type. The reason for looking at effects at the energy mentioned is that at this energy the sharp break occurs for $\theta_{\text{c.m.}}=150°$. The possibility of shifts in energy scale during an experiment is present and judgment must be reserved regarding the proper interpretation of the data. It appears likely that ascertaining the nature of the effect experimentally will lead to a more certain interpretation of scattering data, particularly because the two branches of the (σ,E) plot can be used as a measure of the magnitudes of two parts of u which have a 90° phase difference between them. Reasonably restrictive information on u is thus obtainable through the examination of the phenomenon.

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Dr. S. Bashkin has independently noticed that the arguments concerning the cusp apply only to the case $L=0$, $Z_1Z_2=0$.[14] The writer would like to express his appreciation to Dr. Bashkin in this connection as well as for a discussion of some current experimental work on the subject. He would like to thank Dr. M. H. Hull, Jr. for checking the transformation leading to Eq. (6.3) in the text.

The material presented in this note has been systematized in connection with a longer paper for a review article, the publication of which has been delayed. Since completing these considerations the writer has learned from Dr. Roger G. Newton that very similar conclusions concerning the alternative to the cusp for $L=0$, $Z_1Z_2=0$ case have been reached by him through somewhat different considerations; he would like to express his indebtedness to Dr. Newton for this information.[15]

[12] Hemmendinger, Jarvis, and Taschek, Phys. Rev. **76**, 1137 (1949).

[13] M. E. Ennis and A. Hemmendinger, Phys. Rev. **95**, 722 (1954).

[14] S. Bashkin (private communication).

[15] Roger G. Newton (private communication).