Magnetoresistance of Holes in Germanium and Silicon with Warped Energy Surfaces*

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The series-expansion method which has been described in an earlier paper has been used to obtain expressions for the magnetoconductivity coefficients in the case of warped energy surfaces and low magnetic fields. Using the experimental values of Hall coefficient and conductivity, and assuming a relaxation time of the form $\tau=l\epsilon^{-\lambda}$, where l is a temperature-dependent coefficient, ϵ is the energy of the carriers, and λ is a parameter which most suitably represents the scattering processes, we have calculated the magnetoresistance coefficients in p-type germanium and silicon. The results of these calculations are used to interpret the experimentally derived directional magnetoresistance effects and also the variation of magnetoresistance with magnetic field in these materials.

I. INTRODUCTION

IN an earlier paper[1] we described a method for carrying out calculations of the statistical properties and galvanomagnetic effects of germanium and silicon which is valid for weak magnetic fields and takes into account the warping of the energy surfaces. In that paper we dealt specifically with problems related to the hole densities, conductivity, effective masses, intrinsic carrier concentration, and Hall coefficient. We shall now apply similar techniques to the interpretation of the directional effects of magnetoresistance in p-type germanium and silicon and also to the variation of the magnetoresistance with magnetic field.

II. MAGNETOCONDUCTIVITY COEFFICIENTS

The calculation of magnetoresistance requires a knowledge of the magnetoconductivity coefficients. The expressions for these coefficients which are defined in the manner of Abelès and Meiboom[2] are summarized in Appendix A. We need two sets of such coefficients, one set for the heavy holes and the other, for light holes. We shall assume that the total conductivity is the simple sum of these two contributions. For our purposes we shall ignore interband scattering except in that it may be described by a suitable energy dependence of the relaxation time.

Calculations of magnetoconductivity coefficients have been made assuming that the energy ϵ can be expressed as a function of the wave vector $\mathbf{k}$ in the following form[3]:

$$\epsilon=-\frac{\hbar^2}{2m_0}\{Ak^2\pm[B^2k^4+C^2(k_x^2k_y^2+k_x^2k_z^2+k_y^2k_z^2)]^{\frac{1}{2}}\}, \quad (1)$$

and that the expressions under the radical can be expanded to give

$$\epsilon=-\frac{\hbar^2k^2}{2m_0}(A\pm B')\times\left\{1-\gamma\left[\frac{k_x^2k_y^2+k_x^2k_z^2+k_y^2k_z^2}{k^4}-\tfrac{1}{6}\right]+\cdots\right\}. \quad (2)$$

Here the plus sign is associated with the light holes and the minus sign, with the heavy holes. The $\mathbf{k}$ coordinate system is coincident with the cubic axes, m_0 is the mass of the free electrons and A, B, and C are constants determined experimentally[4,5] to have the following values:

For germanium: $A=13$, $B=8.7$, and $C=11.4$;
for silicon: $A=4.1$, $B=1.4$, and $C=3.7$.

B' and γ are defined in terms of A, B, and C in the following manner:

$$B'\equiv(B^2+\tfrac{1}{6}C^2)^{\frac{1}{2}},\qquad \gamma\equiv\mp\tfrac{1}{2}C^2/B'(A\pm B'). \quad (3)$$

For the relaxation time τ we have assumed the form

$$\tau=l\epsilon^{-\lambda},$$

where l is a constant and λ is a parameter which is chosen to represent the type of scattering. Since the scattering is actually due to a combination of processes, one needs a combination of terms. For example, when both lattice and impurity scattering need be considered, the relaxation time is given by

$$1/\tau=1/\tau_l+1/\tau_i, \quad (4)$$

where τ_l and τ_i are the lattice and impurity scattering terms, respectively.

