Plasma Oscillations in a Steady Magnetic Field: Circularly Polarized Electromagnetic Modes*†

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Van Kampen's method of treating the singularity which occurs in the solution of Boltzmann's equation for plasma oscillations is applied to the corresponding problem in the presence of a uniform external magnetic field. The propagation of electromagnetic waves in the direction of the field is considered in this paper. Expressions for the refractive index and damping of the waves have been obtained which show striking departures from the Appleton-Hartree electromagneto-ionic theory near the cyclotron resonance frequency.

I. INTRODUCTION

In the presence of an external magnetic field the motions of electrons in a plasma are very complex. The electrostatic and the electromagnetic modes of electron-oscillations of the plasma, which exist ordinarily as independent modes,[1–3] are in general, coupled due to the presence of the external magnetic field.[4,5] Only in the special case of a steady and uniform magnetic field, whose direction coincides with the direction of propagation of the waves, are the electromagnetic and the electrostatic modes independent; the electromagnetic modes in this case are circularly polarized. When propagation in directions other than that of the magnetic field are considered, the coupling of the electrostatic and the electromagnetic modes tends to zero in the limit of very large propagation constant,[4,5] i.e., when $ck \gg \Omega$ and $ck \gg \omega_0$, Ω and ω_0 being, respectively, the Larmour and the plasma frequency. In this limiting case one can use Poisson's equation (instead of the complete set of Maxwell's equations) together with Boltzmann's equation to obtain the dispersion formula for the electrostatic modes.[5] The present paper (Paper I) is concerned with the case when the direction of propagation is in the direction of the magnetic field. In a subsequent paper (Paper II) the case of the arbitrary direction of propagation will be treated.

It is known that for plasma oscillations in the absence of external magnetic fields, the method of stationary solutions of the linearized Boltzmann equation without the short-range collision term, gives dispersion formulas which contain certain singular integrals.[1,6] Recently Van Kampen[6] has shown that the prescription for integrating across the pole in these integrals is determined uniquely by the initial conditions of the problem. Exactly the same type of singular integrals occur in the dispersion formulas for the plasma oscillations in the presence of an external magnetic field. The application of Van Kampen's technique to the problem of electromagnetic oscillations propagated in the direction of the magnetic field shows that the refractive index of the plasma does not tend to infinity at the electron cyclotron resonance and that these oscillations are damped, the damping being maximum at the cyclotron frequency. This is in contrast to the ordinary Appleton-Hartree theory (hereinafter referred to as AH theory) which does not taken into account the thermal motion of electrons.

II. DISPERSION FORMULA BY THE METHOD OF STATIONARY SOLUTION OF THE BOLTZMANN EQUATION

In the absence of short-range collisions, the Boltzmann equation describing electrons in the plasma can be written as

$$\frac{\partial f}{\partial t}+\mathbf{u}\cdot\nabla_r f-\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{u}\times\mathbf{B}\right)\cdot\nabla_u f=0, \tag{1}$$

where $f=f(\mathbf{r},\mathbf{u},t)$ is the distribution function, $\mathbf{u}$ is the velocity vector, $\mathbf{B}$ is the steady, uniform magnetic field vector, and ∇_r and ∇_u are the gradient operators in the coordinate and the velocity space, respectively. The electric field $\mathbf{E}$ arises from the charges and currents that develop in the plasma in consequence of departures from the equilibrium state. Consequently $\mathbf{E}$ obeys Maxwell's equation. Thus we must have

$$\nabla_r\times(\nabla_r\times\mathbf{E})=-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}-\frac{4\pi}{c^2}\frac{\partial\mathbf{j}}{\partial t}, \tag{2a}$$

where $\mathbf{j}$ denotes the current density vector; this can be expressed as an integral over the distribution function f:

$$\mathbf{j}=-e\int\mathbf{u}f d\mathbf{u}. \tag{2b}$$

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It will be seen that in virtue of Eqs. (2a) and (2b), Eq. (1) is nonlinear in f. However, if one is dealing with very small departures from equilibrium, as obtain in oscillations with small amplitudes, Eq. (1) can be made linear by the following procedure. We split the distribution function f into two parts:

