Proton-Nucleus Scattering at 17 Mev*†

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Previous studies of proton-nucleus scattering have been extended to 17 Mev. As at 10 Mev, it is difficult to obtain unique sets of parameters because of the similar effects of the real part of the potential V and the interaction radius R, on the one hand, and the imaginary part of the potential W and the diffuseness parameter a, on the other. If the best half-way radius for each element is expressed as $R=r_0A^{\frac{1}{3}}\times10^{-13}$ cm, r_0 decreases from $r_0=1.29$ for light and medium elements to $r_0=1.22$ for heavy elements. Values for the diffuseness parameter near $a=0.5\times10^{-13}$ cm are used. Normalized to $r_0=1.30$ the average value of the optical model potential at this energy is $-(50+i8)$ Mev. Values of the reaction cross section are given. An analysis of proton-carbon scattering from 14 to 20 Mev is presented. It is also shown that proton-nucleus scattering at this energy is insensitive to the precise details of the potential in the central and surface regions. In addition the scattering is independent of the shape of the nuclear charge density for energies up to 100 Mev.

INTRODUCTION

AN intensive analysis,[1] based on the optical model, has recently been presented of angular distributions[2] of 10-Mev protons elastically scattered from nuclei. The present paper extends such studies to the measurements by Dayton and Schrank[3] of proton-nucleus scattering at 17 Mev. The energy variation of proton-carbon scattering, observed by Peelle,[4] is also discussed.

A fair measure of success was achieved in understanding Hintz's 10-Mev data. An important feature of that analysis was that neither the radius R nor the strength V of the interaction could be determined uniquely, but only the combination VR^2. Of course this invariance under VR^2 is only approximate so that unreasonably small or unreasonably large radii were not allowed. On the other hand, the theorem may be exactly true at zero energy, since it holds for 1–3-Mev neutrons[5] but begins to break down for 17 Mev protons. In any case, it was possible to fit the 10-Mev scattering quite well with a value of $|V|r_0^2\simeq89$ Mev-$(10^{-13}$ cm$)^2$, values near 0.5×10^{-13} cm for the diffuseness, and absorptions in the range from $W=-7$ to -9 Mev.[6] In most cases it was possible to use a radius parameter of $r_0=1.20$, with $V\simeq-62$ Mev.

The data of Dayton and Schrank have been the subject of detailed investigation with the optical model by Saxon and associates at the University of California at Los Angeles (U.C.L.A.).[7] These authors also note an "ambiguity" between V and R but consider the best over-all fits to be obtained with $r_0=1.33$. In this paper, a quantitative criterion is used to determine the best radius. From a set of solutions for a range of radii (each solution having been obtained by least-squares analysis) the best radius is chosen as that giving the minimum deviation from the observed cross sections. Although it is usually possible to find reasonable fits for $r_0=1.33$, the best fit always occurs for a radius parameter smaller than $r_0=1.33$. These best solutions range from $r_0=1.29$ for light elements to $r_0=1.22$ for gold.

The angular distributions of 17-Mev protons elastically scattered from nuclei are more complicated than for 10-Mev protons. There is usually an additional oscillation and the amplitudes of all the oscillations are larger. Furthermore, the average value of the ratio of the cross section to the Rutherford cross section either increases (light elements) or decreases (heavy elements) more rapidly with angle than for 10-Mev protons. As a result, the agreement of the optical-model calculations reported here is somewhat inferior to that previously achieved at 10 Mev.[1] As at 10 Mev, it is impossible to fit the large-angle scattering from light elements. This difficulty, as well as others, now appears for some heavier elements. For example, although several acceptable fits were easily found for copper at 10 Mev, at 17 Mev this case is particularly hard to analyze, and it is impossible to get agreement for large scattering angles. On the other hand, the scattering from gold agrees extremely well with the optical model and for this case the most thorough investigation of the interaction radius is carried out.

A brief summary of developments in procedure is given in Sec. II. The analysis of some of the 17-Mev data of Dayton and Schrank[3] is given in Sec. III, and Sec. IV is devoted to a study of the energy variation of

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[1] Glassgold, Cheston, Stein, Schuldt, and Erickson, Phys. Rev. **106**, 1207 (1957).
[2] N. Hintz, Phys. Rev. **106**, 1201 (1957).
[3] I. E. Dayton and G. Schrank, Phys. Rev. **101**, 1358 (1956).
[4] R. W. Peelle, Phys. Rev. **105**, 1311 (1957).
[5] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954).
[6] The terminology for the optical-model parameters is the same as used in reference 1. Energies will be in Mev and lengths in 10^{-13} cm. The formula, $R=r_0A^{\frac{1}{3}}\times10^{-13}$ cm, is used for the interaction radius.
[7] R. W. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954); Melkanoff, Moszkowski, Nodvik, and Saxon, Phys. Rev. **101**, 507 (1956); Melkanoff, Nodvik, Saxon, and Woods, Phys. Rev. **106**, 793 (1957).

proton-carbon scattering, as measured by Peelle.[4] The basic analysis is done with the Woods-Saxon potential,[7] previously used at 10 Mev.[1] In Sec. V modifications of this potential are considered. These are a "wine-bottle" shape emphasizing the nuclear surface, a Gaussian instead of an exponential "tail," and the Hill-Ford potential with a smooth charge density.[8]

PROCEDURE

The calculations are carried out with the optical-model scattering program previously described.[1,9] The machine time required has been significantly reduced by using larger integration intervals[10] which, however, are consistent with the desired accuracy. In addition the least-squares analysis has also been improved in many details.[11]

For most of this work, the Woods-Saxon potential[7] was used. The complex potential is written as a form factor,

$$f(r)=\left[1+\exp\left(\frac{r-R}{a}\right)\right]^{-1}, \tag{1}$$

times the complex number $(V+iW)$; R is the half-way radius, a is the diffuseness parameter, while V and W are called the strengths of the real and imaginary parts of the nuclear potential, respectively. In addition, the Woods-Saxon potential contains an electrostatic term, taken to be the potential of a uniform spherical charge distribution of radius R. As discussed in Sec. V, this approximation is valid for energies at least as high as 100 Mev.