The method of carrying out the integrations which are necessary to evaluate the conductivity coefficients has already been described in our previous paper,[1] where expressions are given for the hole density p,

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† Staff members.
[1] B. Lax and J. G. Mavroides, Phys. Rev. **100**, 1650 (1955).
[2] B. Abelès and S. Meiboom, Phys. Rev. **95**, 31 (1954).
[3] Dresselhaus, Kip, and Kittel, Phys. Rev. **95**, 568 (1954).
[4] Dexter, Zeiger, and Lax, Phys. Rev. **95**, 557 (1954).
[5] R. N. Dexter and B. Lax, Phys. Rev. **96**, 223 (1954).

zero-field electrical conductivity σ_{ij}, and Hall conductivity σ_{ijl} coefficients. The magnetoconductivity coefficients are evaluated by using the same general procedure. From symmetry considerations it can be shown that there may be a maximum of four different nonvanishing magnetoconductivity components for any cubic crystal.[2] Actually, however, measurements depend only on the sum of σ_{xyyx} and σ_{xyxy} so that measurements are determined by at most three independent parameters. For the holes, whose energy surfaces are warped spheres centered at $k=0$, these become the following:

$$\begin{aligned}
\sigma_{xxxx}&=-0.296a\gamma^2(1-0.4295\gamma+0.0188\gamma^2+0.0103\gamma^3\\
&\quad+0.00249\gamma^4+0.000474\gamma^5+0.000085\gamma^6+\cdots),\\
\sigma_{xxyy}&=-0.213a(1-0.2214\gamma+0.3838\gamma^2-0.0167\gamma^3\\
&\quad+0.00755\gamma^4+0.000661\gamma^5-0.000190\gamma^6-\cdots),\\
\sigma_{xyyx}&=0.148a\gamma^2 \text{ (same } \gamma \text{ series as } \sigma_{xxxx}),\\
\sigma_{xyxy}&=0.213a(1-0.0500\gamma-0.0469\gamma^2+0.0040\gamma^3\\
&\quad-0.00063\gamma^4+0.000114\gamma^5+0.000004\gamma^6+\cdots),
\end{aligned} \tag{5}$$

where

$$a=\frac{\pi e^4\exp(\epsilon_v-\epsilon_F)/KT}{h^3c^2(2m_0)^{\frac{3}{2}}}l^3(KT)^{\frac{3}{2}-3\lambda}\Gamma(\tfrac{5}{2}-3\lambda)(A\pm B')^{\frac{3}{2}},$$

ϵ_v is the energy at the top of the valence band, ϵ_F is the Fermi energy level, K is Boltzmann's constant, T is the temperature, and $\Gamma(n)$ represents the gamma function.

The number of terms in the above series that is necessary in any particular case depends on the value of the anisotropy parameter γ. For the heavy holes in germanium, where γ $(=2.1)$ is relatively large, terms above the sixth term contribute less than two percent.

Once these conductivity coefficients have been found for both the heavy and light holes, the current density J_i can be written as

$$J_i=\Sigma_{ij}E_j+\Sigma_{ijl}E_jH_l+\Sigma_{ijlm}E_jH_lH_m\cdots, \tag{6}$$

where the Σ coefficients represent the sum of the individual σ coefficients for the two kinds of holes, and E_j and H_l are the electric and magnetic field components; respectively. Then following Abelès and Meiboom, Eq. (6) may be inverted to give the usual Hall and magnetoresistance coefficients in the inverse expansion.

$$E_i=\Lambda_{ij}J_j+\Lambda_{ijl}J_jH_l+\Lambda_{ijlm}J_jH_lH_m+\cdots. \tag{7}$$

The Λ coefficients in Eq. (7) are determined in terms of the Σ coefficients by substituting Eq. (7) into Eq. (6) and equating terms of like orders in H. For reference these Λ's, although already published,[2] are given in Appendix B.

The magnetoresistance coefficients are defined as[6]

$$\frac{\rho-\rho_0}{\rho_0H^2}=\frac{1}{H^2}\left[\frac{(E_iJ_i)}{(E_iJ_i)_{H=0}}-1\right],$$

where ρ and ρ_0 are the resistivities with and without the magnetic field. For the case in which the current and magnetic field are along cubic axes, examination of Eq. (7) shows that the longitudinal magnetoresistance coefficient is

$$M_{100}/H^2=-\Sigma_{xxxx}/\sigma_0, \tag{8}$$

where $\sigma_0=1/\rho_0$, and that the transverse magnetoresistance coefficient is

$$\frac{M_{100}{}^{010}}{H^2}=-\frac{\Sigma_{xxyy}}{\sigma_0}-\left(\frac{\Sigma_{xyz}}{\sigma_0}\right)^2. \tag{9}$$

