Use of Field Theory Techniques in Quantum Statistical Mechanics

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It is shown that the normal product in quantum statistical mechanics, as defined by Matsubara, has zero expectation value. Another method for the treatment of the Bose particles is given.

MATSUBARA[1] has shown that the formalism of quantum field theory can be adapted for the calculation of the grand partition function in quantum statistical mechanics. It will be shown here that his computation rules are exact, and do not neglect terms of order $1/N$ as he supposed. The average of a product of boson operators is calculated by a method different from Matsubara's.

He considers a system of fermions and bosons whose interaction can be expressed in terms of the field operators

$$\begin{aligned}\psi^*(x)&=V^{-\frac{1}{2}}\sum_{\mathbf{k}}a_{\mathbf{k}}^*\exp(-i\mathbf{k}\cdot\mathbf{x}+\epsilon_{\mathbf{k}}t),\\ \psi(x)&=V^{-\frac{1}{2}}\sum_{\mathbf{k}}a_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{x}-\epsilon_{\mathbf{k}}t),\\ \varphi(x)&=\sum_{\mathbf{w}}(\hbar ws/2V)^{\frac{1}{2}}[b_{\mathbf{w}}\exp(i\mathbf{w}\cdot\mathbf{x}-\hbar wst)\\ &\qquad+b_{\mathbf{w}}^*\exp(-i\mathbf{w}\cdot\mathbf{x}+\hbar wst)],\end{aligned}\tag{1}$$

with

$$\{a_{\mathbf{k}},a_{\mathbf{k}'}^*\}=\delta_{\mathbf{k}\mathbf{k}'}\quad[b_{\mathbf{w}},b_{\mathbf{w}'}^*]=\delta_{\mathbf{w}\mathbf{w}'},\tag{2}$$

where x denotes the position $\mathbf{x}$ and a variable t which is related to the temperature. Denoting averages over the free-particle grand canonical ensemble by $\langle\cdots\rangle$, he shows that it is necessary to evaluate quantities like $\langle T[\mathfrak{H}(\psi^*\psi)]\rangle$ and $\langle P[\mathfrak{F}(\varphi)]\rangle$, where T is Wick's[2] chronological product, and $\mathfrak{H}$ and $\mathfrak{F}$ are functionals of the operators $\psi^*(x)\psi(x)$ and $\varphi(x)$.

The fermion operators are rewritten as

$$\begin{aligned}\psi^*(x)&=u_1^*(x)+u_2(x),\\ \psi(x)&=u_1(x)+u_2^*(x),\end{aligned}\tag{3}$$

$$\begin{aligned}u_1(x)&=\sum_{\mathbf{k}}(1-g_{\mathbf{k}})\alpha_{1\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{x}-\epsilon_{\mathbf{k}}t),\\ u_2(x)&=\sum_{\mathbf{k}}g_{\mathbf{k}}^*\alpha_{2\mathbf{k}}\exp(-i\mathbf{k}\cdot\mathbf{x}+\epsilon_{\mathbf{k}}t),\end{aligned}\tag{4}$$

$$\begin{aligned}a_{\mathbf{k}}&=\alpha_{1\mathbf{k}}=\alpha_{2\mathbf{k}}^*,\\ a_{\mathbf{k}}^*&=a_{1\mathbf{k}}^*=\alpha_{2\mathbf{k}}.\end{aligned}\tag{5}$$

The normal product of $\mathfrak{H}(\psi^*\psi)$ is defined by making the substitution (3) in $\mathfrak{H}$, and reordering each term so that all the u_1^* operators come first, followed by u_2^*, u_2 and u_1, with a sign $\pm$ according to whether the permutation is even or odd. A chronological product can be reduced to a sum of such normal products in the way demonstrated by Wick,[2] but with the propagator

$$S(x-x')=\begin{cases}V^{-1}\sum_{\mathbf{k}}|g_{\mathbf{k}}|^2\\ \quad\times\exp[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')+\epsilon_{\mathbf{k}}(t-t')],\ t\geq t'\\ V^{-1}\sum_{\mathbf{k}}(|g_{\mathbf{k}}|^2-1)\\ \quad\times\exp[-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')+\epsilon_{\mathbf{k}}(t-t')],\ t<t'.\end{cases}\tag{6}$$

