densities in the crystals studied were about $10^{17}-10^{18}$ cm^{-3}.

The data for sample 2 do not agree too well in the intrinsic range. This is the result of a decrease in sample cross section and contact area due to the vapor attack mentioned above, causing the conductivity calculated from the original dimensions to be low.

ACKNOWLEDGMENTS

The present work was suggested by a preliminary conductivity experiment performed by R. N. Hall and R. I. Scace of this Laboratory which indicated that intrinsic measurements of silicon carbide were feasible. The use of Mo, W, and their alloys to make ohmic contacts to SiC was suggested by Dr. Hall.

PHYSICAL REVIEW VOLUME 107, NUMBER 6 SEPTEMBER 15, 1957

Range Straggling of High-Energy Electrons in Carbon*

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(Received May 13, 1957)

Approximate range straggling curves for the slowing down of high-energy electrons in carbon have been calculated by a Monte Carlo technique using the University of Illinois digital computer. The effects of ionization straggling, radiation, and multiple scattering have been included. Calculations were made for incident electron energies from 5 to 55 Mev.

INTRODUCTION

IN connection with calculating counter efficiencies in an experiment involving the counting of positrons from the decay of μ^+ mesons,[1] it has been necessary to know the range straggling of high-energy electrons and positrons in low-Z materials. To determine this, a Monte Carlo calculation has been performed using the University of Illinois digital computer. A previous more approximate calculation in this energy region has been made by Steinberger[2] for a special geometry.

Since the single-collision cross sections of electrons with matter are rather well known, the most accurate way to perform this random sampling calculation is to follow a statistical sample of electrons through the material, collision by collision, until the energy of the electrons has been reduced to a sufficiently low value. With the speeds of currently available digital computers, such a procedure is impractical because of the extremely large number of collisions. It is thus necessary to adopt a more appropriate calculational procedure.

While analytical solutions are not available to the general problem of electron penetration through matter, at least at high energies where catastrophic[3] processes are reasonably probable, partial solutions to the diffusion problem have been made for thin foils.[4–6] These distributions have the general form of a "head" due to the addition of multiple collisions involving small but reasonably probable changes, and a "tail" due primarily to rare single collisions involving large changes.

The procedure adopted for the calculations presented here is to break up the path of each particle into a large number of thin foils and to determine the ionization loss, radiation, and multiple scattering by random sampling of the thin-foil distribution functions. By following the path for each incident particle chosen, we can determine the penetration along the original direction of incidence.

CALCULATIONAL PROCEDURE

We adopt spherical coordinates (r,θ,φ) and assume monoenergetic electrons incident in the $+Z$ direction $(\theta=0)$ upon a semi-infinite slab of material. Consider an increment Δ along the true path of the electron. In this increment of path the electron will suffer energy loss by ionization and by radiation and will suffer deflection due to multiple scattering, the amount of energy loss and of scattering being subject to statistical fluctuations. It is assumed for this calculation that these processes may be considered as independent. While this is a good assumption for small energy losses and scatterings, the validity of this assumption is questionable for large energy losses or scatterings since they correspond to discrete events and some degree of correlation exists between the different processes considered.

It is assumed that the true path length Δ may be approximated by the chord connecting the two ends of this increment of path. Thus the path of the electron is thought of as a series of straight-line segments of length Δ connected end to end. Yang[4] has estimated the

* Supported in part by the joint program of the U. S. Atomic Energy Commission and the Office of Naval Research, and by the National Science Foundation.

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[1] J. E. Leiss, Ph.D. thesis, University of Illinois, 1954 (unpublished); Leiss, Robinson, and Penner, Phys. Rev. **98**, 201 (1955).

[2] J. Steinberger, Phys. Rev. **75**, 1139 (1949).

[3] By a catastrophic process we mean one in which a single collision can radically influence the subsequent history of the particle.

[4] C. N. Yang, Phys. Rev. **84**, 599 (1951).

[5] L. Landau, J. Phys. U.S.S.R. **8**, 201 (1954).

[6] L. Eyges, Phys. Rev. **76**, 264 (1949).

error in using the length Δ rather than the true path. For these calculations this error is less than one percent and is neglected.

