In particular this gives for the average normal product

$$\langle b_{\mathbf{w}}^{*n} b_{\mathbf{w}}^{n}\rangle = (1-p_{\mathbf{w}})p_{\mathbf{w}}^{n}\frac{d^n}{dp_{\mathbf{w}}^{n}}(1-p_{\mathbf{w}})^{-1} = n!p_{\mathbf{w}}^{n}(1-p_{\mathbf{w}})^{-n} = n!N_{\mathbf{w}}^{n}, \tag{11}$$

where $N_{\mathbf{w}}$ is the average occupation number of the state $\mathbf{w}$. It can be seen that the average normal product of an even functional $\mathfrak{F}(\varphi)$ is obtained by taking all possible pairings of the φ factors, and replacing the pair $\varphi(x)\varphi(x')$ by

$$\sum_{\mathbf{w}}(\hbar ws/2V)N_{\mathbf{w}}\{\exp[i\mathbf{w}\cdot(\mathbf{x}-\mathbf{x}')-\hbar ws(t-t')] +\exp[-i\mathbf{w}\cdot(\mathbf{x}-\mathbf{x}')+\hbar ws(t-t')]\}.$$

This result shows that the average of a chronological product is obtained by contracting the product in all possible ways, using the propagator

$$D(x-x')=\sum_{\mathbf{w}}(\hbar ws/2V) \times\{(N_{\mathbf{w}}+1)\exp[i\mathbf{w}\cdot(\mathbf{x}-\mathbf{x}')-\hbar ws|t-t'|] +N_{\mathbf{w}}\exp[-i\mathbf{w}\cdot(\mathbf{x}-\mathbf{x}')+\hbar ws|t-t'|]\}. \tag{12}$$

This is also in agreement with Matsubara's rules.

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Some Theoretical Consequences of a Particle Having Mass Zero

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When the mass is zero, the operator γ_5 commutes with the Hamiltonian of a noninteracting spinor field. This leads to the possibility of a two-component neutrino that has been employed in connection with parity-nonconserving neutrino reactions. Every representation of the full inhomogeneous Lorentz group describing a free particle of arbitrary nonzero spin can be split in the same way when the mass is zero. In particular, a reduction of the free electromagnetic field from six components to three, is exhibited in a way exactly analogous to the reduction of four-component massless spinors to two components. This illustrates the fact that parity nonconservation, when it occurs, cannot be a result of any intrinsic property of a free field, but must, instead, be ascribed to particular interactions occurring in nature.

The possibility of relating the γ_5 degeneracy of the massless spin one-half field to its invariance under conformal coordinate transformations is discussed. The two-component free Dirac particle is invariant under the conformal point transformations, and also under the reciprocal radius transformations. Some different definitions of the conformal group are distinguished.

I. INTRODUCTION

THE study of strange particles has, for two reasons, stimulated interest in processes involving neutrinos. In the first place, the lifetimes of hyperons and of K-mesons imply interaction constants corresponding remarkably well with those found in the β decay of nucleons and of muons and in pion decay.[1] In the second place, its zero mass suggested for the neutrino a special role with respect to parity.[2,3]

When the mass is zero, the operator γ_5 commutes with the Hamiltonian of a free Dirac particle, so that a free neutrino is describable by a projection $\frac{1}{2}(1+\gamma_5)\psi$ or $\frac{1}{2}(1-\gamma_5)\psi$ of the four component ψ. The present paper investigates the question whether this projection, and what it implies, is a special property of massless neutrinos, or whether a corresponding reduction is also possible with spins other than one-half when the mass is zero. The theory of spin-zero and spin-one particles of vanishing mass is developed in Sec. II in a way entirely parallel to that for spin one-half.

In Sec. IV the γ_5 degeneracy of the massless Dirac particle is related to another consequence of having zero mass, to invariance under conformal transformations. These transformations, which have been discussed in more than one way in the literature, are defined in Sec. III. Only the interaction-free case is considered.

Notation

$$\boldsymbol{\partial}\equiv\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z\equiv i\mathbf{p};$$
$$\partial_t\equiv\partial/c\partial t\equiv i\partial_4;$$
$$\mathbf{a}\cdot\mathbf{b}\equiv a_1b_1+a_2b_2+a_3b_3;$$
$$a\cdot b\equiv\mathbf{a}\cdot\mathbf{b}+a_4b_4.$$

Repeated indices are summed. Dummy indices take on all values $\mu=1, 2, 3, 4$. The asterisk designates the

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1 S. A. Bludman and M. A. Ruderman, Phys. Rev. 101, 910 (1956).