To begin the analysis at this energy, a number of calculations were carried out to determine the effect of each of the above parameters on the cross section, or rather on the ratio of the cross section to Rutherford scattering. Regularities similar to those discovered at 10 Mev[1] hold at this energy but with some differences. Although increasing V or R shifts the diffraction pattern towards smaller angles, keeping VR^2 constant no longer determines the positions of maxima and minima, except for the lightest element under study, carbon. Nevertheless, changes in V and R are still related, but at 17 Mev the radius is slightly more effective in determining the positions of maxima and minima than the real part of the potential. Thus a power larger than the second must be used. It is difficult to determine or define this power with any precision since it depends on the particular element and the range of optical-model parameters under consideration. In any case, the parameter essential for positioning maxima or minima is somewhere between VR^2 and VR^3.

When the calculated maxima and minima have been lined up with the data, the remaining parameters are varied to give the oscillations the proper amplitudes. The changes caused by a and W are most important for this purpose. The effects of increasing these parameters are somewhat similar, as was the case at 10 Mev. For 17-Mev protons there is a reduction in the average value of $\sigma(\theta)/\sigma_R(\theta)$, which increases with increasing angle, in addition to a simple damping of the diffraction pattern. Furthermore, although the majority of minima are made shallower by increases in a and W, there are minima which behave anomalously in this respect. It is worth noting that most sets of parameters give too large an amplitude to the oscillations of the diffraction pattern, and the usual problem is to find that rather restricted set of parameters which does not make most of the minima too deep.

The above statements are, of course, extremely qualitative and not very susceptible to generalization. They depend greatly on the particular element and parameters involved. Fortunately, the analysis reported here does not depend on a detailed knowledge of these variations for all the cases studied. Instead a systematic least-squares analysis of the data is carried out. This procedure is described briefly in the 10-Mev paper[1] and more fully by Schuldt.[11] In practice, the method varies three of the optical-model parameters[12] (or three independent combinations of them) to minimize the mean-square relative deviation,

$$\Delta^2=\frac{1}{N}\sum_{i=1}^{N}\left[\frac{\sigma(\theta_i)-\sigma_{\exp}(\theta_i)}{\sigma_{\exp}(\theta_i)}\right]^2, \tag{2}$$

where $\sigma(\theta_i)$ is the calculated optical-model cross section at θ_i. As previously remarked, this procedure allows a systematic investigation of the optical-model parameters to be made and, in addition, a quantitative definition of best fit.

III. ANALYSIS OF 17-MEV PROTON-NUCLEUS SCATTERING

A. Aluminum

Two values of the radius parameter were considered in studying the scattering from aluminum. The best fits for $r_0=1.20$ and $r_0=1.29$ are given in Fig. 1 and the pertinent parameters summarized in Table I. The very deep and narrow minimum at 40° is a good example of the detail that can be reproduced with the present

[8] D. L. Hill and K. W. Ford, Phys. Rev. **94**, 1617 (1954).

[9] The authors are indebted to Remington Rand Univac for the use of their Univac Scientific Computer (E.R.A. 1103) in St. Paul, Minnesota.

[10] It is also unnecessary to use a smaller interval near the nuclear surface. It is usually possible to get cross sections accurate at all angles to 1% with an integration interval equal to $\frac{3}{8}$ except near the origin.

[11] S. B. Schuldt, M.A. thesis, University of Minnesota, 1957 (unpublished).

[12] It has also been established that a three-way least-squares analysis is sufficient for studying the Saxon potential. G. W. Erickson has discovered that, for potentials of the form $(V+iW)\times f(r)$, the scattering amplitude F is an analytic function of the complex variable $V+iW$, so that $\partial F/\partial W=-i(\partial F/\partial V)$. [University of Minnesota Linear Accelerator Progress Report, 1956–1957 (unpublished).]

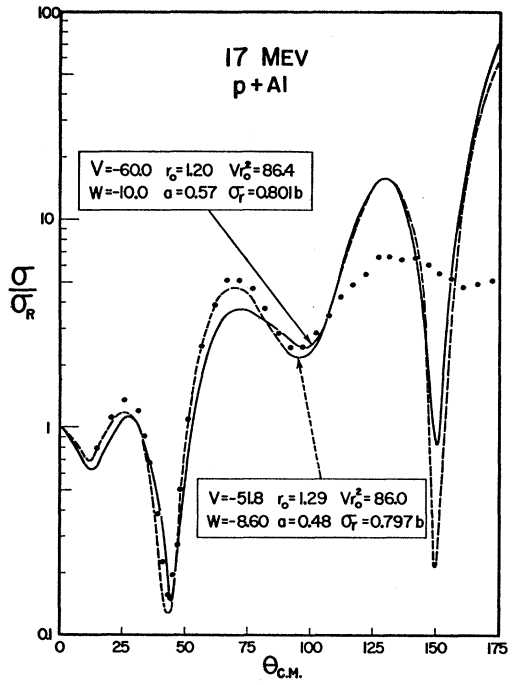

FIG. 1. Best solutions for 17-Mev proton-aluminum scattering. The optical model is quite successful for scattering angles less than 110°. For larger scattering angles, however, there is pronounced disagreement of a type which is characteristic of light elements at this energy.

version of the optical model, provided the scattering angle is not too large. In finding these two particular fits the experimental points beyond $\theta_N=125°$ were excluded from the least-squares analysis, and the agreement in this region is very poor. All attempts to improve the agreement at such large scattering angles failed.