For the transverse case superscripts on the magnetoresistance coefficient denote the magnetic-field direction and the subscripts the current direction.[2] The coefficients for directions other than cubic may be found by transforming the Σ's to other coordinates. In this manner we find, for example, that

$$\begin{aligned}
\frac{M_{110}}{H^2}&=-\frac{1}{2\sigma_0}[\Sigma_{xxxx}+\Sigma_{xxyy}+(\Sigma_{xyyx}+\Sigma_{xyxy})],\\
\frac{M_{110}{}^{\bar{1}10}}{H^2}&=-\frac{1}{2\sigma_0}[\Sigma_{xxxx}+\Sigma_{xxyy}-(\Sigma_{xyyx}+\Sigma_{xyxy})]\\
&\quad-\left(\frac{\Sigma_{xyz}}{\sigma_0}\right)^2,
\end{aligned} \tag{10}$$

and in general, independent of specific model, $M_{100}{}^{010}/H^2=M_{110}{}^{010}/H^2$.

III. COMPARISON WITH EXPERIMENT

The experimental data available at the present time for comparison with theory are somewhat limited, particularly in the region of low fields for which the theory is most suitable. Consequently, we shall compare the results of our analysis with the measurements reported by Pearson and Suhl[6] on p-type germanium at 77°K and 300°K and with the data of Pearson and Herring[7] on p-type silicon at 78°K.

Let us consider the magnetoresistance coefficients in germanium. As indicated previously, these coefficients depend strongly on the relaxation time τ. Assuming that $\tau=l\epsilon^{-\lambda}$, it is evident that for calculations one needs to know four quantities, namely l and λ for the two kinds of holes. The simplest assumption that can be made concerning the relaxation time is that it is independent of energy and the same for both the light and heavy holes. One might use this as an approximate representation of the situation at 77°K where actually the scattering consists of both lattice and impurity. In order to determine a suitable value of τ for describing the situation, the relaxation time was determined by matching the largest coefficient, namely $M_{100}{}^{010}/H^2$, with the experimental data; this gave a value for

[6] G. L. Pearson and H. Suhl, Phys. Rev. **83**, 768 (1951).

[7] G. L. Pearson and C. Herring, Physica **20**, 975 (1954).

TABLE I. Summary of low-field magnetoresistance calculations in germanium at 77°K.

Coefficient	Experimental (Pearson and Suhl)	Energy independent relaxation time[a]	Both holes impurity scattered[b]	Light hole, impurity scattered. Heavy hole, lattice scattered[c]
$(M_{100}/H^2)\times10^9$	0.14[d]	0.093	0.098	0.096
$(M_{110}/H^2)\times10^9$	2.0	3.25	3.95	3.49
$(M_{100}^{010}/H^2)\times10^9$	30.4	30.4	30.4	30.4
$(M_{110}^{110}/H^2)\times10^9$	27.0	27.4	27.0	27.0
$R_0\times10^3$	4.4	2.95	2.2	2.1

[a] $\tau_L=\tau_H=1.03\times10^{-12}$ sec.
[b] $\bar{\tau}_L=4.0\times10^{-13}$ sec, $\bar{\tau}_H=4.9\times10^{-13}$ sec.
[c] $\bar{\tau}_L=4.7\times10^{-13}$ sec, $\bar{\tau}_H=0.86\times10^{-13}$ sec.
[d] See reference 2, Table II.

$\tau=1.03\times10^{-12}$ sec. Using this value of relaxation time, the other coefficients were calculated. The results are given in Table I; it is seen that the fit in this case is reasonable and that semiquantitatively the anisotropy features of the magnetoresistance coefficient are fairly well described. Included in Table I are results for other assumptions concerning the type of scattering. In these cases it is assumed that the relaxation times for the two holes are not necessarily equal. In order to obtain the appropriate values of relaxation parameters the theoretical expressions for the two transverse magnetoresistance coefficients are equated to the experimental values. This was the scheme utilized when impurity or lattice scattering was the principal mechanism for either one or both carriers. The energy-independent scattering time which in essence is a phenomenological approximation to the true situation seems to give a slightly better description than the others when only one type of scattering is considered for each of the holes.