$$f(\mathbf{r},\mathbf{u},t)=n_0 f_0(\mathbf{u})+f_1(\mathbf{r},\mathbf{u},t), \tag{3}$$

where n_0 is the electron density, $f_0(\mathbf{u})$ is the equilibrium distribution function when the system is not oscillating, and $f_1(\mathbf{r},\mathbf{u},t)$ represents the perturbation caused by the oscillations. We assume that $f_1 \ll f_0$, and neglect terms quadratic in f_1 in Eq. (1). We then get a linear Boltzmann equation for f_1:

$$\frac{\partial f_1}{\partial t}+\mathbf{u}\cdot\nabla_r f_1-\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{u}\times\mathbf{B}\right)\cdot\nabla_u f_1 =\frac{e}{mc}(\mathbf{u}\times\mathbf{B})\cdot\nabla_u f_0. \tag{4}$$

Since in the absence of oscillations f_1 should vanish, we have the following differential equation for $f_0(\mathbf{u})$ from Eq. (4):

$$(\mathbf{u}\times\mathbf{B})\cdot\nabla_u f_0(\mathbf{u})=0, \tag{5}$$

whose solution is immediately found to be of the form

$$f_0(\mathbf{u})=f_0[(u_1^2+u_2^2)^{\frac{1}{2}}, u_3]. \tag{6}$$

Equation (4) now becomes, as a result of Eq. (6),

$$\frac{\partial f_1}{\partial t}+\mathbf{u}\cdot\nabla_r f_1-\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{u}\times\mathbf{B}\right)\cdot\nabla_u f_1=0. \tag{7}$$

This equation has to be solved simultaneously with Eqs. (2a) and (2b) which can be combined in the manner

$$\nabla_r\times(\nabla_r\times\mathbf{E})=-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}+\frac{4\pi e}{c^2}\frac{\partial}{\partial t}\int \mathbf{u} f_1 d\mathbf{u}, \tag{8}$$

since the first moments of $f_0(\mathbf{u})$ vanish [see Eq. (6)]. Equations (7) and (8) have stationary solution of the form

$$f_1=f_1^{(k,\omega)}(\mathbf{r},\mathbf{u},t)=g^{(k,\omega)}(\mathbf{u})e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{9a}$$

and

$$\mathbf{E}=\mathbf{E}^{(\mathbf{k},\omega)}=\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}; \tag{9b}$$

these form a complete set. We shall see below that there *does not* exist a dispersion formula, i.e., a relation between k and ω, for these stationary oscillations of the plasma.[6] Oscillations which are special linear superpositions of these stationary oscillations, and are governed by the initial conditions on f_1, *do have* a dispersion formula.

We shall first use the stationary solutions (9) to obtain, by a formal procedure, a dispersion formula. The resulting formula however contains (as we shall presently see) a singular integral and as such is not useful without an interpretation.

As mentioned in Sec. 1, we shall consider propagation in the direction of the steady magnetic field. We shall take both $\mathbf{B}$ and $\mathbf{k}$ to be along the z axis. We also introduce the new variables: $u_1=v\cos\theta$, $u_2=v\sin\theta$, $u_3=u$, $\Omega=-eB/mc$, $2F_1^{(k,\omega)}=E_1^{(k,\omega)}-iE_2^{(k,\omega)}$, $2F_2^{(k,\omega)}=E_1^{(k,\omega)}+iE_2^{(k,\omega)}$, $F_3^{(k,\omega)}=E_3^{(k,\omega)}$, $2J_1^{(k,\omega)}=j_1^{(k,\omega)}-ij_2^{(k,\omega)}$, $2J_2^{(k,\omega)}=j_1^{(k,\omega)}+ij_2^{(k,\omega)}$, and $J_3^{(k,\omega)}=j_3^{(k,\omega)}$. In terms of these new variables Eqs. (7) and (8) reduce to