A significantly better fit is achieved with the larger radius parameter, $r_0=1.29$. It is interesting that the solutions give the same reaction cross section. The quality of the best fit is about the same as the 10-Mev result for aluminum, where agreement could also only be obtained for scattering angles less than 110°. The main discrepancy is that at 10 Mev it was essential to use an extremely small and anomalous diffuseness, $a\simeq 0.2$.

TABLE I. Summary of 17-Mev analysis. The best fits for particular values of the radius parameter r_0 are listed for each element studied. In addition to the optical-model parameters, the reaction cross section σ_r (in barns), the rms deviation Δ for the angular distribution, and the cutoff angle θ_N in the least-squares analysis, are given. The best fit is that having the smallest value of Δ.

Element	Energy	r_0	a	V	W	σ_r	Δ	θ_N
C	17.4	1.24	0.46	−53.1	− 7.5	0.40	0.12	125°
C	17.4	1.28	0.43	−51.1	− 6.9	0.40	0.13	125°
Al	17.6	1.20	0.57	−60.0	−10.0	0.80	0.22	115°
Al	17.6	1.29	0.48	−51.8	− 8.6	0.80	0.17	115°
Cu	17.3	1.20	0.54	−57.8	− 7.7	0.98	0.28	130°
Cu	17.3	1.29	0.51	−47.6	− 7.6	1.11	0.22	130°
Cu	17.3	1.33	0.54	−43.8	− 7.3	1.14	0.24	130°
Ag	17.1	1.10	0.57	−70.7	−17.3	0.95	0.18	172°
Ag	17.1	1.20	0.56	−56.3	−12.8	1.06	0.14	172°
Au	17.0	1.13	0.60	−72.0	−13.0	0.83	0.108	172°
Au	17.0	1.20	0.55	−63.1	− 8.2	0.88	0.070	172°
Au	17.0	1.23	0.55	−60.3	− 7.9	0.93	0.068	172°
Au	17.0	1.33	0.54	−48.6	−10.4	1.13	0.107	172°
Au	17.0	1.42	0.52	−40.5	− 9.5	1.28	0.134	172°

B. Copper

A great deal of difficulty was encountered in understanding the data for copper. First of all, the experimental points beyond 135° present the same difficulties as the large-angle scattering from aluminum. After it was determined that there was no hope for this part of the data, it was eliminated from consideration.

A thorough investigation of the optical-model parameter space was made to understand the problems which still remained. As long as V and R are constrained to position properly the maxima and minima, it is possible to restrict the discussion to the a, W plane. Along and near the line given roughly by $W=-4.9+30a$, the calculated values of the minimum at 110° are too deep by a factor of 1000. There are also parallel regions on *each* side of this line for which the second minimum has approximately the observed depth. The least-squares analysis can then converge to two *isolated* points in the a, W plane. The parameters for these two solutions are given in Table II for the particular choice of $r_0=1.20$. The solution with large W and small a is too *shallow* at the first minimum at 67° by a factor of two, whereas the other solution with smaller W and larger a is too deep by a factor of two. The latter solution actually has a significantly smaller standard deviation, and is included in Table I as the best fit for copper for a radius of $r_0=1.20$.[13]

TABLE II. Two types of solutions possible for copper. The second set of parameters with small a and large W is discarded because of its large deviation Δ from the experimental values.

r_0	V	a	W	Δ
1.20	−57.8	0.539	− 7.77	0.29
1.20	−58.4	0.391	−12.4	0.43

This choice of the best fit is reasonable if one considers the many ways in which a diffraction minimum can be filled in. There are, to begin with, the finite energy and angle resolution of the apparatus. For copper inelastic scattering presents no problem, but a mixture of isotopes was used for the target foil. The minima for the two isotopes occur at slightly different angles and the minimum in the average cross section is shallower than for either isotope. This effect was investigated by actually calculating the scattering for the two isotopes and performing the proper average. With reference to the second minimum at 112°, it was

[13] The values of the parameters for this solution are also more reasonable for this solution, i.e., close to those obtained for the other elements.

concluded that the scattering at this minimum for either isotope cannot be deeper than the observed value for the mixture by more than a factor of 1.5 if the agreement with experiment is to be preserved.

Two other values of the interaction radius were also studied using the normal solution chosen above for which $a \simeq 0.5$ and $W \simeq -8$. The best parameters for the three radii are given in Table I and the angular distributions are compared with experiment in Fig. 2. The parabola obtained by plotting Δ against r_0 has a minimum close to $r_0 = 1.29$. This determination of the interaction radius for copper is not very precise since the minimum deviation is rather large (roughly 20%) and the deviation changes rather slowly with radius.