In the increment of path Δ_i the electron is first allowed to lose energy by ionization at random according to a probability distribution to be described, the initial energy of the electron upon entering this path-length bin being used in the calculation. Using the energy at the end of the bin, one then allows the electron to radiate at random by using a different probability distribution. The average of the initial and final energies in this bin is then used to compute a random scattering at angle (θ_i,φ_i) according to a Gaussian probability distribution[7] in θ_i and a uniform distribution in φ_i. The (θ_i,φ_i) are measured relative to the direction (θ_0,φ_0) in which the electron was moving upon entering the path-length bin Δ_i. The direction θ_0' in which the electron is moving after this deflection, relative to the Z direction, is given by

$$\cos\theta_0' = \cos\theta_0 \cos\theta_i + \sin\theta_0 \sin\theta_i \cos\varphi_i. \quad (1)$$

The new angle (θ_0',φ_0') then becomes the incident angle (θ_0,φ_0) for the next path-length bin, etc. The motion along the Z direction in path-length increment Δ_i is given by

$$(\delta Z)_i = \Delta_i \cos\theta_0'. \quad (2)$$

In this manner the electron is followed through the material from bin to bin, the energy and angle with which the electron leaves one bin becoming the input parameters for the next bin. The total distance of travel Z_j along the Z direction at any one bin j is given by

$$Z_j = \sum_{i=1}^{j} (\delta Z)_i. \quad (3)$$

For calculational simplicity we do not follow electrons that achieve a total deflection beyond 75°. Trial calculations indicate that the results are insensitive to the value of this cutoff. Similarly, we do not follow electrons below 1 Mev. The results do depend on the value of this energy cutoff, but the range of 1-Mev electrons is sufficiently small in comparison to the total distance achieved that the electrons may be considered as essentially stopped.

In the present calculations the above procedure was carried out one thousand times for each incident electron energy, to obtain the range straggling distributions. To give some idea of the computing times involved, consider the case of 50-Mev electrons. The average distance $(\bar{Z})$ traveled by electrons of this energy in carbon of density 1.697 g/cm³ is about 13 cm. For $\Delta_i = \frac{1}{4}$ cm there are about 50 path-length bins along the path. For each path-length bin four quantities were chosen at random and six sines or cosines calculated, or a total of about 2.5×10^5 operations for 1000 incident electrons. For the code written, each of these processes could be done in an average time of about 3.5 milliseconds. The total time is thus about 28 minutes for 50-Mev electrons, or an average of about 15 minutes per energy for the energies computed.

It should be noted that this relatively short time per operation of 3.5 milliseconds was achieved by consulting probability tables previously placed in the computer. To do this it was necessary to use thin-foil distributions which could be represented to a large extent by universal probability tables. Sines and cosines were also obtained by consulting tables placed in the computer rather than using the usual series type of calculations. The saving in computing time from this procedure is about a factor of ten.

Thin-Foil Straggling Distributions

The three thin-foil straggling distributions used in these calculations are for ionization loss, radiation, and multiple scattering. The predominant one of these is ionization loss. For the case of carbon which was considered here, radiation is not a predominant effect; however, the effects of radiation are quite noticeable on the straggling distributions, being primarily responsible for the loss of electrons early in their range. For these fairly high energies, multiple scattering has the primary effect of shortening the range of the electrons without greatly increasing the width of the straggling distribution.

The straggling distributions will now be considered.

Ionization Loss

The ionization straggling distribution used was that given by Landau[5] with some modification. Landau gave a universal straggling distribution which is a function of the most probable energy loss ΔE_p, a parameter S_0 which is proportional to the number of electrons in the foil and which is chosen such that the probability for a collision of energy loss S_0 is unity, and the actual energy loss ΔE. The path-length bins Δ used were $\frac{1}{4}$ cm carbon of density 1.697 g/cm³. For this case $\Delta E_p = 0.588$ Mev and $S_0 = 0.0325$ Mev, where ΔE_p is calculated by using the expression,

$$\Delta E_p = 0.1537(\sum Z/\sum A)D[19.43+\ln(D/C)], \quad (4)$$

given by Goldwasser, Mills, and Hanson[8] who have experimentally confirmed the distribution of Landau for not too large energy losses and an incident electron energy of 15.7 Mev. Here D is the thickness of the foil in g/cm² and C is the density in g/cm³. The above expression for ΔE_p is corrected for the density effect.