To show that the average of a normal product is zero, consider one Fourier component of the functional $\mathfrak{H}(\psi^*\psi)$, which will consist of a product of a and a^* operators. The normal product is obtained by replacing these by α_1,α_2^* and α_1^*,α_2 and reordering. Since the occupation numbers of different free-particle states are statistically independent, the average of a product of operators factorizes into the product of the averages of those operators referring to a single state. If such an average is nonvanishing, there must be an equal number of creation and annihilation operators, and two similar operators must not be adjacent. The average normal product of two operators is zero if[1]

$$|g_{\mathbf{k}}|^2=[\exp(\alpha+\beta\epsilon_{\mathbf{k}})+1]^{-1},\tag{7}$$

and the only nonvanishing normal product of higher order is $\alpha_{1\mathbf{k}}^*\alpha_{2\mathbf{k}}^*\alpha_{2\mathbf{k}}\alpha_{1\mathbf{k}}$, and antisymmetry makes its coefficient zero in the normal product of $a_{\mathbf{k}}^*a_{\mathbf{k}}a_{\mathbf{k}}^*a_{\mathbf{k}}$. Therefore Matsubara's decomposition of the average chronological product of fermion operators is exact.

A Fourier component of $\mathfrak{F}(\varphi)$ will also vanish in the average unless there are an equal number of creation and annihilation operators referring to each state. If there are equal numbers, it can be shown that

$$\langle\mathfrak{F}(b_{\mathbf{w}}^*,b_{\mathbf{w}'})\rangle=\prod_{\mathbf{w}''}(1-p_{\mathbf{w}''})\mathfrak{F}\left(p_{\mathbf{w}},\frac{\partial}{\partial p_{\mathbf{w}'}}\right)\times\prod_{\mathbf{w}''}(1-p_{\mathbf{w}''})^{-1},\tag{8}$$

where

$$p_{\mathbf{w}}=\exp(-\beta\hbar ws)\tag{9}$$

since $p_{\mathbf{w}}$, $\partial/\partial p_{\mathbf{w}}$ satisfy the same commutation relations as $b_{\mathbf{w}}^*$, $b_{\mathbf{w}}$, and since

$$\langle(b_{\mathbf{w}}^*b_{\mathbf{w}})^n\rangle=(1-p_{\mathbf{w}})\left(p_{\mathbf{w}}\frac{d}{dp_{\mathbf{w}}}\right)^n(1-p_{\mathbf{w}})^{-1}.\tag{10}$$

[1] T. Matsubara, Progr. Theoret. Phys. Japan 14, 351 (1955).
[2] G. C. Wick, Phys. Rev. 80, 268 (1950).

In particular this gives for the average normal product

$$\begin{aligned}\langle b_{\mathbf{w}}^{*n} b_{\mathbf{w}}^{n}\rangle &= (1-p_{\mathbf{w}})p_{\mathbf{w}}^{n}\frac{d^n}{dp_{\mathbf{w}}^{n}}(1-p_{\mathbf{w}})^{-1} \\ &= n!p_{\mathbf{w}}^{n}(1-p_{\mathbf{w}})^{-n} \\ &= n!N_{\mathbf{w}}^{n},\end{aligned} \tag{11}$$

where $N_{\mathbf{w}}$ is the average occupation number of the state $\mathbf{w}$. It can be seen that the average normal product of an even functional $\mathfrak{F}(\varphi)$ is obtained by taking all possible pairings of the φ factors, and replacing the pair $\varphi(x)\varphi(x')$ by

$$\sum_{\mathbf{w}}(\hbar ws/2V)N_{\mathbf{w}}\{\exp[i\mathbf{w}\cdot(\mathbf{x}-\mathbf{x}')-\hbar ws(t-t')] +\exp[-i\mathbf{w}\cdot(\mathbf{x}-\mathbf{x}')+\hbar ws(t-t')]\}.$$