2 S. A. Bludman, Phys. Rev. 102, 1420 (1956).

3 T. D. Lee and C. N. Yang, Phys. Rev. 106, 1671 (1957); L. Landau, Nuclear Phys. 3, 127 (1957); A. Salam, Nuovo cimento 5, 299 (1957).

complex conjugate of a c-number or the Hermitian adjoint of an operator.

$\epsilon^{\mu\nu\lambda\rho} \equiv$ alternating symbol on 1, 2, 3, 4;

$\epsilon^{ijk} \equiv$ alternating symbol on 1, 2, 3;

$\beta_{\lambda\mu\rho} \equiv \beta_\lambda \beta_\mu \beta_\rho$, etc. $\hbar = c = 1$.

II. REDUCED REPRESENTATIONS FOR MASS-ZERO FIELDS

A. Dirac Field

What will be called the γ_5 degeneracy of the massless spin one-half field reveals itself in the following way. In covariant form the Dirac equation is

$$(\gamma_\mu \partial_\mu + m)\psi = 0. \quad (1)$$

When transformed to Hamiltonian form, the Hamiltonian,

$$\begin{aligned} H &= \boldsymbol{\alpha} \cdot \mathbf{p} + \gamma_4 m \\ &= \gamma_5 \boldsymbol{\sigma} \cdot \mathbf{p} + \gamma_4 m, \end{aligned} \quad (2)$$

commutes with $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$, if and only if $m=0$. When the mass does vanish, the state vector ψ decomposes according to the eigenvalues ± 1 of γ_5:

$$i\partial_t \psi_\pm = \pm \boldsymbol{\sigma} \cdot \mathbf{p} \psi_\pm, \quad (3)$$

where

$$\psi_\pm \equiv \tfrac{1}{2}(1 \pm \gamma_5)\psi.$$

A particle described by Eq. (3) involving ψ_+ alone, and not ψ_- or ψ_+^*, will be called a Weyl particle. For ψ_+, the positive-frequency solutions of Eq. (3) correspond to $\boldsymbol{\sigma} \cdot \mathbf{p}/|E| = 1$ or spin along the direction of propagation $m_s = \frac{1}{2}$; the negative-frequency solutions, to $\boldsymbol{\sigma} \cdot \mathbf{p}/|E| = -1$ or $m_s = -\frac{1}{2}$. Space reflection must, therefore, involve particle-antiparticle conjugation, defined as the interchange of positive and negative frequencies. Indeed, since

$$i\partial_t \psi_+^* = \boldsymbol{\sigma}^* \cdot \mathbf{p} \psi_+^*, \quad (4)$$

the choice[4]

$$\psi_+^{\text{ref}}(\mathbf{x}) = \epsilon^{-1} \psi_+^*(-\mathbf{x}), \quad (5)$$

where $\epsilon \boldsymbol{\sigma}^* \epsilon^{-1} = -\boldsymbol{\sigma}$, makes ψ_+^{ref} obey the same interaction-free equation as ψ_+.

Under gauge transformations of the first kind, however, ψ^* and therefore ψ_+^{ref} transform oppositely to ψ_+. Hence, if a Weyl particle had charge $e \neq 0$, the operation (5) would combine reflection and charge conjugation. If, on the other hand, the particle has no coupling to the electromagnetic field, a distinction can be drawn between (1) particle-antiparticle conjugation, which *is* involved in space reflection in the Weyl theory, and (2) charge conjugation for which the identity operation can be trivially chosen.[4]

[4] J. A. McLennan, Jr., Phys. Rev. **106**, 821 (1957); K. M. Case, Phys. Rev. **107**, 307 (1957).

B. Scalar and Vector Fields

In order to see whether with bosons a development parallel to that for spinors is possible, the scalar and vector meson equations will be cast in a form like the Dirac equation, the Hamiltonian found, and the limit $m=0$ taken, and a decomposing operator like γ_5 sought.