In Table II we give a comparison between theory and experiment for germanium at 300°K. Following the argument proposed by Brooks[8] we have assumed in the case of lattice scattering that the relaxation time for both holes is the same. We have for purposes of comparison also used an energy-independent relaxation time. Both sets of calculations again account semiquantitatively for the anisotropy, with lattice scattering as one might expect giving a better value for R_0, the Hall coefficient.

TABLE II. Summary of low-field magnetoresistance calculations in germanium at 300°K.

Coefficient	Experimental (Pearson and Suhl)	Both holes lattice scattered[a]	Energy-independent relaxation time[b]
$(M_{100}/H^2)\times10^9$	0.04	0.004	0.004
$(M_{110}/H^2)\times10^9$	0.21	0.12	0.14
$(M_{100}^{010}/H^2)\times10^9$	1.3	1.3	1.3
$(M_{110}^{110}/H^2)\times10^9$	1.3	1.2	1.2
$R_0\times10^4$	8.4	8.8	10.3

[a] $\bar{\tau}_L=\bar{\tau}_H=1.51\times10^{-13}$ sec.
[b] $\tau_L=\tau_H=2.11\times10^{-13}$ sec.

[8] H. Brooks, *Advances in Electronics and Electron Physics* (Academic Press, Inc., New York, 1955), Vol. VII, p. 152.

We have made a similar comparison for silicon at 78°K; the results are given in Table III. Without a knowledge of the Hall coefficient, it is difficult to calculate the scattering parameters for each of the carriers separately. Partly because of this and partly because of the experience in germanium where it was possible to make calculations and the τ's came out nearly equal, we have made the simplifying assumption that even for the impurity case the average relaxation times for light and heavy holes are equal. Here again the most suitable value for the relaxation time was obtained by equating the experimental value of M_{100}^{010}/H^2 to the theoretical expression. We have obtained again fairly good semiquantitative agreement between theory and experiment with a slightly better fit for the assumption of an energy-independent relaxation time.

Pearson and Suhl also give the variation of magnetoresistance with magnetic field and this variation for the coefficient M_{100}^{010}/H^2 at 77°K is given in Fig. 1. It is seen that M_{100}^{010}/H^2 starts out at some value at $H=0$ and then decreases, finally leveling off when the magnetic field reaches about 4000 oersteds. At low fields both the light and heavy holes are effective in the magnetoresistance. As the field increases the light holes move in tighter orbits and their contribution to the magnetoresistance decreases. We have made calculations of the changes in the transverse magnetoresistance coefficient with magnetic field considering that this change is due primarily to the light holes. In our first calculation we assumed spherical energy surfaces for the light holes and for both holes a constant relaxation time $\tau=1.03\times10^{-12}$ sec (the value used in the calculation of Table I). This value of τ gave the curve labeled 1 in Fig. 1. A better fit (curve 2) was obtained by choosing a lower value for $\tau(=0.88\times10^{-12}$ sec). The third curve (curve 3) shows the effect of an approximate calculation which considers the slight warping of the energy surfaces of the light holes ($\tau=0.88\times10^{-12}$ sec). It will be noted that the warping of the light hole effects primarily the low-field end of the magnetoresistance curve, the high-field end being determined principally by the heavy holes.

Calculations of M_{100}^{010}/H^2 as a function of the magnetic field have also been made assuming constant τ for the light hole and either lattice or impurity scattering

TABLE III. Summary of magnetoresistance calculations in silicon at 78°K.

Coefficient	Experimental Pearson and Herring ($B=4400$ gauss)	Energy-independent relaxation time[a]	Both holes impurity scattered[b]	Both holes lattice scattered[c]
$(M_{100}/H^2)\times10^9$	0.5	0.018	0.012	0.014
$(M_{110}/H^2)\times10^9$	0.5	0.44	0.3	0.35
$(M_{100}^{001}/H^2)\times10^9$	1.6	1.6	1.6	1.6
$(M_{110}^{001}/H^2)\times10^9$	2.0	1.6	1.6	1.6