The large energy-loss tail on the Landau distribution was approximated by a function proportional to

[7] In a previous but similar unpublished calculation we found that the use of the multiple scattering theory of H. S. Snyder and W. T. Scott [Phys. Rev. **76**, 220 (1949)] rather than the Gaussian approximation did not greatly influence the straggling distribution.

[8] Goldwasser, Mills, and Hanson, Phys. Rev. **88**, 1137 (1954).

TABLE I. Collision probabilities for $\frac{1}{4}$-cm carbon for 20-Mev electrons and positrons.

E' (Mev)	ϕ_{e^-} (theory)	ϕ_{e^+} (theory)	ϕ (high tail)	ϕ (low tail)
3	38.4×10^{-4}	27.8×10^{-4}	18.7×10^{-4}	9.91×10^{-4}
5	15.3×10^{-4}	9.56×10^{-4}	5.94×10^{-4}	3.15×10^{-4}
7	9.39×10^{-4}	5.12×10^{-4}	3.27×10^{-4}	1.73×10^{-4}
10	7.33×10^{-4}	3.66×10^{-4}	2.82×10^{-4}	1.45×10^{-4}

$(\Delta E-\Delta E_p)^{-2}$ in the region $\Delta E=0.9$ Mev to E_0, where E_0 is the incident electron energy. Energy losses greater than E_0 were forbidden. If the energy loss ΔE is greater than $E_0/2$, the energy of the secondary electron is approximately $E_0-(\Delta E+\Delta E_p)$. In these cases the secondary electron was followed for the remainder of the path in the material rather than the original particle.

Two fits to this straggling tail were made, the first called the "high tail" being matched smoothly to the Landau distribution at $\Delta E=0.9$ Mev; the second called the "low tail" being only 53% as large as the high tail. Although the use of the low tail came about as a computing blunder, it is not obvious which tail should give the better approximation to the distribution which one should use for this problem. The probability per g/cm^2 for a positron of energy E to undergo a single collision from which either the electron or the positron emerges with energy in dE' at E', for $\beta^2\approx1$, is[9]

$$\phi_{\text{col}}(E,E')dE'=2Cm_ec^2E^2(E-E')^{-1}(E')^{-2}dE' \times[1-E'/E+(E'/E)^2][1-2E'/E+2(E'/E)^2], \quad (5)$$

where $C=0.150Z/A$ g^{-1} cm^2. For electrons the same expression is obtained except that the factor $[1-2E'/E+2(E'/E)^2]$ is not present. Table I presents the probability of energy loss E' for $E_0=20$ Mev in one $\frac{1}{4}$-cm carbon bin, as calculated by the expressions above, for the low and high tail. Thus for this particular case, the high tail is closer to the expected value of the tail.

Another useful comparison is to consider the average energy loss for these two distributions compared to that predicted theoretically. This comparison is given in Table II where the density correction of Halpern and Hall[10] was applied to the theoretical values. The average energy loss per g/cm^2 (k_{col}) used is that given by Rossi[9] with $I(Z)=13.5Z$:

$$k_{\text{col}}=2Cm_ec^2\{\ln[\pi^2(m_ec^2)^2/(1-\beta^2)^{\frac{3}{2}}I^2(Z)]-a\},\ \beta\approx1 \quad (6)$$

where $a=2.9$ for electrons, $a=3.6$ for positrons. From this table it is seen that neither of the two distributions gives the correct average energy loss at all energies; however, the distribution with the low tail gives a more nearly correct average energy loss, especially for the positron case for which these calculations were primarily made.

[9] B. Rossi, *High Energy Particles* (Prentice-Hall, Inc., New York, 1952).

[10] O. Halpern and H. Hall, Phys. Rev. **73**, 477 (1948); see also reference 9.