This result shows that the average of a chronological product is obtained by contracting the product in all possible ways, using the propagator

$$\begin{aligned}D(x-x') = \sum_{\mathbf{w}}(\hbar ws/2V) &\times\{(N_{\mathbf{w}}+1)\exp[i\mathbf{w}\cdot(\mathbf{x}-\mathbf{x}')-\hbar ws|t-t'|] \\ &+N_{\mathbf{w}}\exp[-i\mathbf{w}\cdot(\mathbf{x}-\mathbf{x}')+\hbar ws|t-t'|]\}.\end{aligned} \tag{12}$$

This is also in agreement with Matsubara's rules.

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Some Theoretical Consequences of a Particle Having Mass Zero

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When the mass is zero, the operator γ_5 commutes with the Hamiltonian of a noninteracting spinor field. This leads to the possibility of a two-component neutrino that has been employed in connection with parity-nonconserving neutrino reactions. Every representation of the full inhomogeneous Lorentz group describing a free particle of arbitrary nonzero spin can be split in the same way when the mass is zero. In particular, a reduction of the free electromagnetic field from six components to three, is exhibited in a way exactly analogous to the reduction of four-component massless spinors to two components. This illustrates the fact that parity nonconservation, when it occurs, cannot be a result of any intrinsic property of a free field, but must, instead, be ascribed to particular interactions occurring in nature.

The possibility of relating the γ_5 degeneracy of the massless spin one-half field to its invariance under conformal coordinate transformations is discussed. The two-component free Dirac particle is invariant under the conformal point transformations, and also under the reciprocal radius transformations. Some different definitions of the conformal group are distinguished.

I. INTRODUCTION

THE study of strange particles has, for two reasons, stimulated interest in processes involving neutrinos. In the first place, the lifetimes of hyperons and of K-mesons imply interaction constants corresponding remarkably well with those found in the β decay of nucleons and of muons and in pion decay.[1] In the second place, its zero mass suggested for the neutrino a special role with respect to parity.[2,3]

When the mass is zero, the operator γ_5 commutes with the Hamiltonian of a free Dirac particle, so that a free neutrino is describable by a projection $\frac{1}{2}(1+\gamma_5)\psi$ or $\frac{1}{2}(1-\gamma_5)\psi$ of the four component ψ. The present paper investigates the question whether this projection, and what it implies, is a special property of massless neutrinos, or whether a corresponding reduction is also possible with spins other than one-half when the mass is zero. The theory of spin-zero and spin-one particles of vanishing mass is developed in Sec. II in a way entirely parallel to that for spin one-half.

In Sec. IV the γ_5 degeneracy of the massless Dirac particle is related to another consequence of having zero mass, to invariance under conformal transformations. These transformations, which have been discussed in more than one way in the literature, are defined in Sec. III. Only the interaction-free case is considered.

Notation

$$\boldsymbol{\partial}\equiv\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z\equiv i\mathbf{p};$$

$$\partial_t\equiv\partial/c\partial t\equiv i\partial_4;$$

$$\mathbf{a}\cdot\mathbf{b}\equiv a_1b_1+a_2b_2+a_3b_3;$$

$$a\cdot b\equiv\mathbf{a}\cdot\mathbf{b}+a_4b_4.$$

Repeated indices are summed. Dummy indices take on all values $\mu=1, 2, 3, 4$. The asterisk designates the

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[1] S. A. Bludman and M. A. Ruderman, Phys. Rev. **101**, 910 (1956).

[2] S. A. Bludman, Phys. Rev. **102**, 1420 (1956).

[3] T. D. Lee and C. N. Yang, Phys. Rev. **106**, 1671 (1957); L. Landau, Nuclear Phys. **3**, 127 (1957); A. Salam, Nuovo cimento **5**, 299 (1957).