[a] $\tau_L=\tau_H=7.7\times10^{-13}$ sec.
[b] $\bar{\tau}_L=\bar{\tau}_H=2.63\times10^{-13}$ sec.
[c] $\bar{\tau}_L=\bar{\tau}_H=5.14\times10^{-13}$ sec.

for the heavy hole, which throughout this range of magnetic field is in the low-field region. The results in either of these two cases, however, were not as good as the case where an energy-independent τ was assumed for the heavy hole. This seems to indicate that the scattering is due actually to a combination of mechanisms where the relaxation time is given as in Eq. (4). Assuming that the heavy holes are scattered mainly by the impurities but with some small contributions from the lattice, the expression for the relaxation time becomes

$$\tau = l_i \epsilon^{\frac{3}{2}} - (l_i^2/l_l)\epsilon^{7/2}. \quad (11)$$

Curve 4 in Fig. 1 is a plot of the results by assuming values for l_i and l_l which give $\bar{\tau}_i = 1.03\times10^{-14}$ sec and $\bar{\tau}_l = 1.43\times10^{-11}$ sec for the heavy holes, and spherical energy surfaces and constant $\tau = 1.05\times10^{-12}$ sec for the light holes. It is seen that curve 4 gives the best fit to the experimental data. The assumption of energy-independent relaxation time for the light hole is only an approximation. A more exact representation for the light holes would be to assume that they are partly impurity and partly lattice scattered and to include the warping which although small is quite important as may be seen by a comparison of curves 2 and 3. Such a calculation may give a good quantitative agreement with experiment but is quite formidable and we have not attempted it.

The variation of magnetoresistance with magnetic field at 300°K is also given by Pearson and Suhl. Their

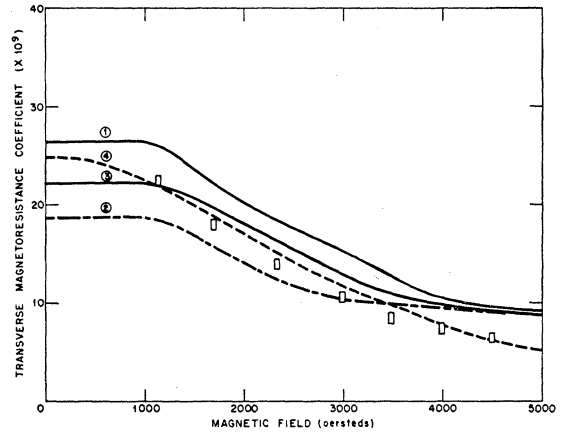

FIG. 1. Variation of the transverse magnetoresistance coefficient with magnetic field at 77°K in Ge. Points are taken from data of Pearson and Suhl. Curves are obtained from our calculations and are labeled (1), (2), (3), and (4). (1) Spherical energy surfaces for light holes, warped energy surfaces for heavy holes, and energy-independent relaxation time ($\tau = 1.03\times10^{-12}$ sec) for both. (2) Spherical energy surfaces for light holes, warped energy surfaces for heavy holes, and energy-independent relaxation time ($\tau = 0.88\times10^{-12}$ sec). (3) Warped energy surfaces and energy-independent relaxation time ($\tau = 0.88\times10^{-12}$ sec) for both light and heavy holes. (4) Spherical energy surfaces and energy-independent relaxation time ($\tau = 1.05\times10^{-12}$ sec) for light holes, but warped energy surfaces with impurity scattering ($\bar{\tau}_i = 1.03\times10^{-14}$ sec) and some lattice contribution ($\bar{\tau}_l = 1.43\times10^{-11}$ sec) for the heavy holes.