1. Duffin-Kemmer Formalism and the Mass-Zero Limit

Bosons are described[5] by the covariant equation

$$(\beta_\mu \partial_\mu + M)\psi = 0, \quad (6)$$

where the β_μ obey the Duffin-Kemmer algebra determined by

$$\beta_{\mu\nu\lambda} + \beta_{\lambda\nu\mu} = \beta_\mu \delta_{\nu\lambda} + \beta_\lambda \delta_{\nu\mu}, \quad (7)$$

special instances of which are

$$(1 - \beta_\mu{}^2)\beta_\nu = \beta_\nu \beta_\mu{}^2, \quad \mu \neq \nu \quad (8)$$

$$(1 - \beta_\mu{}^2)\beta_\mu = 0. \quad (9)$$

Although only four β's occur in Eq. (6), these relations are also satisfied by a fifth,

$$\beta_5 = (1/4!)\epsilon^{\mu\nu\lambda\rho} \beta_{\mu\nu\lambda\rho}, \quad (10)$$

just as in the case of the Dirac algebra. The algebra of the β_μ is distinguished from that of the γ_μ in that the β_μ have more than one irreducible representation and are singular. The representations by 10×10 and by 5×5 matrices describe spin-one and spin-zero fields, respectively. The vector meson field, for example, is described by a ten-component quantity ψ which, in a particular choice of the β_μ, is given by $\psi = (\mathbf{E}, \mathbf{H}, \mathbf{A}, \phi)$.

Plane-wave solutions are assured if the second-order equation $(\Box - M^2)\psi = 0$ is derivable from the first-order Eq. (6). It can be shown generally[6] that to describe massless particles, M must be a singular matrix. Also, for a given irreducible representation, M must transform as a scalar under Lorentz transformations. The Dirac algebra has only one inequivalent irreducible representation, from which M must be a multiple of unity: for M to be singular it must be zero. In the Duffin-Kemmer algebra, the representation of the Lorentz transformations is reducible, and M need not be a multiple of unity. As will be seen in Eq. (15), M is indeed nonzero, and if Eq. (6) is to describe massless particles then M must be a nonvanishing singular matrix γ. In order to develop,[7] in as parallel a way as possible, the cases of mass $m \neq 0$ and of mass zero, we shall proceed from Eq. (6) as far as possible with M unspecified. Only in those equations that are different for mass m and mass zero will M be replaced by m or γ.

A second-order wave equation can be derived by multiplying the first-order equation (6) by $\partial_\rho \beta_{\rho\nu}$ on the

[5] N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939).

[6] H. J. Bhabha, Revs. Modern Phys. **21**, 451 (1949).

[7] The Hamiltonian formulation for nonzero mass has been obtained by Kemmer, reference 5, and later authors. Massless particles have been treated in the abstract Γ formalism by Harish-Chandra, Proc. Roy. Soc. (London), **A186**, 502 (1946). A somewhat different treatment has been presented in order to develop, in as parallel a way as possible, the cases of finite mass and zero mass and to find an actual realization of the zero-mass case.

left and using the basic relation (7) to obtain

$$\partial_\nu(M\psi)=\partial_\rho\beta_{\rho\nu}(M\psi). \tag{11}$$

Then, upon differentiating, one obtains

$$\partial_\nu^2(M\psi)=\partial_{\rho\nu}\beta_{\rho\nu}(M\psi). \tag{12}$$

If the particle has mass, $M=m\neq 0$ can be divided through in Eq. (12). Applying Eq. (6) twice then gives

$$(\Box-m^2)\psi=0 \quad (\text{mass } m\neq 0). \tag{13}$$

If the particle is massless then Eq. (6), on left multiplication by $(1-\gamma)$, leads to

$$\beta_\mu\partial_\mu(\gamma\psi)=0, \tag{14}$$

provided

$$(1-\gamma)\beta_\mu=\beta_\mu\gamma, \quad (1-\gamma)\gamma=0. \tag{15}$$

Then, in Eq. (12),

$$\Box(\gamma\psi)=0 \quad (\text{mass zero}). \tag{16}$$

The first of Eqs. (15) shows that γ is not zero; the second of Eqs. (15) shows that it must be a singular matrix, an idempotent projection operator.