This phase of these calculations has been left in a somewhat unsatisfactory state; however, it should be said that the effect of uncertainties in the high-energy-loss tail on the range straggling distributions calculated should not be too great because of the small probability of having energy losses in this tail region. The tail region of the ionization straggling distribution used starts at $\Delta E=0.9$ Mev. The probability of having energy loss in this region is about 12% in any one foil, while the probability of having an energy loss greater than 3 Mev is only about 0.2%. Thus a large preponderance of the energy losses in this tail region are relatively small energy losses and should not be considered as catastrophic events for these calculations. This leads to the conclusion that one should adjust the tail so as to attempt to keep the average ionization loss correct. For this reason most of these calculations have been made using the low tail.

Radiation

The radiation probability distribution used was the function $\pi_{02}(E_0,E,t)$ given by Eyges[6] for the values of the parameters $a=0.25$, $b=\frac{4}{3}$. For the bin width

TABLE II. Average energy loss, Mev/(g/cm^2).

E_0 (Mev)	$\langle\Delta Ee^-\rangle_{\text{Av}}$ (predicted)	$\langle\Delta Ee^+\rangle_{\text{Av}}$ (predicted)	$\langle\Delta E\rangle_{\text{Av}}$ (high tail)	$\langle\Delta E\rangle_{\text{Av}}$ (low tail)
5	1.58	1.53	1.74	1.63
10	1.67	1.62	1.81	1.67
20	1.74	1.69	1.87	1.70
50	1.83	1.78	1.91	1.73

$\Delta=\frac{1}{4}$-cm carbon, this function becomes

$$\pi_{02}(\nu)d\nu=0.01278(1-\nu)^{\frac{1}{4}}[\ln(1-\nu)^{-1}]^{-0.9874}d\nu, \quad (7)$$

where ν=fractional energy lost by radiation. The quantity $\pi_{02}(\nu)d\nu$ is the probability that an electron loses between ν and $\nu+d\nu$ of its initial energy. Therefore, the energy remaining to the electron is $(1-\nu)E_0$. This function is considered accurate enough for these calculations since in carbon radiation is not the major effect, although it is the primary process by which electrons are lost during the first portion of their range. As an example, the probability of a 20-Mev electron in one path-length bin having ionization loss >3 Mev is about 0.2% while the probability of radiation loss >3 Mev is 1.7%. Making the same comparison for energy loss >0.5 Mev, the probability of ionization >0.5 Mev is 100%, while the probability of radiation loss >0.5 Mev is only 3.6%. The primary effect of radiation is thus to remove a small fraction of the electrons in each path-length bin.

Multiple Scattering

The multiple-scattering distribution assumed was that given by Rossi[9]:

$$P(\theta)d\theta=(2\pi X)^{-\frac{1}{2}}\theta_s^{-1}\exp(-\theta^2/2\theta_s^2X), \quad (8)$$

where $\theta_s^2 = (21.2 \text{ Mev})^2/(\beta cp)^2$ and X is the thickness of the foil in radiation lengths (0.00951 for the bins considered). The quantity (βcp) was approximated by E_T, the total energy of the electrons. The $2\theta_s^2$ instead of θ_s^2 in the denominator of the exponential arises because Rossi considers a projected scattering distribution while this calculation does not.

The large-angle scattering tail, such as appears in the theories of Snyder and Scott[11] or of Molière,[12] is not included in this distribution. This corresponds to the neglect of electron-nuclear collisions and should produce no large error.[7] A proper way to include this type of event would be to consider the angle-energy relations of such collisions in detail. The probability of events which need to be considered in this way is sufficiently small that the extra effort was not considered justified.

Results

Figure 1 shows integral straggling distributions, $R(Z,E_0)$, calculated for an incident electron or positron