$$i(ku-\omega)f_1^{(k,\omega)}-\Omega\frac{\partial f_1^{(k,\omega)}}{\partial\theta} =-\frac{n_0 e}{m}\left[\frac{df_0}{dv}(F_1^{(k,\omega)}e^{+i\theta}+F_2^{(k,\omega)}e^{-i\theta})+\frac{df_0}{du}F_3^{(k,\omega)}\right], \tag{10}$$

and

$$(k^2c^2-\omega^2)F_n^{(k,\omega)}=4\pi i\omega J_n^{(k,\omega)}, \quad n=1,2 \tag{11a}$$

$$-\omega^2 F_3^{(k,\omega)}=4\pi i\omega J_3^{(k,\omega)}. \tag{11b}$$

In these equations f_1 and $\mathbf{E}$ have the stationary forms given by Eq. (9). The solution of Eq. (10) gives

$$f_1^{(k,\omega)}=\frac{ku_0^2}{4\pi i}\left[\frac{df_0}{dv}\left(\frac{F_1^{(k,\omega)}e^{+i\theta}}{u_P+u_L-u}+\frac{F_2^{(k,\omega)}e^{-i\theta}}{u_P-u_L-u}\right) +\frac{df_0}{du}\left(\frac{F_3^{(k,\omega)}}{u_P-u}\right)\right], \tag{12}$$

where $u_P=\omega/k$, $u_L=\Omega/k$, and $u_0=\omega_0/k=(4\pi n_0 e^2/mk^2)^{\frac{1}{2}}$. Using this value of $f_1^{(k,\omega)}$ to construct $J_n^{(k,\omega)}$ $(n=1, 2, 3)$ in accordance with Eq. (2b) and substituting the result in Eqs. (11) we obtain the following dispersion formulas:

$F_1^{(k,\omega)}$ mode:

$$\frac{c^2-u_P^2}{\pi u_0^2 u_P}=\int_{-\infty}^{+\infty}\int_0^{+\infty} v^2 du dv\frac{df_0}{dv}\left(\frac{1}{u_P+u_L-u}\right), \tag{13a}$$

$F_2^{(k,\omega)}$ mode:

$$\frac{c^2-u_P^2}{\pi u_0^2 u_P}=\int_{-\infty}^{+\infty}\int_0^{+\infty} v^2 du dv\frac{df_0}{dv}\left(\frac{1}{u_P-u_L-u}\right), \tag{13b}$$

and

$F_3^{(k,\omega)}$ mode:

$$\frac{-u_P}{2\pi u_0^2}=\int_{-\infty}^{+\infty}\int_0^{+\infty} uv du dv\frac{df_0}{du}\left(\frac{1}{u_P-u}\right). \tag{13c}$$

We observe that the integral expressions on the right-hand side of Eq. (13) contain singular integrands. Postponing the discussion of these singularities, we note that substitution of $F_n^{(k,\omega)}$ $(n=1, 2, 3)$ from Eqs. (11) in (12) leads to the following integral equation for

$g^{(k,\omega)}(\mathbf{u})$:

$$g^{(k,\omega)}(\mathbf{u})=\frac{u_0^2}{2\pi k}\left[\frac{df_0}{dv}\left(\frac{C_1^{(k,\omega)}e^{+i\theta}}{u-u_P-u_L}+\frac{C_2^{(k,\omega)}e^{-i\theta}}{u-u_P+u_L}\right)+\frac{df_0}{du}\left(\frac{C_3^{(k,\omega)}}{u_P-u}\right)\right], \quad (14)$$

where

$$C_1^{(k,\omega)}=\frac{\pi u_P k}{u_P^2-c^2}\int_{-\infty}^{+\infty}\int_0^{+\infty}\int_0^{2\pi} v'^2 e^{-i\theta'} g^{(k,\omega)}(\mathbf{u}')du'dv'd\theta', \quad (15a)$$

$$C_2^{(k,\omega)}=\frac{\pi u_P k}{u_P^2-c^2}\int_{-\infty}^{+\infty}\int_0^{+\infty}\int_0^{2\pi} v'^2 e^{+i\theta'} g^{(k,\omega)}(\mathbf{u}')du'dv'd\theta', \quad (15b)$$

and

$$C_3^{(k,\omega)}=\frac{\pi k}{u_P}\int_{-\infty}^{+\infty}\int_0^{+\infty}\int_0^{2\pi} u'v' g^{(k,\omega)}(\mathbf{u}')du'dv'd\theta'. \quad (15c)$$