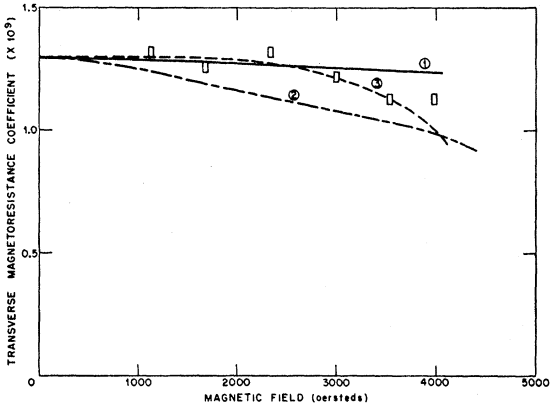

FIG. 2. Variation of the transverse magnetoresistance coefficient with magnetic field at 300°K in Ge. Points are taken from data of Pearson and Suhl. Curves are obtained from our calculations and are labeled (1), (2), and (3). (1) Spherical energy surfaces for light holes, warped energy surfaces for heavy holes, and energy-independent relaxation time ($\tau = 2.29\times10^{-13}$ sec) for both. (2) Spherical energy surfaces and lattice scattering ($\bar{\tau} = 1.89\times10^{-13}$ sec) for both. (3) Spherical energy surfaces for light holes, warped energy surfaces for heavy holes, and lattice scattering ($\bar{\tau} = 1.64\times10^{-13}$ sec) for both.

experimental points along with three calculated curves are shown in Fig. 2. Curve 1 was obtained by assuming an energy-independent relaxation time $\tau = 2.29\times10^{-13}$ sec, spherical energy surfaces for the light holes and warped energy surfaces for the heavy holes. Using the expressions of Willardson, Harman, and Beer,[9] i.e., assuming lattice scattering and spherical energy surfaces for the two types of holes with $\bar{\tau} = 1.89\times10^{-13}$ sec, curve 2 was found. Curve 3 shows the results for the assumption of spherical energy surfaces for the light hole, warped energy surfaces for the heavy hole, lattice scattering for both holes, and $\bar{\tau} = 1.64\times10^{-13}$ sec. It is seen that curve 3 gives the best fit to the experimental points.

IV. DISCUSSION

We have calculated the directional effects of magnetoresistance in p-type germanium and silicon by assuming anisotropic energy surfaces, whose parameters were determined from microwave cyclotron resonance measurements at 4°K and by making simplifying assumptions about the scattering mechanisms. Thus, we have been able to account semiquantitatively for the anisotropy in the magnetoresistance. At 77°K in both germanium and silicon an energy-independent relaxation time gave a good approximation to the true situation. This may be fortuitous since for these materials in this temperature range the energy dependence due to lattice scattering ($\tau_l = l\epsilon^{-\frac{1}{2}}$)[10] and impurity

[9] Willardson, Harman, and Beer, Phys. Rev. **96**, 1512 (1954).

[10] This assumed energy dependence gives the mobility $\mu \sim T^{-1.5}$. Since experimentally $\mu \sim T^{-2.3}$ in the lattice-scattering region, this assumed energy dependence is not strictly correct. H. Ehrenreich and A. W. Overhauser [Phys. Rev. **104**, 649 (1956)] are able to account for the correct temperature dependence in germanium by including optical as well as acoustical modes if they assume a Debye temperature $\theta = 300$°K.

scattering ($\tau_i = l\epsilon^{\frac{3}{2}}$) may compensate one another. In principle one should really consider a combination of impurity and lattice scattering for each of the carriers; this, however, is a tedious calculation and was not our immediate objective. Similar calculations in p-type germanium at 300°K were carried out with lattice scattering providing a slightly better fit than constant τ as one might expect.

Using the assumptions discussed above, we have also calculated the variation of magnetoresistance with magnetic field at both 77°K and 300°K in germanium. At 77°K we found the best fit by assuming an energy-independent relaxation time for the light holes, a mixture of impurity and lattice scattering for the heavy holes, and that the change in the magnetoresistance with magnetic field is due primarily to the light holes. A good fit at 300°K was obtained on the assumption of lattice scattering for both types of holes. In most of these calculations of the variation of magnetoresistance with magnetic field we have assumed spherical energy surfaces for the light hole. This introduces an error at very low fields. Even though the warping of the light holes is small, neglecting it in germanium lowers the transverse magnetoresistance in the limit of zero field by about fifteen percent.