A Hamiltonian formulation is obtained by choosing $\nu=4$ in Eq. (11) and writing

$$\partial_4(1-\beta_4^2)(M\psi)-\boldsymbol{\partial}\cdot\boldsymbol{\beta}\beta_4(M\psi)=0. \tag{17}$$

Multiplying Eq. (6) by β_4 on the left, one obtains

$$\partial_4\beta_4^2\psi+\boldsymbol{\partial}\cdot\beta_4\boldsymbol{\beta}\psi+\beta_4M\psi=0. \tag{18}$$

If the particle has mass, then Eq. (17) can be divided by $M=m$ and added to Eq. (18) to give

$$i\partial_t\psi=H\psi\equiv(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta_4 m)\psi \quad (\text{mass } m) \tag{19}$$

where

$$\boldsymbol{\alpha}\equiv i(\beta_4\boldsymbol{\beta}-\boldsymbol{\beta}\beta_4). \tag{20}$$

If the particle has no mass, then Eq. (18) can be multiplied by $M=\gamma$ on the left, and conditions (15) applied before adding to Eq. (17), to give

$$i\partial_t(\gamma\psi)=H(\gamma\psi)\equiv\boldsymbol{\alpha}\cdot\mathbf{p}(\gamma\psi) \quad (\text{mass zero}). \tag{21}$$

The Hamiltonian formulation is completed by the time-independent equations,

$$[\boldsymbol{\beta}\cdot\boldsymbol{\partial}\beta_4^2+(1-\beta_4^2)M]\psi=0, \tag{22}$$

obtained from Eq. (6) on multiplication by $(1-\beta_4^2)$. Equation (22) relates the field quantities to the space derivatives of the potentials, $\mathbf{H}=\nabla\times\mathbf{A}$. If the mass is nonvanishing, Eq. (22) also relates the potentials to the space derivatives of the field quantities. If the mass is zero, the projection operator γ must be chosen so that $M\psi=\gamma\psi=(\mathbf{E},\mathbf{H},0,0)$. Then $0=\nabla\cdot\mathbf{E}$ and the potentials are *not* derivable from the field quantities. This can be accomplished by choosing $\gamma=\beta_5^2$, so that because β_5 satisfies the relations (8) and (9), γ satisfies the conditions (15).

The second-order wave equation (16) and the Hamiltonian equations (21), which read

$$\nabla\times\mathbf{E}+\partial\mathbf{H}/\partial t=0, \quad \nabla\times\mathbf{H}-\partial\mathbf{E}/\partial t=0, \tag{23}$$

are obtained only for the field quantities. For the potentials $(1-\gamma)\psi=(0,0,\mathbf{A},\phi)$, such equations can be obtained only by a special choice of gauge, such as imposing

$$\partial_\nu(1-\gamma)\psi=\partial_\rho\beta_{\rho\nu}(1-\gamma)\psi, \tag{24}$$

in addition to the Eq. (21) derived for $\gamma\psi$.

The equations of motion (6) for finite or zero mass are derivable from the Lagrangian density,

$$\mathcal{L}=\tfrac{1}{2}[\bar{\psi}\beta_\mu\partial_\mu\psi-\partial_\mu\bar{\psi}\beta_\mu\psi]+\bar{\psi}M\psi,$$

where the adjoint is

$$\bar{\psi}=\psi^\dagger\eta_4, \quad \eta_4=2\beta_4^2-1.$$

From this Lagrangian, a conserved energy-momentum tensor and current density are obtained:

$$\Theta_{\mu\nu}=\bar{\psi}(\delta_{\mu\nu}-\beta_{\mu\nu}-\beta_{\nu\mu})M\psi,$$
$$j_\mu=i\bar{\psi}\beta_\mu\psi=[\psi^\dagger(\beta_4\boldsymbol{\alpha}+\boldsymbol{\alpha}\beta_4)\psi,\ \psi^\dagger\beta_4\psi].$$

Because $\eta_4\beta_4=\beta_4$ is an indefinite matrix, the particle density j_0 is indefinite in sign. This indefinite particle or charge density characterizes bosons as compared with fermions. Because of it, an extra factor β_4 appears[5] in the definition of expectation values. In particular, the energy density is given by

$$W=\psi^\dagger\beta_4H\psi=\psi^\dagger M\psi=-\Theta_{44}.$$

The usual expression $\psi^\dagger H\psi$ is indefinite in sign in the c-number theory and is appropriate only where Fermi-Dirac quantization is involved.