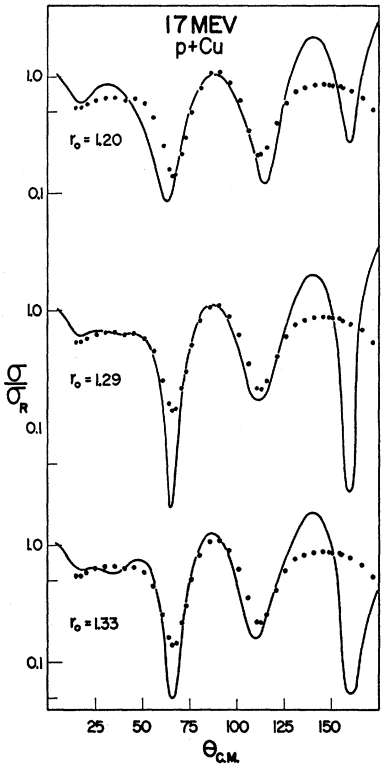

FIG. 2. Best solutions for 17-Mev proton-copper scattering for three values of the interaction radius. The other optical-model parameters for these calculations are given in Table I. The best radius is quite close to $r_0 = 1.29$. Beyond 125° the optical-model calculations disagree with experiment in a manner typical of lighter elements.

In other words, the three solutions do not differ very much on the average. Of course they all disagree with experiment at very large scattering angles. The interaction radius determined for copper at 10 Mev by the same method is $r_0 = 1.26$.[1]

C. Silver

In studying the scattering from silver the rare situation was encountered in which the first two minima in the experimental diffraction pattern were always deeper than calculated. Nevertheless the over-all agreement with experiment is fairly good. (See Fig. 3.) Two radii were considered, and $r_0 = 1.20$ is superior to $r_0 = 1.10$.[14]

[14] Significant corrections were made for these data by Dayton and Schrank (reference 3) because of the large inelastic scattering to low-lying states, particularly beyond 80°. The minima at 90° and 140° have been fitted with a large absorption, probably at the expense of the main forward minimum at 50°. If the two minima at back angles are actually much deeper, then agreement with all three might have been obtained with a smaller absorption.

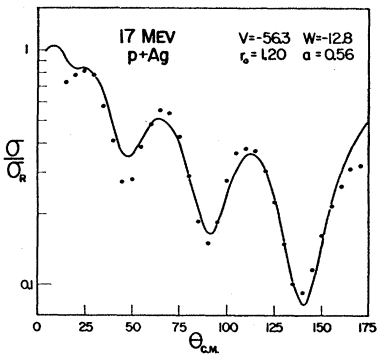

FIG. 3. Best solution for 17-Mev proton-silver scattering. The rather large value for W is the result of trying to get agreement with the osberved depths of the diffraction minima.

D. Gold

The most significant determination of an interaction radius at 17 Mev is possible for gold since it is for this element that the optical model is most successful. The full angular range of the experiment from 25° to 172° can be treated. Although the measured values include some inelastic scattering from low-lying states, this effect is estimated by Dayton and Schrank[3] to be only about 3% at back angles.

Best fits have been obtained for five radii: $r_0 = 1.13$, 1.20, 1.23, 1.33, and 1.42. The angular distributions are compared with experiment in Fig. 4 and the parameters given in Table I. The rms relative deviations are plotted in Fig. 5 as a function of the radius parameter r_0. The minimum value of Δ, i.e., the best fit, occurs for $r_0 = 1.22$. The sensitivity of this determination is not very great since one must change r_0 by 0.13 to increase Δ by a factor of two.

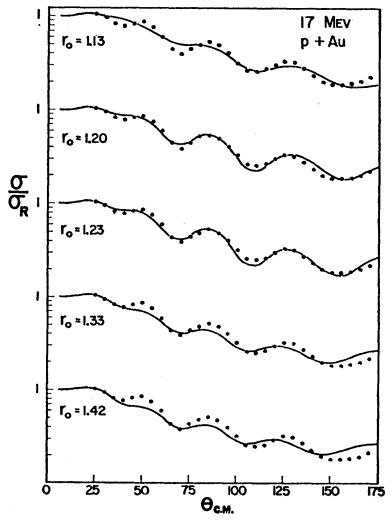

FIG. 4. Best solutions for 17-Mev proton-gold scattering. The optical-model parameters for each radius are given in Table I.

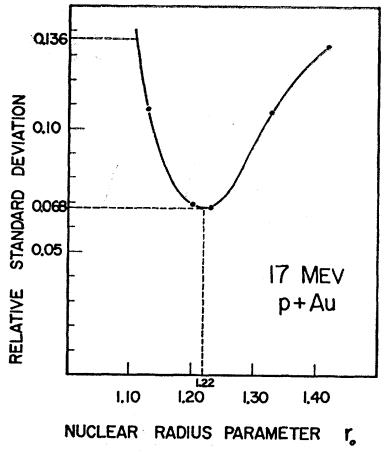

FIG. 5. Determination of the interaction radius for gold. The rms relative deviation Δ is plotted as a function of r_0 for each of the five solutions given in Fig. 4 and Table I. The curve through these five points has a minimum at $r_0=1.22$. This determination is not very sensitive since r_0 must be changed by 0.13 or 10% to increase Δ by a factor of two.

The value of n for the function VR^n which these five solutions have in common is $n=2.5$. The accuracy with which this power is determined is only about 20%.

E. Reaction Cross Sections

It has been emphasized previously[1,2,8] that measurements of reaction cross sections would provide additional checks on the optical model and possibly help determine the interaction radius. The seventh column of Table I gives the reaction cross sections for the elements discussed in this paper. Solutions for at least two radii are given in each case. For aluminum and carbon the two best fits have the same reaction cross section, but differences do occur for higher atomic numbers.[15] For gold, a 10% change in radius means almost a 20% change in reaction cross section, which includes, of course, the combined effects of changes in all the parameters.