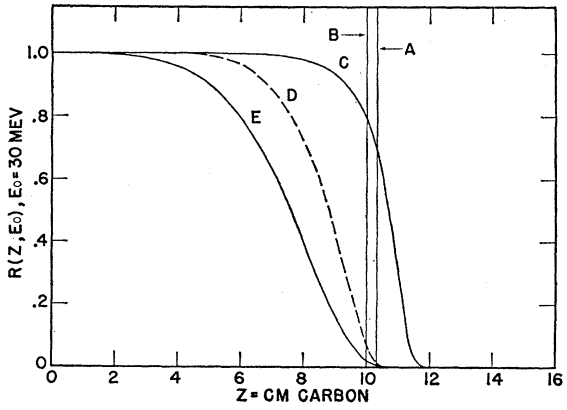

FIG. 1. Integral range-straggling distributions for an incident electron or positron energy of 30 Mev showing the effects of each straggling effect. A and B are positron and electron ranges, respectively, calculated from Eq. (6) and the density correction of Halpern and Hall. C, D, and E include successively ionization, multiple scattering, and radiation straggling.

energy of $E_0 = 30$ Mev, where $R(Z,E_0)$ is the fraction of the incident flux reaching Z or greater. Curves A to E show the effect of including the various straggling effects in the calculation. Curves A and B show the range calculated, neglecting straggling, for positrons and electrons respectively, using Eq. (6) and the density correction of Halpern and Hall[10] in the expression

$$\int_{E_0=30 \text{ Mev}}^{1 \text{ Mev}} \left(\frac{dx}{dE}\right) dE = \int_{E_0=30}^{1 \text{ Mev}} \left(\frac{1}{k_{\text{col}}}\right) dE.$$

The upper limit of integration of 1 Mev was chosen to agree with the energy limit adopted for the straggling calculations. Curves C, D, and E show the effect of adding successively ionization straggling, multiple scattering, and radiation straggling.

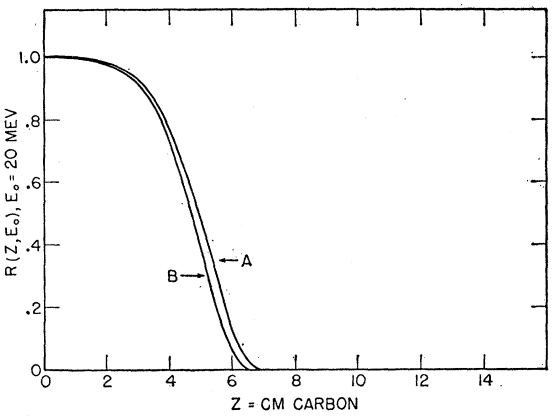

FIG. 2. Integral range-straggling distributions for an incident electron or positron energy of 20 Mev. A and B are calculated using the low and high tails, respectively, of the ionization straggling distributions.

Defining an average energy loss as the initial energy divided by the average distance traveled in a straggling distribution, curves A, B, and C (which includes only ionization straggling) may be used to compare the average energy loss of electrons, positrons, and the ionization straggling distributions used. The average energy loss of curve C is 3.7% less than for curve A (positrons) and 7.0% less than of curve B (electrons). Since these calculations were intended primarily for positron ranges, this difference of 3.7% was not corrected. (It is interesting to note that the experimental straggling data of Goldwasser *et al.*[8] for both polystyrene and aluminum at 15.7 Mev and 9.6 Mev all show the most probable energy loss to be about 2 to 3% less than the most probable energy loss predicted with the Fermi density correction. The average discrepancy is 2.8% although this is near the limits of accuracy of the data.)

Figure 2 shows the effect of using the "low tail" (curve A) on the ionization straggling distribution compared to the use of the "high tail" (curve B) for $E_0 = 20$ Mev. The average energy loss differs in these two cases by about 8%, which is about the difference in the extrapolated ranges. As was expected, the difference between these two tails on the straggling distribution is primarily one of change in average energy loss, i.e., the two curves have essentially the same shape. Thus, from Figs. 1 and 2 one would expect errors in the extrapolated range due to errors in the ionization distribution used to be less than 3% in the extrapolated range for positrons. One would also expect the extrapolated range for electrons to be about 3% less than that for positrons. The latter statement is somewhat incorrect since the ionization straggling distribution for positrons should be slightly narrower than for electrons because the decreased high-energy tail for positron-electron scattering compared to electron-electron scattering.

The authors are not aware of any published experi-

[11] H. S. Snyder and W. T. Scott, Phys. Rev. **76**, 220 (1949).
[12] G. Molière, Z. Naturforsch. **3a**, 78 (1948); **2a**, 133 (1947).