Also $F_n^{(k,\omega)}$ ($n=1, 2, 3$) can be written in terms of the $C_n^{(k,\omega)}$ ($n=1, 2, 3$) with the help of Eqs. (11) as follows:

$$F_n^{(k,\omega)}=-\frac{2i}{k^2}C_n^{(k,\omega)}e^{ik(z-u_Pt)}. \quad (16)$$

Since the electric field vectors $F_n^{(k,\omega)}$ are well-behaved functions without singularities, it follows from Eq. (16) that the $C_n^{(k,\omega)}$ are also free of singularities.

As we have already noted, the dispersion formulas (13) contain singular integrals. For example the integrand on the right-hand side of Eq. (13b) has a pole at $u=u_P-u_L$. Writing $f_0(u,v)=\mathcal{G}(v)F(u)$ and integrating over v, this equation can be written as

$$\frac{u_P^2-c^2}{u_Pu_0^2}=\int_{-\infty}^{+\infty}\frac{F(u)du}{(u_P-u_L)-u}=\mathcal{P}\int_{-\infty}^{+\infty}\frac{F(u)du}{(u_P-u_L)-u}+\lambda(u_P-u_L)F(u_P-u_L), \quad (17)$$

where $\mathcal{P}$ stands for the principal value and λ is an undetermined parameter which is a function of (u_P-u_L). Its value has to be determined from boundary conditions either in space or in time.[6] For instance, in scattering theory where singularities of this type occur, the value of λ is found to be $\pm i\pi$ by demanding that at infinite distance from the scatterer there should be only incoming or outgoing waves.[7] In the case of the plasma waves with which we are dealing, λ should be fixed by suitable choice of the initial perturbation Boltzmann function $f_1(\mathbf{r}, \mathbf{u}, t=0)$. Since in deriving the dispersion formula (17) we have used stationary solutions of the type $f_1^{(k,\omega)}=g^{(k,\omega)}(\mathbf{u})e^{i(kz-\omega t)}$, there is no scope for prescribing initial conditions. This means that by choosing λ appropriately, equation (17) can be fulfilled for every real k and ω, i.e., they are *not* connected by a dispersion formula.[6] However, since the stationary solutions

$$f_1^{(k,\omega)}=g^{(k,\omega)}(\mathbf{u})e^{i(kz-\omega t)}$$

form a complete set, we can construct a general solution $f_1(z,\mathbf{u},t)$ as a linear superposition of the solutions $f_1^{(k,\omega)}(z,\mathbf{u},t)$ for varying k and ω; and we can impose initial conditions on the general solution. We shall show in the following section that by suitable choice of initial conditions this method of linear superposition of the stationary solutions gives plasma waves which are damped and which are characterized by a unique dispersion formula.

III. LINEAR SUPERPOSITION OF STATIONARY SOLUTIONS

We may express the general solution as a linear superposition of $f_1^{(k,\omega)}$ for all possible k and ω in the manner:

$$f_1(z,\mathbf{u},t)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_1^{(k,\omega)}d\omega dk=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g^{(k,\omega)}(\mathbf{u})e^{i(kz-\omega t)}d\omega dk. \quad (18)$$