Measurements of reaction cross sections may not be very helpful in distinguishing equivalent sets of optical-model parameters. For the most favorable case at 17 Mev, gold, the radius is already known to better than 10% from the above analysis of the elastic-scattering angular distribution. To be useful in this connection, the reaction cross sections would have to be known to about 10%. On the other hand, less accurate values will be valuable in establishing the consistency of the optical-model parameters obtained from the elastic scattering.

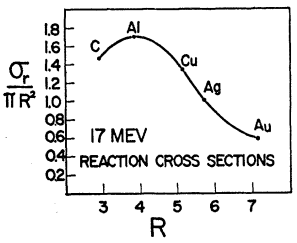

FIG. 6. Variation of the total reaction cross section divided by πR^2 as a function of radius. A resonance effect is observed in the neighborhood of aluminum. In dividing σ_r by πR^2, the same value of r_0 was not used in all cases. The calculations were made using the best sets of parameters given in Table I.

[15] This increase in sensitivity has been explained by Hintz (reference 2) as a barrier effect.

The partial reaction cross sections, $T_L=\sigma_r^{(L)}/(2L+1)\pi\lambda^2$, for each orbital angular momentum are summarized in Table III. Figure 6 shows the variation with interaction radius of the reaction cross section divided by the "geometric" value πR^2. There is a well defined resonance in the neighborhood of aluminum. Size resonances are, of course, characteristic of the complex-potential model[5] and have been observed for high-energy as well as low-energy neutrons.[16,17]

IV. ENERGY VARIATION OF PROTON-CARBON SCATTERING WITH THE SAXON POTENTIAL

In this section an analysis is presented of Peelle's measurements of the elastic scattering of protons from carbon in the energy range from 14 to 20 Mev.[4] Unfortunately it is impossible to understand the large-angle scattering so that some of the most interesting features have to be ignored. Specifically, Peelle observed a marked variation with energy of the depth of the second minimum near 150°. It would be interesting to determine whether this variation can be reproduced without large variations in the optical-model parameters. However, only the first maximum and minimum and part of the second maximum can be fitted. Only at one energy, 18.4 Mev, does the optical-model calculation begin to resemble the scattering beyond 125°.

TABLE III. Reaction cross sections for the best sets of optical-model parameters at 17 Mev. The first seven columns give the partial reaction cross sections for each orbital angular momentum divided by the maximum possible value, i.e., $T_L=\sigma_r^{(L)}/(2L+1)\pi\lambda^2$. The last column gives the total reaction cross section divided by the geometric value. This ratio has a maximum in the neighborhood of aluminum. The sets of parameters used in calculating these cross sections are the ones in Table I having the minimum deviation Δ from experiment. Thus the radius parameter r_0 is not the same for all of the elements.

Element	T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	$\sigma_r/\pi R^2$
C	0.73	0.61	0.58	0.48	0.02				1.47
Al	0.83	0.84	0.85	0.60	0.85	0.05			1.70
Cu	0.90	0.91	0.85	0.97	0.58	0.72	0.06	0.01	1.34
Ag	0.96	0.97	0.94	0.86	0.81	0.33	0.13	0.01	1.04
Au	0.85	0.95	0.54	0.91	0.46	0.42	0.08	0.02	0.58

The analysis was carried out for four energies, 14.0, 17.4, 18.4, and 19.4 Mev. The data beyond 125° were ignored after it was determined that nothing could be done with them. The results for $r_0=1.24$ are plotted as solid curves in Fig. 7. The dash curves given with the fits at 14 and 1.94 Mev are calculations made at these energies with the parameters found at 17.4 Mev. Apparently only slight changes in the parameters are needed, to the extent that the data can be fitted at all. This is borne out by the parameters listed in the first four rows of Table IV. The optical-model parameters, other than the radius which has the same value $r_0=1.24$,

[16] A. E. Taylor and E. Wood, Phil. Mag. 44, 95 (1953).
[17] E. Lampi, University of Minnesota Linear Accelerator Progress Report, 1956–1957 (unpublished).

do not vary monotonically with energy. In fact they show either a peak or a dip at 17.4 Mev. This behavior also occurs for a somewhat larger radius parameter, $r_0=1.29$, as summarized in the fifth and sixth rows of Table IV for 17.4 and 19.4 Mev. The largest effect is the small diffuseness obtained at 17.4 Mev. The last line of Table IV gives the result of a calculation in which this parameter was kept constant at $a=0.50$ and only V and W were varied. The value of V is restored to a more reasonable value but W is depressed even further. For comparison with the fit obtained without this constraint, this calculation is plotted as the dash curve in Fig. 7.

The above analysis indicates a slight anomaly in the optical-model parameters for carbon at 17 Mev. It is not at all clear whether this corresponds to any real physical change at this energy, since the very application of the optical model in this case, especially the simple one used here, is open to serious question. The anomaly may be described as a minimum in either W or a, which indicates a surface effect. A model with a static and central complex potential, such as the one used in this analysis, ignores a number of important surface terms which are particularly important for a light nucleus. Francis and Watson[18,19] have shown that there are corrections to the usual optical model of order A^{-1}, where A is the number of scatterers. This correction is proportional to the gradient of the nuclear density and Kisslinger[20] has used such a term to advantage in studying meson scattering from nuclei with the optical model. Riesenfeld and Watson[21] have also shown that the spin-orbit coupling, expected of course from its important role in the shell model, most likely appears as a surface term, i.e., proportional to the derivative of the central potential. Finally, the velocity dependence of the nuclear potential[22,23] has its biggest effect at the nuclear surface where the incident particle undergoes its greatest change in velocity.