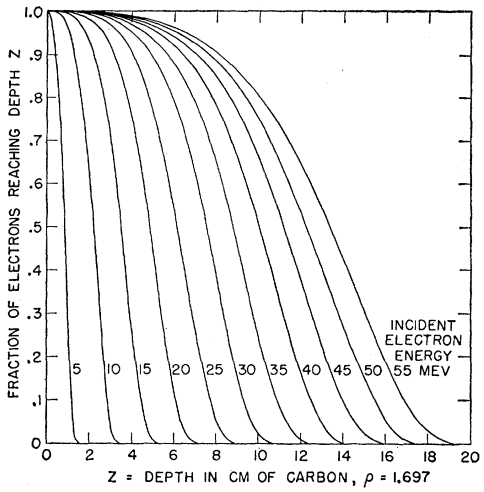

FIG. 3. Final integral range-straggling distributions for incident electron or positron energies from 5 to 55 Mev. Note that these distributions should actually be somewhat different for electrons and positrons.

mental data which can be compared with these calculations directly. An attempt to compare with depth dose measurements was not very successful. This is to be expected since what is measured in depth dose measurements is not at all the same thing calculated here. The calculations presented here give the probability for an electron to reach a certain depth or greater, while depth dose measurements determine the ionization produced at a certain depth.

Figure 3 shows straggling distributions calculated for a series of incident electron energies from 5 Mev to 55 Mev. These curves are all calculated by using the low ionization tail. The distributions for 5 Mev and 10 Mev probably contain appreciable errors due to neglect of the diffusion of the electrons near the end of their range. These effects should become relatively smaller as the energy of the electrons is increased. Note that the electrons are considered to have stopped when their energy falls below 1 Mev.

From the curves of Fig. 3 one may obtain Fig. 4, in which is plotted a curve of extrapolated range *versus* electron energy, which plot is well fitted by a straight line. An average energy loss calculated from the slope of this line is in good agreement with the average energy loss of electrons in this energy region. This is in apparent disagreement with the observations of Katz and Penfold[13] on the extrapolated range of electrons in aluminum. These authors found the average energy loss calculated from a range-energy curve to be too large. That the agreement found in these calculations is purely accidental can be seen by an examination of Fig. 1, where it can be seen that an extrapolated range will depend strongly on the relative sizes of the spread due to ionization straggling and the range shortening due to multiple scattering. These calculations would thus tend to predict the observation of Katz and Penfold since in aluminum the range shortening due to multiple scattering would be relatively larger.

[13] L. Katz and A. S. Penfold, Revs. Modern Phys. **24**, 28 (1952).

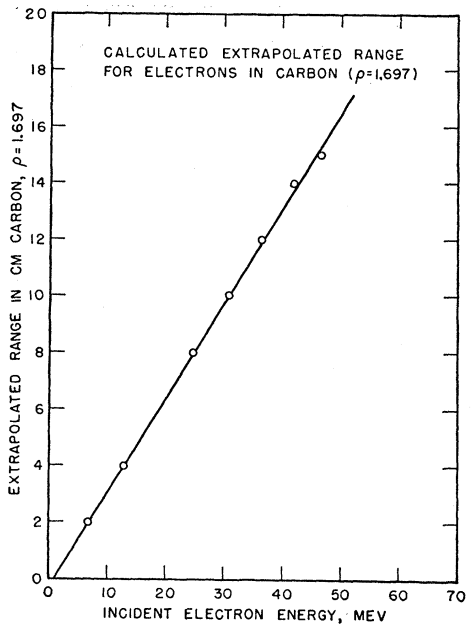

FIG. 4. Extrapolated range in carbon *versus* incident electron energy. The points are derived from Fig. 3. The straight line is a best fit to the points.

Tables of the straggling distributions shown in Fig. 3, in steps of $\frac{1}{4}$ cm and 1 Mev, may be obtained from the last-named author upon request.

ACKNOWLEDGMENTS

We would like to thank Professor A. O. Hanson for many helpful discussions during the course of these calculations. We would also like to extend our thanks to the staff of the University of Illinois Digital Computer who made these calculations possible.