The eigenvalues λ of H may be positive or negative corresponding to positive- or negative-frequency solutions, or zero corresponding to the static solution. Since the total energy,

$$E=\int\psi^\dagger\beta_4H\psi dV=\int\psi^\dagger M\psi dV,$$

is positive, and

$$E=\lambda N,$$

where

$$N=\int\psi^\dagger\beta_4\psi dV$$

is the total number of particles (which is conserved in the absence of interaction), the positive- and negative-frequency solutions correspond to particle and antiparticle, with the energy positive in any case.

2. *Spin-One Analog of the Weyl Particle*

With Eq. (21), a treatment of the vector field $\Phi\equiv\gamma\psi$ can now be given that is analogous to that given for spin one-half at the beginning of this section. Since, by Eq. (7), β_5 commutes with $\boldsymbol{\alpha}$ but not with β_4, β_5

commutes with the Hamiltonian (21) for mass zero but not with that (19) for finite mass. In fact,

$$\boldsymbol{\alpha}\gamma=\beta_5\mathbf{S}\gamma, \tag{25}$$

where

$$\mathbf{S}=\frac{1}{i}\boldsymbol{\beta}\times\boldsymbol{\beta} \tag{26}$$

obeys the angular-momentum commutation rules

$$[S_i,S_j]=i\epsilon_{ijk}S_k \tag{27}$$

and has the eigenvalues $+1, 0, -1$ of a spin-one particle.

When the mass vanishes, Φ decomposes according to the eigenvalues ±1 of β_5:

$$\Phi_\pm\equiv\tfrac{1}{2}(1\pm\beta_5)\Phi, \tag{28}$$

$$i\partial_t\Phi_\pm=\pm\mathbf{S}\cdot\mathbf{p}\Phi_\pm. \tag{29}$$

(In the five-dimensional representation $\beta_5=0$, and no such decomposition takes place for the spin-zero field.)

For the state Φ_+, the positive- and negative-frequency solutions in Eq. (29) correspond to $m_S=\pm1$ along the direction of propagation. The eigenvalue $m_S=0$ belongs to the static solution of Eq. (29) and might be eliminated by a supplementary condition. In the abbreviated theory involving Φ_+, as in the original theory involving Φ, the energy is positive for both signs of the frequency.[8] The abbreviated theory corresponds to the same association between spin direction and the sign of the frequency that was discussed in the Weyl theory.

The law of transformation under space reflection,

$$\Phi_+^{\mathrm{ref}}(\mathbf{x})=\eta_4\Phi_+^*(-\mathbf{x}), \tag{5$'$}$$

involves particle-antiparticle conjugation, as in Eq. (5). [Eq. (5′) is especially clear in the representation in which $\psi=(\mathbf{E},\mathbf{H},\mathbf{A},\phi)$. Here one finds $\Phi(\mathbf{x})=(\mathbf{F}(\mathbf{x}), i\mathbf{F}(\mathbf{x}), 0, 0)$, where $\mathbf{F}=\mathbf{E}+i\mathbf{H}$, and $\Phi^{\mathrm{ref}}(\mathbf{x})=(-\mathbf{F}^*(-\mathbf{x}), -i\mathbf{F}^*(-\mathbf{x}), 0, 0)$, corresponding to the conventional behavior of $\mathbf{E}$ and $\mathbf{H}$ under space reflections.]

The particle-antiparticle conjugation involved would generate opposite changes in Φ and Φ^{ref} under gauge transformations of the first kind unless the vector particle had $e=0$ as well as $m=0$. When the massless vector particle *is* neutral, a charge conjugation operation can be defined and is, up to a phase factor, the identity operation. This is actually the situation for the electromagnetic field.

This discussion indicates that when the mass is zero and in the absence of interaction, a vector field Φ_+ can be constructed that is related to Φ as two-component spinors ψ_+ are related to four-component spinors ψ. The free-particle equations are, in both cases, invariant under space reflection with a law of transformation involving the interchange of positive and negative frequencies. The particles involved must then be of zero charge as well as zero mass if the theory is to be invariant under charge conjugation.

The free neutrino is not intrinsically parity-nonconserving[4] any more than is the free photon. Indeed, while mass and spin are intrinsic properties of free particles, parity (like electric charge) is meaningful only relative to the fields with which a particle interacts. The difference between the neutrino and the photon is in their couplings. In β-decay interactions, apparently neutrinos (or antineutrinos) of only one spin polarization are emitted, while in electromagnetic interactions, photons of both polarizations (called particle and antiparticle in the Weyl-like formulation) are emitted and absorbed.