Thus,

$$f_1(z,\mathbf{u},0)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g^{(k,\omega)}(\mathbf{u})e^{ikz}d\omega dk=\int_{-\infty}^{+\infty}\Psi_k(\mathbf{u})e^{ikz}dk, \quad (19)$$

where

$$\Psi_k(\mathbf{u})=\int_{-\infty}^{+\infty} g^{(k,\omega)}(\mathbf{u})d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty} f_1(z,\mathbf{u},0)e^{-ikz}dz \quad (20)$$

is the Fourier transform of $f_1(z,\mathbf{u},0)$. Equation (20) can be written, after performing a suitable averaging over v and θ, as follows:

$$\langle\Psi(\mathbf{u})\rangle=\int_{-\infty}^{+\infty}\langle g^{(k,\omega)}(\mathbf{u})\rangle d\omega=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\langle f_1(z,\mathbf{u},0)\rangle e^{-ikz}dz, \quad (21)$$

where

$$\langle\Psi(\mathbf{u})\rangle=\int_0^{+\infty}\int_0^{2\pi}\Psi(\mathbf{u})v^2e^{+i\theta}dvd\theta, \quad (22)$$

and $\langle g^{(k,\omega)}(\mathbf{u})\rangle$ and $\langle f(z,\mathbf{u},0)\rangle$ are defined in the same way. In Eq. (21) and thereafter, the index k has been suppressed. Now using $g^{(k,\omega)}(\mathbf{u})$ from Eq. (14) in Eq. (21) we get

$$\langle\Psi(\mathbf{u})\rangle=u_0^2F(u)\int_{-\infty}^{+\infty}du_P\frac{C_2}{u_P-u_L-u}. \quad (23)$$

[7] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), third edition, p. 195.

The integrand on the right-hand side of the above equation has a singularity of the kind we encountered in Eq. (17). Let us assume that C_2 is a function of the difference (u_P-u_L), i.e.,

$$C_2=C_2(u_P-u_L). \tag{24}$$

In that case $\langle\Psi(\mathbf{u})\rangle$ becomes a function of u alone, so that $\langle f_1(z,\mathbf{u},0)\rangle$ is independent of the presence of the magnetic field. This choice for the form of C_2 is arbitrary. One could assume any other form for C_2, but this would make the treatment very complex; and the final result would be unaffected in any case. With the aid of (24) we have, from Eq. (23),

$$\langle\Psi(\mathbf{u})\rangle=u_0^2F(u)\left[\mathcal{P}\int_{-\infty}^{+\infty}\frac{C_2(u_P-u_L)}{(u_P-u_L)-u}\times d(u_P-u_L)+\lambda(u)C_2(u)\right]; \tag{25}$$

we can rewrite this as

$$\langle\Psi(\mathbf{u})\rangle/u_0^2F(u)=i\pi C_2^*(u)+\lambda(u)C_2(u), \tag{26}$$

where C_2^* is the Hilbert transform of C_2. In the same notation, Eq. (17) can be written as

$$[(u+u_L)^2-c^2]/u_0^2(u+u_L)=-i\pi F^*(u)+\lambda(u)F(u). \tag{27}$$

We can now eliminate λ from Eqs. (26) and (27) by decomposing $C_2(u)$, $C_2^*(u)$, $F(u)$, $F^*(u)$, and $\langle\Psi(\mathbf{u})\rangle$ into their positive and negative-frequency parts. Thus, writing

$$F^{(+)}(u)=\int_0^{\infty}\Phi(p)e^{+ipu}dp$$

and

$$F^{(-)}(u)=\int_{-\infty}^{0}\Phi(p)e^{+ipu}dp, \tag{28a}$$

where

$$\Phi(p)=(1/2\pi)\int_{-\infty}^{+\infty}F(u)e^{-ipu}du$$

is the Fourier transform of $F(u)$, we have

$$F(u)=F^{(+)}(u)+F^{(-)}(u)$$

and

$$F^*(u)=F^{(+)}(u)-F^{(-)}(u). \tag{28b}$$

$F^{(+)}(u)$ and $F^{(-)}(u)$ are the "positive frequency part" and the "negative frequency part" of $F(u)$, respectively. $F^{(+)}(u)$ has an analytic continuation without singularities in the upper half of the complex u-plane while $F^{(-)}(u)$ has a similar continuation in the lower half-plane. The above relations hold good for $C_2(u)$, $C_2^*(u)$ and $\langle\Psi(\mathbf{u})\rangle$. Now, following the method of Van Kampen,[6] we obtain, by eliminating $\lambda(u)$ from Eqs. (26) and (27):