The one disturbing feature of these calculations is that the calculated values of the longitudinal coefficient M_{100}/H^2 are consistently lower than the experimental values both for germanium and silicon. Similar results have been found for germanium by Goldberg, Adams, and Davis.[11] As the temperature increases the agreement is less satisfactory. We have also made comparisons with recent experimental results of Long[12] on higher resistivity silicon in which all the coefficients give good agreement again with the exception of M_{100}/H^2. As can be seen from an examination of Eqs. (5) and (8), it is the longitudinal magnetoresistance coefficient along a cubic axis which is most sensitive to anisotropy since its leading term goes as γ^2. The difficulty here is that magnetoresistance is really a measure of ratio τ/m rather than the mass m, or curvature of the energy surfaces, so that one is not really able to separate the anisotropy of the scattering from that of the energy surfaces. In other words, we may have an anisotropy in the scattering as well as in the energy surfaces. Apparently the anisotropy of τ/m is increasing with temperature.

The significance of the calculations carried out in this paper is the fact that one can semiquantitatively account for the anisotropy of the magnetoresistance in p-type germanium and silicon by using the parameters which define the degenerate warped energy surfaces. In this instance we have used the results of cyclotron resonance measurements to demonstrate the situation. However, in principle, one could perform similar calculations from magnetoresistance data on new materials with the zinc blende structure, when cyclotron resonance experiments are not feasible, and determine the parameters of the energy surfaces within the limitations of the theory.

[11] Goldberg, Adams, and Davis, Phys. Rev. **105**, 865 (1957).
[12] G. D. Long (private communication).

ACKNOWLEDGMENTS

We would like to express our appreciation to Mrs. Virginia Johnson and Mrs. Janet B. Sisler for performing the difficult computations necessary in the evaluation of the magnetoresistance coefficients. We also would like to thank G. L. Pearson for discussions concerning his experimental data, W. H. Kleiner, W. E. Krag, and H. J. Zeiger for critical review of the manuscript, and G. D. Long for communicating some of his results prior to publication.

APPENDIX A. CONDUCTIVITY COEFFICIENTS

The single-energy-surface conductivity coefficients are defined in the following manner: The current density j_i per energy surface may be expressed as

$$j_i = \sigma_{ij}E_j + \sigma_{ijl}E_jH_l + \sigma_{ijlm}E_jH_lH_m + \cdots,$$

where

$$\sigma_{ij} = -\frac{e^3}{4\pi^3\hbar^2}\int \tau\frac{\partial f_0}{\partial\epsilon}\frac{\partial\epsilon}{\partial k_i}\frac{\partial\epsilon}{\partial k_j}dV_k,$$

$$\sigma_{ijl} = \frac{e^3}{4\pi^3\hbar^4 c}\int \tau\frac{\partial f_0}{\partial\epsilon}\frac{\partial\epsilon}{\partial k_i}\frac{\partial\epsilon}{\partial k_r}\frac{\partial}{\partial k_s}\left(\tau\frac{\partial\epsilon}{\partial k_j}\right)\epsilon_{lrs}dV_k,$$

$$\sigma_{ijlm} = -\frac{e^4}{4\pi^3\hbar^6 c^2}\int \tau\frac{\partial f_0}{\partial\epsilon}\frac{\partial\epsilon}{\partial k_i}\frac{\partial\epsilon}{\partial k_r}\frac{\partial}{\partial k_s} \times\left[\tau\frac{\partial\epsilon}{\partial k_t}\frac{\partial}{\partial k_u}\left(\tau\frac{\partial\epsilon}{\partial k_j}\right)\right]\epsilon_{mrs}\epsilon_{ltu}dV_k.$$

In the above, τ is the relaxation time, f_0 is the unperturbed distribution function, ϵ is the energy, dV_k is an element of volume in k space, and the remaining symbols have their usual meaning; the Einstein summation notation is used.

APPENDIX B. MAGNETORESISTANCE COEFFICIENTS

The magnetoresistance coefficients obtained by substituting Eq. (7) into Eq. (6) and equating like orders in H, but neglecting terms higher than H^2, are

$$\Lambda_{ij} = \delta_{ij}/\sigma_0,$$

$$\Lambda_{ijl} = -\Sigma_{ijl}/\sigma_0^2 = -R_0\epsilon_{ijl},$$

$$\Lambda_{ijlm} = -\frac{1}{\sigma_0^2}\left[\Sigma_{ijlm} - \frac{1}{\sigma_0}\Sigma_{i\alpha l}\Sigma_{\alpha jm}\right].$$