Incidentally, β_5 is not the only operator for the spin-one field that commutes with H if the mass is zero. A second operator is the scalar $\eta_5=2\beta_5^2-1$. In fact, the decomposition according to $\frac{1}{2}(1+\eta_5)=\beta_5^2=\gamma$ and $\frac{1}{2}(1-\eta_5)=1-\beta_5^2=1-\gamma$ is precisely the separation of ψ into gauge-independent and gauge-dependent parts discussed after Eq. (22) above. In the spin one-half case $1-\gamma=0$, and in the spin-zero case $\gamma=0$, and no gauge group appears in the zero-mass limit. For mass zero, the spin-one field can be decomposed into both Φ_+ and Φ_- and into gauge-independent and gauge-dependent parts. The spin one-half field decomposes into ψ_+ and ψ_- parts but not into gauge-independent and gauge-dependent parts. The spin-zero field decomposes neither way.

Considerable detail has been gone into to show that the Weyl 2-component spin one-half particle is not unique, but that an entirely analogous possibility exists theoretically for the massless vector field. The result may be stated quite generally for any representation $\psi(m,s)$ of the inhomogeneous Lorentz group. If the mass is zero and the quantized spin $s\neq0$, each representation $\psi(0,s)$ of the group including space reflections splits into two $\psi_\pm(0,s)$ that are inequivalent except under reflections. These two representations correspond to the two spin states, parallel and antiparallel to the direction of propagation, that are possible for a massless particle of discrete spin. The zero-spin representation is exceptional because no such splitting of the one-dimensional representation is possible. If the mass is not zero, the various states of spin polarization are equivalent under proper Lorentz transformations.

III. CONFORMAL TRANSFORMATIONS

The γ_5 degeneracy, or the possibility of a reduced representation has been seen to arise, with nonzero spin, whenever the mass is zero. It is also known[9] that wave

[8] With this understanding the difficulties of interpretation and quantization disappear from the otherwise identical electrodynamics discussed by J. R. Oppenheimer, Phys. Rev. **38**, 725 (1931).

[9] J. A. McLennan, Jr., Nuovo cimento **10**, 1360 (1956) and thesis, Lehigh University, 1952 (unpublished). I am indebted to Dr. McLennan for the loan of this thesis. McLennan actually proves the conformal invariance of an entire class of homogeneous wave equations, including the neutrino, in a very general but formal way. The conformal invariance of the massless Dirac wave equation was also proven in the somewhat unphysical six-dimensional formalism by P. A. M. Dirac, Ann. Math. **37**, 429 (1936).

equations for zero-mass particles—or at least those, including the neutrino and Maxwell equations, that can be written so as to be homogeneous in the space-time derivatives—are invariant under the conformal group. In this section several definitions of the conformal group are distinguished in order to see what connection there is, if any, between the conformal invariance of the massless Dirac equation and its γ_5 degeneracy.

A. Reciprocal-Radius Transformations and Acceleration Transformations

The second-order zero-mass wave equation,

$$\square V=0, \tag{30'}$$

is invariant with respect to the transformations by reciprocal radii:

$$x_\mu'=x_\mu/x^2, \tag{30}$$

or $r'=1/r$, where $r=(x_\mu^2)^{\frac{1}{2}}$. In two and three dimensions this is well known and is the basis for the treatment of certain problems in potential theory. In n dimensions, from the form of the Laplacian in spherical coordinates, it follows that a law of transformation can be devised for $V(r)$,

$$V'(r)=r^{(n-2)}V(r), \tag{31}$$

so that with $\square(r')\equiv\square'(r)$, the transformed equation $\square'(r)V'(r)=0$ is a consequence of the original equation $\square(r)V(r)=0$. That such a law of transformation on the field variables can be found so that the transformed equation holds if the original holds, is the sense in which an equation is said to be invariant under a certain point transformation.