$$C_2(u)=C_2^{(+)}(u)+C_2^{(-)}(u)=\frac{\langle\Psi(\mathbf{u})\rangle^{(+)}}{\dfrac{(u+u_L)^2-c^2}{(u+u_L)}+2\pi iu_0^2F^{(+)}(u)}+\frac{\langle\Psi(\mathbf{u})\rangle^{(-)}}{\dfrac{(u+u_L)^2-c^2}{(u+u_L)}-2\pi iu_0^2F^{(-)}(u)}. \tag{29}$$

This result is correct provided the denominators on the right-hand side do not vanish in their respective half-planes of regularity.[6] That this is actually the case will be seen from later Eqs. (37) and (38) which show that the only zero of the denominator of the first term in Eq. (29) occurs at $u=u_S=u_R-iu_I$, with $u_I>0$.

With the value of C_2 thus determined, we now examine the nature of oscillations that can arise by the use of the perturbed Boltzmann function f_1 expressed as the linear superposition of the stationary solutions. This will of course depend on the initial conditions imposed on f_1. Let us choose the initial conditions such that

$$\langle f_1(z,\mathbf{u},0)\rangle=g_0(u)e^{ikz}. \tag{30}$$

It then follows from Eq. (21) that

$$\langle\Psi(\mathbf{u})\rangle=g_0(u)\delta(k-k_0), \tag{31}$$

so that

$$C_2(u)=\left[\frac{g_0^{(+)}(u)}{Z(u)}+\frac{g_0^{(-)}(u)}{Z^*(u)}\right]\delta(k-k_0), \tag{32}$$

where

$$Z(u)=\frac{(u+u_L)^2-c^2}{(u+u_L)}+2\pi iu_0^2F^{(+)}(u), \tag{33a}$$

$$Z^*(u)=\frac{(u+u_L)^2-c^2}{(u+u_L)}-2\pi iu_0^2F^{(-)}(u). \tag{33b}$$

With the aid of Eqs. (16), (32), and (33) we obtain for F_2:

$$\begin{aligned}F_2&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_2^{(k,\omega)}dkd\omega\\&=-2i\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{k}C_2^k(u_P-u_L)e^{ik(z-u_Pt)}dkdu_P\\&=-\frac{2i}{k_0}e^{ik_0z}\int_{-\infty}^{+\infty}du_P\left[\frac{g_0^{(+)}(u_P-u_L)}{Z(u_P-u_L)}+\frac{g_0^{(-)}(u_P-u_L)}{Z^*(u_P-u_L)}\right]e^{-ik_0u_Pt}.\end{aligned} \tag{34}$$

For $k_0>0$ and $t>0$, the second term in the integrand in Eq. (34) does not contribute so that

$$F_2=\frac{-2i}{k_0}e^{ik_0z}\int_{-\infty}^{+\infty}du_P\frac{g_0^{(+)}(u_P-u_L)}{Z(u_P-u_L)}. \qquad (35)$$

Suppose now we assume that $g_0^{(+)}(u)$ is as smooth as $F(u)$ and that the only zero of $Z(u)$ occurs at $u=u_S-u_L$. It then follows from Eq. (35) that

$$F_2\sim e^{ik_0(z-u_st)}, \qquad (36)$$

where u_S is determined from the equation

$$Z(u_S-u_L)=0. \qquad (37)$$

From Eqs. (33a) and (37) one finds that

$$\frac{u_S^2-c^2}{u_S}+i\pi u_0^2F^*(u_S-u_L)+i\pi u_0^2F(u_S-u_L)=0. \qquad (38)$$