The fact that the large-angle scattering is not at all reproduced by the optical model should lessen any surprise in finding an anomaly in the optical-model parameters at 17.4 Mev. The same explanation may well serve for both difficulties. The anomaly in the optical-model parameters may be roughly summarized as a minimum in the absorption. This could be due to a local variation in the level density in N^{13}, the appropriate compound nucleus. Recently Greenlees *et al.*[24] have suggested such an explanation for the difference in the energy variation of proton scattering between aluminum and magnesium in the neighborhood of 10 Mev. Unfortunately little is known about

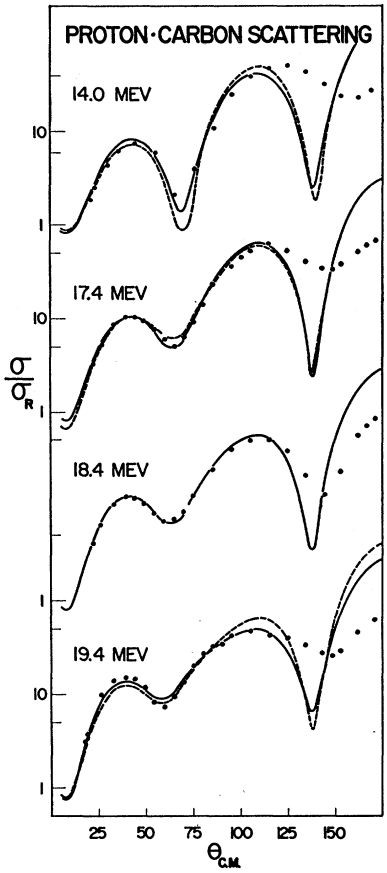

FIG. 7. Analysis of the energy variation of proton-carbon scattering between 14 and 20 Mev. The solid curves at each energy are the best fits with a fixed radius parameter, $r_0=1.24$. The dash curves at 14 and 19.4 Mev were made using the parameters obtained in the 17-Mev analysis. They indicate that very little energy variation is required to get the limited agreement possible for this element. The most important changes observed by Peelle[4] have to do with the second minimum, but in this region the optical model is completely unsuccessful. The least-squares analysis yields parameters which have maxima or minima near 17.4. The dash curve at 17.4 Mev was obtained from a least squares analysis which kept a as well as r_0 fixed. The main anomaly in the optical-model parameters at 17.4 Mev is the depression in the absorption to $W=-5.9$ Mev. The parameters for all these calculations are given in Table II.

TABLE IV. Optical-model parameters for proton-carbon scattering from 14 to 20 Mev. The first four rows give the best fits at four energies obtained for a fixed value of $r_0=1.24$. Equivalent solutions at 17.4 and 19.4 Mev are also given for a radius parameter of $r_0=1.28$. The optical-model parameters have a minimum or a maximum near 17.4 Mev. The last solution at 17.4 Mev, keeping a as well as r_0 fixed, has only an anomalous value for W.

E_0	r_0	a	V	W
14.0	1.24	0.51	−49.2	−8.5
17.4	1.24	0.46	−53.1	−7.5
18.4	1.24	0.48	−49.9	−7.8
19.4	1.24	0.54	−48.9	−8.0
17.4	1.28	0.43	−51.1	−6.9
19.4	1.28	0.50	−46.9	−7.6
17.4	1.24	0.50	−50.7	−5.9

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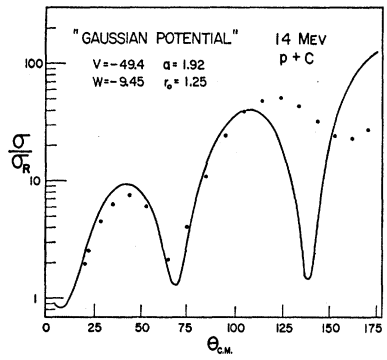

FIG. 8. Best solution for 14-Mev proton-carbon scattering using the Gaussian modification of the Saxon potential of Eq. (3). Comparison with the first panel of Fig. 7 shows that no improvement has been achieved. This is an indication of the insensitivity of the scattering at this energy to the details of the diffuse nuclear surface.

the levels in N^{13}, the compound nucleus in this case, at the appropriate excitation energy.

V. FORM-FACTOR STUDIES

Modifications of the Saxon potential have been investigated for the following reasons. First of all, there is the possibility for improvement in the large-angle scattering from light elements. Next, the validity of using a uniform charge density with a sharp edge has to be better established, particularly for wavelengths comparable to the surface thickness of the nucleus. Finally, it is important to relate the parameters for equivalent models, i.e., potentials which give the same scattering but have different shapes. In this way some idea of the general features of the interaction may be obtained, in contrast to parameters for specific models.

The first modification of Eq. (1) is a Gaussian tail which preserves the uniform behavior at the center of the nucleus:

$$f_G=\left[1+\exp\left(\frac{r^2-R_G^2}{a_G^2}\right)\right]^{-1}. \tag{3}$$

Applied to Peelle's 14-Mev carbon data,[4] very little improvement is obtained with this form factor, as is shown by a comparison of Fig. 8 with the first panel of Fig. 7. The main difference is in the large diffuseness parameter a_G which is needed to give the Gaussian shape roughly the same surface thickness as the Saxon form factor.