The result of two reciprocal-radius transformations, the first $x_\mu''=x_\mu/x^2$ about the origin, and the second $x_\mu'=(x_\mu''+\alpha_\mu)/(x+\alpha)^2$ about another point α_μ [symbolically $1/x'=(1/x)+\alpha$] is the nonlinear transformation

$$x_\mu'=(x_\mu+\alpha_\mu x^2)/(1+2\alpha\cdot x+\alpha^2x^2). \tag{32}$$

This will be called an "acceleration transformation" because it carries a point at rest into uniform accelerated motion, just as a Lorentz transformation carries a point at rest into motion with uniform velocity.

Under an acceleration transformation,

$$x'^2=x^2/(1+2\alpha\cdot x+\alpha^2x^2),$$

from which it follows that "circles" generally, meaning hyperspheres, hyperhyperboloids and hyperplanes in four dimensions, go into other "circles" under conformal transformations. Plane waves do not generally go into plane waves.

This kinematic interpretation of the acceleration transformation leads simply[10] to an interesting result in electrodynamics. The conformal group is known[11] to be the widest group of transformations leaving Maxwell's equations invariant. The radiation damping force vanishes for a charged particle at rest, and under the conformal group transforms as a vector density. Motion with uniform acceleration, including motion with uniform velocity, is obtained from rest by conformal transformation. Uniformly accelerated motion is the entire class of motions of a charged particle, for which there is no reaction of radiation on its motion. So far as electromagnetic fields are concerned, not only systems moving with constant velocity with respect to one another, but also systems uniformly accelerated with respect to one another, are equivalent.[12]

B. The Conformal Group and the Proper Conformal Group

The reciprocal-radius transformations, together with the uniform dilatations

$$x_\mu'=\lambda x_\mu, \tag{33}$$

and the ten inhomogeneous Lorentz transformations

$$x_\mu'=a_{\mu\nu}x_\nu+b_\mu, \tag{34}$$

generate the conformal group, C_4. Since a reciprocal-radius transformation inverts the orientation of the three space axes, the elements of C_4 consist of the space reflections and reciprocal-radius transformations along with fifteen continuous transformations: six homogeneous Lorentz transformations, four translations, four accelerations, and the dilatation.

The acceleration transformations introduced in Eq. (32) by *two* reciprocal-radius transformations are proper transformations. By omitting the individual reciprocal-radius transformations and the reflections, a completely continuous fifteen-parameter group, C_{4+}, consisting of the proper inhomogeneous Lorentz transformations, the accelerations, and the dilatations, is defined.[9]

All relativistic equations homogeneous in the space-time derivatives admit the similarity group. The second-order wave equations (30′), because they admit the reciprocal-radius transformations, admit the full conformal group C_4. In the next section the invariance of the first-order massless Dirac equation will be investigated under both C_4 and C_{4+}.

C. Conformal Transformations on the Metric

The conformal transformations have to this point been considered as point transformations $x_\mu\rightarrow x_\mu'$ with the metric $g_{\mu\nu}\rightarrow g_{\mu\nu}$ unaltered. Since, under the acceleration transformation (32),

$$(dx_\mu')^2=[(1-2\alpha\cdot x)^2+4(\alpha\times x)^2](dx_\mu)^2, \tag{35}$$

and under the dilatation (33),

$$(dx_\mu')^2=\lambda^2(dx_\mu)^2, \tag{36}$$

[10] S. A. Bludman, Phys. Rev. **95**, 654(A) (1954).

[11] E. Cunningham, Proc. London Math. Soc. **8**, 77 (1910); H. Bateman, Proc. London Math. Soc. **8**, 223, 469 (1910).

[12] This is the basis of the extension of special relativity for electromagnetism devised by L. Page, Phys. Rev. **49**, 254 (1936). See also L. Page and N. I. Adams, *Electrodynamics* (D. Van Nostrand Company, Inc., New York, 1940), and H. P. Robertson, Phys. Rev. **49**, 755 (1936).

the conformal group does not preserve lengths [other than the light cone $(dx_\mu)^2=0$]. Instead, $ds^2=(dx_\mu)^2$ is carried into $ds'^2=\sigma(x)(dx_\mu)^2$, where σ depends on x in a definite way prescribed by the parameters α_μ, λ in Eqs. (35) and (36).