Since $F(u)$ is real and $F^*(u)$ is purely imaginary for real u, u_S must be complex in order to satisfy Eq. (38). Equation (36) therefore represents a damped wave whose phase velocity is $\mathrm{Re}(u_S)$ and whose decay time $\tau=1/k_0\,\mathrm{Im}(u_S)$. If we assume that $\mathrm{Im}(u_S)\ll\mathrm{Re}(u_S)$, then we get from Eq. (38) the following approximate identity, with $u_S=u_R-iu_I$,

$$\frac{u_R^2-c^2}{u_R}-2iu_I+i\pi u_0^2F^*(u_R-u_L)+i\pi u_0^2F(u_R-u_L)=0. \qquad (39)$$

Equating the real and imaginary parts in Eq. (39), we obtain

$$(u_R^2-c^2)/u_R=-i\pi u_0^2F^*(u_R-u_L), \qquad (40a)$$

and

$$1/\tau(u_R)=k_0u_I=\tfrac{1}{2}\pi k_0u_0^2F(u_R-u_L). \qquad (40b)$$

Equations (40a) and (40b) give, respectively, the dispersion formula and the damping of the oscillations of the F_2 mode given by Eq. (36). One finds from these equations that at $u_R=u_L$, i.e., at the electron cyclotron resonance frequency, $u_R=c$; this means that the refractive index is unity, and the damping $1/\tau$ is the maximum. The damping decreases on either side of $u_R=u_L$; in this respect it is similar to the behavior of the function $F(u)$ on either side of $u=0$.

IV. REFRACTIVE INDEX WHEN THE DISTRIBUTION IS MAXWELLIAN

The refractive index can be found from the dispersion formula (40a). For this we shall evaluate $F^*(u_R-u_L)$ for a Maxwellian distribution function

$$F(u)=(m/2\pi\kappa T)^{\frac{1}{2}}\exp(-mu^2/2\kappa T).$$

Since $F^*(u_0)=F^{(+)}(u_0)-F^{(-)}(u_0)$, and

$$F^{(+)}(u_0)=\int_0^\infty\Phi(p)e^{ipu_0}dp,\quad F^{(-)}(u_0)=\int_{-\infty}^0\Phi(p)e^{ipu_0}dp,$$

where

$$\Phi(p)=\Phi(-p)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(u)e^{-ipu}du=\frac{1}{2\pi}\exp(-p^2/4\alpha),\quad \alpha=m/2\kappa T$$

we have[8]

$$i\pi F^*(u_0)=\int_0^\infty\exp(-p^2/4\alpha)\sin pu_0dp=-2\alpha u_0\mathfrak{F}(1,\tfrac{3}{2},-\alpha u_0^2),$$

where $\mathfrak{F}$ is the Pochhamer-Kummer function. Using this result in Eq. (40a), we obtain the following formula for the refractive index:

$$n^2-1=\frac{c^2(\Omega-\omega)}{n^2\omega^3\lambda_D^2}\mathfrak{F}\left(1,\tfrac{3}{2},-\frac{c^2(\omega-\Omega)^2}{2\omega_0^2n^2\omega^2\lambda_D^2}\right), \qquad (41)$$

where $\lambda_D=(\kappa T/4\pi n_0e^2)^{\frac{1}{2}}=(1/2\alpha\omega_0^2)^{\frac{1}{2}}$ is the so-called Debye wavelength. In the limit $T=0$ when $\lambda_D=0$, formula (41) reduces to the AH formula[9]:

$$n^2-1=\omega_0^2/\omega(\Omega-\omega). \qquad (42)$$

For finite, but small T we get by making use of the asymptotic expansion of the Kummer function[8] and keeping only the lowest order terms in T,

$$n^2=\left(1+\frac{\omega_0^2}{\omega(\Omega-\omega)}\right)\Big/\left(1-\frac{\omega_0^4\omega\lambda_D^2}{c^2(\Omega-\omega)^2}\right). \qquad (43)$$

It will be seen that while formula (42) gives $n=\infty$ at $\omega=\Omega$, formula (43) gives $n=1$ at this same frequency. It is thus clear that the AH formula is inadequate for plasmas with finite temperatures and formula (41) must be used. For $\omega\to0$ Eq. (41) reduces to the AH formula for all temperatures. The same is true for $\omega\gg\Omega$. The departures between formulas (41) and (42)