The next modification of the Saxon potential,

$$f_w=\frac{1+b(r/R_w)^2}{1+b\beta}\left[1+\exp\left(\frac{r-R_w}{a_w}\right)\right]^{-1}, \tag{4}$$

is one which either raises ($b<0$) or depresses ($b>0$) the central region, i.e., a wine-bottle potential. In its application to light elements, it is not intended to represent the effect of the Coulomb repulsion on the proton distribution, discussed, for example, by Feenberg.[25] Rather the parameter b is chosen so that a peaking is obtained at the nuclear surface.[26,27] This should give an indication of the dependence of the scattering on the local conditions in the nuclear surface. As discussed in the previous section, this is the region where the present simple optical model will most likely fail.

In studying the effects of the wine-bottle modification for 14-Mev protons scattered from carbon, the starting point is the best fit with the Saxon potential illustrated in Fig. 7. Keeping V and R_w fixed, the parameter β is chosen so as not to alter the positions of maxima and minima. This choice is practically independent of the peaking parameter b, and for this case, $\beta=1$. The effect of a fairly large b is illustrated in Fig. 9. The parameters $b=\beta=1$ are such that f_w goes from a value of $\frac{1}{2}$ to roughly one and then back to $\frac{1}{2}$ again as r ranges from $r=0$ to $r=R_w$. In attempting to improve the fit with Peelle's data, least-squares analyses were carried out varying a_w W, and b, which are the parameters most intimately related to the nuclear surface. Values of b were obtained in the range from -0.09 to $+0.08$ depending on how the various angles were weighted. The smallness, together with the uncertainty in the sign of b, implies that no improvement can be obtained by using a central elevation or depression in the potential.

The last form factor considered is the one used by Hill and Ford[8,28] in their studies of the electromagnetic size of nuclei[29]:

$$f_H(r)=\frac{1}{1-\frac{1}{2}\exp(-R_H/a_H)} \times\begin{cases}1-\frac{1}{2}\exp[(r-R_H)/a_H], & r<R_H\\ \frac{1}{2}\exp[-(r-R_H)/a_H], & r>R_H.\end{cases} \tag{5}$$

Unlike the function in Eq. (1), the electrostatic potential for a charge density of this shape (but with different parameters R_H', a_H') can be written in closed form.[8] Thus this form factor was used to investigate the validity of using a sharp charge density of the same radius as the nuclear interaction potential, which is the assumption made in the Saxon potential and the modifications just introduced.

Calculations were carried out in which a_H' (the diffuseness of the nuclear charge density) was changed from a finite value to one close to zero, and in which R_H' (the radius of the nuclear charge density) was made 10% smaller than R_H (the radius of the nuclear potential). There were no appreciable changes over the entire range from 10 to 100 Mev.[30] Therefore the de-

[25] E. Feenberg, Phys. Rev. **59**, 593 (1941).

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[27] W. S. Emmerich and H. J. Amster, Bull. Am. Phys. Soc. Ser. I, **2**, 71 (1957).

[28] K. W. Ford and D. L. Hill, *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1955), Vol. 5, p. 25.

[29] The authors wish to thank David Sowle for coding this form-factor subroutine.

[30] Using a uniform charge density of radius R_c, Woods and Saxon (reference 7) previously showed that the scattering was insensitive to small differences between R and R_c.

tails of the nuclear *charge* distribution are not important for proton-nucleus scattering at these energies. The only distribution of importance is that for the nuclear interaction.

Although both the Saxon and Hill form factors are practically constant in the nuclear interior and have exponential tails outside the nucleus, their behavior in the surface region does differ. For example, if all the other parameters except the diffusenesses are the same, the two slopes at the half-way radius are $-(4a_S)^{-1}$ and $-(2a_H)^{-1}$ for the Saxon and Hill potentials, respectively. Therefore it is important to determine whether the two form factors are equivalent, i.e., given the Saxon potential characterized by the parameters (V_S,W_S,R_S,a_S), does there exist a set of parameters for the Hill potential (V_H,W_H,R_H,a_H) which gives the same scattering. If the answer is negative, then the new feature in the Hill potential must be investigated to determine whether it gives better or worse agreement with experiment.

By a modification of the least-squares procedure[31] it was established that, for energies up to 30 Mev,[32] equivalence could be obtained by simply adjusting the diffuseness parameters, i.e., $V_H=V_S$, $W_H=W_S$, $R_H=R_S$, but $a_H=1.25a_S$. This is practically the same result obtained from the analysis of high-energy electron scattering with these two form factors.[28,33] The prescription $a_H=1.25a_S$ does not make the slopes of the form factors equal at the common half-way radius, but rather some average of the slope taken over the surface region. It has become customary in electron scattering studies[34] to characterize this average by the surface thickness t, defined as the distance for the form factor to fall from 0.9 to 0.1 the value at the origin. If the Saxon and Hill form factors are to have the same surface thickness, then $a_H/a_S=1.37$ with $t=4.40a_S$. In fact the ratio $a_H/a_S=1.25$, determined from this analysis and from electron scattering,[28,33] implies that these two form factors have in common the somewhat larger distance $6.5a_S$, for which the form factors decrease from 0.96 to 0.04 their interior values.