An alternative point of view is to regard the conformal transformations as ones in which $x_\mu \to x_\mu$ but the metric is transformed $g_{\mu\nu} \to g_{\mu\nu}'=\sigma(x)g_{\mu\nu}$. So long as $\sigma(x)$ is determined by parameters of the transformation in the same way as before, these two points of view are fairly equivalent. However, the second point of view can be generalized by allowing the $g_{\mu\nu}$ to transform arbitrarily so long as their ratios are unchanged. This generalization leads to a group of transformations on the metric depending on an arbitrary *function* $\sigma(x)$ rather than on arbitrary parameters. This is the conformal group considered by Schouten and Haantjes[13] and by Pauli.[14] In the next section the conformal invariance of the neutrino wave equation will be considered under the first definition, the conformal group as a group of point transformations depending on fifteen parameters.

IV. CONFORMAL INVARIANCE OF THE MASSLESS DIRAC EQUATION

Since the massless Dirac equation

$$\gamma_\mu \partial_\mu \psi = 0 \tag{37}$$

admits the dilatation and the inhomogeneous Lorentz transformations, only its invariance under the acceleration transformations (32) and the reciprocal-radius transformations (31) must be shown, in order to prove invariance under the proper or improper conformal group.

The acceleration transformation may be regarded infinitesimally. Then Eq. (37) implies $\gamma_\mu\partial_\mu'\psi'=0$ provided

$$\gamma_\mu \delta(\partial_\mu)\psi + \gamma_\mu\partial_\mu\delta\psi = 0, \tag{38}$$

where

$$\delta(\partial_\mu) = 2[\alpha\cdot x\partial_\mu + \alpha_\mu x\cdot\partial - x_\mu\alpha\cdot\partial], \tag{39}$$

and $\delta\psi=\psi'-\psi$ is the infinitesimal change in ψ to be found. The form (39) suggests a law of transformation in which $\delta\psi$ is linear in α_μ and x_μ as well as ψ. Lorentz invariance almost completely restricts the form of the spinor transformation and

$$\delta\psi = 2[\alpha\cdot x + \gamma_{\mu\nu}\alpha_\mu x_\nu]\psi \tag{40}$$

is found, together with Eq. (39), to satisfy Eq. (38).

Under the reciprocal-radius transformation,

$$\partial_\mu' = 2x_\mu x\cdot\partial + x^2\partial_\mu, \tag{41}$$

an even fewer number of Lorentz invariants can be formed out of x_μ and the γ's. The form of Eq. (31) suggests the occurrence of some power of x^2, and in fact one obtains

$$\psi' = x^2\gamma\cdot x\psi. \tag{42}$$

The free-particle Dirac equation with zero mass is thus invariant under both C_4 and C_{4+}.[9]

Projection by γ_5 as in Eq. (3) gives, for the acceleration transformations,

$$\delta\psi_\pm = 2[\alpha\cdot x + \gamma_{\mu\nu}\alpha_\mu x_\nu]\psi_\pm, \tag{43}$$

and for the reciprocal-radius transformation,

$$\psi_\pm' = x^2\gamma\cdot x\psi_\mp. \tag{44}$$

Because $\epsilon^{-1}\psi_+^*$ satisfies the same equation as ψ_-,

$$\psi_+' = x^2\gamma\cdot x\epsilon^{-1}\psi_+^* \tag{45}$$

satisfies the Weyl equation after the transformation by reciprocal radii. This transformation therefore includes the same particle-antiparticle conjugation that was involved in the reflection transformation (5). By Eqs. (43) and (45), the free-particle Weyl equation is invariant under both C_4 and C_{4+}.

The interpretation of an interaction is related basically to the presence of an *extended* gauge group, i.e., a group depending on arbitrary functions rather than arbitrary parameters. For this reason, it is especially interesting that the invariance of the massless spin one and spin one-half, but not spin zero, equations extends to the extended conformal metric transformations defined in Sec. III C.[14] Does this suggest a basic role for the massless neutrino field along with the massless electromagnetic and gravitational fields as the seat of a universal, especially primordial, and relatively weak, interaction between all matter? The neutrino interaction is universal, in the sense that the pion or K-meson interactions are not, in that possibly all particles,[15] fermion and boson, interact directly with neutrinos with comparable strength.

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[13] J. A. Schouten and J. Haantjes, Proc. Koninkl. Ned. Akad. Wetenschap. **39**, 1059 (1936).
[14] W. Pauli, Jr., Helv. Phys. Acta **13**, 204 (1940).
[15] There is no evidence at present that neutrinos appear in the decay of hyperons.