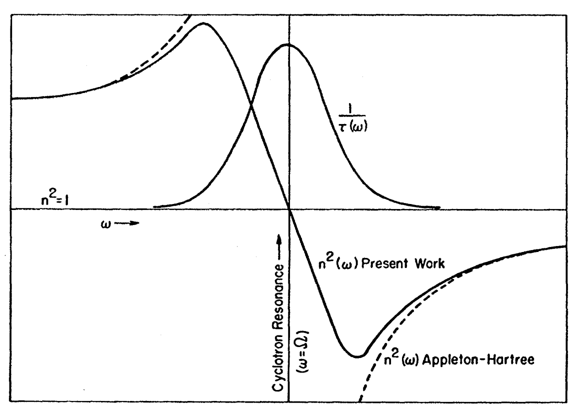

FIG. 1. Behavior of refractive index and damping near cyclotron resonance frequency.

[8] W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1954), p. 35 and p. 87.

[9] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), p. 53.

are great in the neighborhood of $\omega=\Omega$. All these results are qualitatively indicated in Fig. 1. The damping predicted by (40b) is also illustrated in the same figure. These results are similar to those of optical dispersion theory for neutral gases where the infinities in the refraction index at the resonance frequencies of the atoms are smoothed out, almost exactly the same way as ours, by adding a damping term to the equation of motion of the electron. In our treatment, however, we have no such damping terms and yet we get an attenuation of the waves. The reason for this apparent paradox has been discussed by Van Kampen[6] and need not be reapeated here.

V. KRAMERS-KRONIG DISPERSION RELATION

In this section we shall show that relations (40a) and (40b) satisfy the condition

$$n(\omega)=1+\frac{1}{\pi}\mathcal{P}\int_0^{\infty}d\omega'\frac{\gamma(\omega')}{\omega'^2-\omega^2}, \tag{44}$$

which is called the "Kramers-Kronig dispersion relation." Here γ is the attenuation coefficient. This condition is a general requirement that must be satisfied by a genuine dispersion formula just as the law of conservation of energy is a general requirement for a problem in classical mechanics. It serves as a check on the correctness of a dispersion formula. We shall presently show that this condition is obeyed by our dispersion formula. There is another reason for invoking this relation here. This is to suggest an experimental method of determining the refractive index of the electromagnetic waves in a plasma which could check the theoretical relations (40a) and (40b). If one could measure the attenuation coefficient as a function of frequency, then it would be quite easy to compute the refraction index by the use of the relation (44). We shall now show that this relation holds good for (40a) and (40b). From Eq. (40b) we get

$$\frac{1}{\tau(\omega+\Omega)}=\frac{1}{2k_0}\pi\omega_0^2F\left(\frac{\omega}{k_0}\right), \tag{45}$$

and from Eq. (40a)

$$n^2(\omega)-1=\frac{2}{\pi}\mathcal{P}\int_{-\infty}^{+\infty}d\omega'\frac{1/\tau(\omega'+\Omega)}{\omega(\omega'-\omega+\Omega)}$$
$$=\frac{2}{\pi}\mathcal{P}\int_{-\infty}^{+\infty}d\omega''\frac{1/\tau(\omega'')}{\omega(\omega''-\omega)},$$

and since $1/\tau(\omega)=\frac{1}{2}\gamma(\omega)/n(\omega)$, we have, for $(n-1)\ll 1$,

$$n(\omega)=1+\frac{1}{\pi}\mathcal{P}\int_0^{\infty}d\omega'\frac{\gamma(\omega')}{\omega'^2-\omega^2}. \tag{46}$$

Owing to the smallness of ω_0^2 our assumption that $(n-1)\ll 1$ is in keeping with Eq. (41) except for $\omega\rightarrow 0$. Thus our results check with the "Kramers-Kronig relation," except for the region $\omega\rightarrow 0$. This latter discrepancy for $\omega\rightarrow 0$ must arise from the fact that the approximate relation (40b) for $1/\tau$ does not hold good in this region.

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