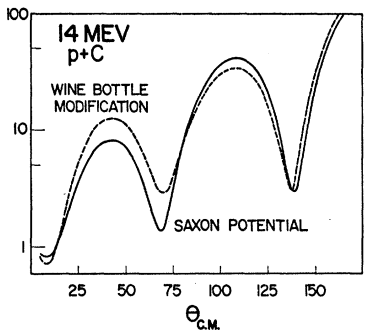

FIG. 9. Effect of the wine-bottle modification to the Saxon potential on proton-carbon scattering at 14 Mev. The solid curve is the best fit for $r_0=1.20$ using the Saxon potential. The dash curve for the wine-bottle potential was calculated using a value of β in Eq. (4) which kept the positions of maxima and minima fixed. The value used for b in Eq. (4) is such that an increase by a factor of two over the central value is obtained for the nuclear form factor. A least-squares analysis of the 14-Mev carbon data led to the conclusion that best agreement is obtained with essentially no elevation or depression of the form factors in the central region.

VI. SUMMARY

The results of the preceding section show that proton-nucleus scattering at 17 Mev is insensitive to the details of the nuclear potential in the surface and central regions.[35] Furthermore the scattering is quite independent of the shape of the nuclear charge density, so that a uniform density may be used. Thus the Saxon potential is sufficiently general for this analysis.

The optical-model parameters obtained in the analysis of Dayton and Schrank's 17-Mev data[3] are summarized in Table I. The best set of parameters for each element is the one with the minimum least-squares deviation Δ. These results agree with those obtained by Saxon and his associates at U.C.L.A.[7] in the following restricted sense. For a particular choice of the radius parameter r_0, both analyses of a 17-Mev angular distribution yield the same optical-model parameters to within 1 or 2 Mev for V and W and to within 0.05 $\times10^{-13}$ cm for a. The one exception is aluminum where the U.C.L.A. group uses a much larger a and a smaller V than this work. The essential difference, however, is in the choice of a best set of optical-model parameters.

Realizing the difficulty in determining a unique solution to this problem, Saxon *et al.* have deliberately tried to find a radius parameter r_0 in the formula $R=r_0A^{\frac{1}{3}}$ $\times10^{-13}$ cm which is independent of atomic number A. Their choice was $r_0=1.33$, although this was only strongly indicated in one case, cobalt, and not possible for light elements.

In the present analysis each nucleus is treated on an individual basis to find the set of parameters which minimizes the rms deviation from the experimental angular distribution. This point is of particular importance in the determination of the interaction radius. Indeed the analyses at both 10 and 17 Mev have *never* yielded a best interaction radius as large as $r_0=1.33$. The present study, in fact, finds that the best radius parameter is $r_0=1.29$ for aluminum and copper and $r_0=1.22$ for gold. The analysis for carbon and silver is not as complete in this respect but is in agreement with a larger parameter for light elements near $r_0=1.29$ and a smaller parameter for heavier elements near $r_0=1.22$.

[31] The calculation for the Saxon potential is treated as "experimental data" and the parameters for the Hill potential are varied to minimize the mean relative difference between the two angular distributions.

[32] The situation at higher energies has not yet been investigated.

[33] Hahn, Ravenhall, and Hofstadter, Phys. Rev. **101**, 220 (1956).

[34] R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).

[35] Investigations of the nuclear charge density with 183-Mev electrons,[29,33] which have the same wavelength as 17-Mev protons, have given a very similar result.

This increase in the half-way radius parameter for the Saxon potential is of a different sort than has been discussed previously in the literature.[36–40] The radius R' of the equivalent square-well, which has the same rms radius as Eq. (1), is

$$R'=R\left[1+\frac{7}{3}\left(\frac{a}{R}\right)^2\right]^{\frac{1}{2}}. \tag{6}$$

Thus, even if R satisfies a simple $A^{\frac{1}{3}}$ law, as is the case so far for electron scattering,[34] the equivalent square-well radius R' will not.§

The energy variations of V and W are of considerable interest in the theory of nuclear structure. Of course it is somewhat premature to discuss this question until the analysis, now in progress, of Hintz's 40-Mev[41] and Strauch's[42] 95-Mev experiments is completed. Thus, there is hardly any significance to the fact that the average absorption at 17 Mev is roughly the same as at 10 Mev,[1] i.e., $\bar{W}=-8$ Mev. To give an average for V, the radius is normalized to $r_0=1.30$ which yields $\bar{V}=-50$ Mev. Both these averages are only good to within 10% although more precise values for each case are given in Table I. This value for the real part of the potential should be compared with -56 Mev used in the nuclear shell model[43] and -52 Mev obtained at 10 Mev.[1] The average decrease in absolute value is in agreement with current ideas of the velocity dependence of this nuclear potential.[44]

The entire analysis of this paper is based on a single-particle model, which is not entirely adequate to describe the actual many-body problem. Furthermore, a simplified optical-model potential has been used which neglects the spin and velocity dependence of the potential. Feshbach, Porter, and Weisskopf[5] have derived a particular correction to the single-particle model, which they call compound elastic scattering, and which is incoherent with the optical-model scattering. The difficulties at back angles for light elements do not seem to be due to neglect of compound elastic scattering since the optical-model cross section is sometimes too high. Furthermore, these difficulties increase with energy, whereas compound elastic effects are expected to decrease with energy.

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§ *Note added in proof.*—Recent analyses of neutron scattering [H. Feshbach and V. F. Weisskopf (private communication) and Schrank, Beyster, Walt, and Selmi (Los Alamos Report No. 2099)] also find that r_0 increases with A.
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