Analyticity of the Schrödinger Scattering Amplitude and Nonrelativistic Dispersion Relations

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The Fredholm theory of integral equations is used to give a rigorous proof of the analyticity and boundedness of the ordinary nonrelativistic scattering amplitude for a fixed momentum transfer. The results follow from ordinary quantum mechanics and certain conditions on the potentials. These conditions are stated explicitly, and the bound states are treated with rigor. It is shown that the amplitude vanishes in the limit of large momenta, and thus simple dispersion relations are derived. Finally, it is proved that the partial-wave expansion is convergent in the unphysical region, provided the potentials satisfy the same conditions as above.

I. INTRODUCTION

IN relativistic field theory it has been rigorously proved by Bogoliubov[1] that the S-matrix has the required analytic properties necessary to derive the dispersion relations for scattering at a fixed energy-momentum transfer. These relations were first heuristically derived by Goldberger in the forward case, and by Goldberger and others for fixed momentum transfer. It is of both theoretical and practical interest to see precisely under what conditions similar relations hold in ordinary quantum mechanics for the scattering of particles from potentials.

Such relations have been conjectured by Goldberger.[2] They follow easily if one assumes the uniform convergence of the Born series for the scattering amplitude, since one can see by inspection that every term in the perturbation series is analytic in the momentum for a fixed momentum transfer.[3] However, this is a very strong assumption and amounts to assuming the result.

In this paper we shall consider the ordinary nonrelativistic Schrödinger scattering amplitude, written as a function of the magnitude of the momentum, k, and the magnitude of the momentum transfer, τ. Under certain explicit assumptions on the potentials, which are quite general, we shall rigorously prove that the scattering amplitude for a fixed τ is analytic in the upper half-plane of k, and uniformly bounded there as well as on the real axis. Our approach is inspired by Jost and Pais' successful application of the Fredholm theory to the scattering integral equation.[4] We shall follow their notation rather closely, and denote by J.P. their paper which we shall refer to often below.

In Sec. II we write out the Fredholm solution of the scattering integral equation and from that get the corresponding expression for the scattering amplitude, writing it explicitly as a function of k and τ, the momentum and momentum transfer respectively. The expression will have the form of a quotient of two functions. In Sec. III we shall prove that for a large class of potentials, essentially those for which the integrals, $\int_0^\infty r|V(r)|dr$, and $\int_0^\infty \exp(\alpha r)r^2|V(r)|dr$ are finite, the numerator of the scattering amplitude is analytic in k in the upper-half complex k-plane, and uniformly bounded for $\operatorname{Im} k \geqslant 0$, if τ is real and $\frac{1}{2}\tau \leqslant \alpha$ $(\alpha \geqslant 0)$.

The use of the Fredholm theory enables us to handle the bound states rigorously and easily. In Sec. IV we shall show that the Fredholm denominator is analytic and bounded as above. We shall also prove that for real potentials the zeros of this denominator occur only on the positive imaginary axis of k, and that, except for the zero at $k=0$, all the other zeros correspond to bound states. Thus our scattering amplitude will have poles on the positive imaginary axis whenever we have bound states.

In Sec. V we show that, except for the Born term, the scattering amplitude actually vanishes on a large semicircle in the upper half-plane of k, in the limit as the radius approaches infinity. We also show that restricting the singularity of the potential at the origin makes the scattering amplitude vanish faster for large $|k|$. In Sec. VI we apply the Cauchy integral formula and get the dispersion relations.

Finally, in Sec. VII, we prove rigorously that in the unphysical region, $|k| \leqslant \frac{1}{2}\tau$, the partial-wave expansion written in terms of k and τ converges absolutely if $\tau < \alpha$. This gives us a method of getting the value of the scattering amplitude in the unphysical region from the phase shifts.

In all our proof we do not use anything which is foreign to the usual formulation of quantum mechanics. In the field-theoretic case the microcausality condition, expressed in terms of the vanishing of the commutator outside the light cone, plays a major role. Below we make no such explicit assumptions, and we do not have a precise definition of what causality means for a potential of infinite range. But the analog to the commutator condition is certainly built into quantum mechanics.

Certainly the choice of the Green's function that we

* Lockhead Fellow, 1956–57.
[1] N. N. Bogoliubov, mimeographed notes (unpublished).
[2] M. L. Goldberger (private communication).
[3] After this work was completed a preprint of a paper by D. Y. Wong was brought to the attention of the author. In it these dispersion relations were written down on the basis of such an assumption.
[4] R. Jost and A. Pais, Phys. Rev. 82, 840 (1951).

use in our integral equation has quite a lot to do with our results. But that is not all, and the fact that we use a local potential is quite important. For even if we had used the same Green's function, but an interaction of the form $\int V(\mathbf{x},\mathbf{y})\psi(y)d^3y$, our results would not have followed. Of course, if $V(\mathbf{x},\mathbf{y})$ is zero outside a finite region, we can make our scattering amplitude bounded by multiplying by $\exp(ika)$, a being the range of V. But including functions with essential singularities at infinity in the dispersion relations is rather meaningless.

We just mention here for completeness that if $V(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}+\mathbf{y})V(|\mathbf{y}|)$ i.e., the pure exchange potential, all our results, except for the first Born term, will still hold. But this case is rather an exception, and the results are probably due to its form being very similar to the local case where $V(\mathbf{x},\mathbf{y})=\delta(\mathbf{x}-\mathbf{y})V(|\mathbf{y}|)$.

In what follows we shall restrict ourselves to central potentials, but the generalization to noncentral potentials is not difficult. We have also applied the same techniques to the case of the Dirac equation. The Green's function is similar and the results follow easily except for stronger restrictions on the behavior of the potential at the origin. The amplitude in this case will not vanish for large E, and we get dispersion relations with one subtraction. The integral in these relations will extend from $E=m$, to infinity, and we have to deal with only one unphysical region, namely $m<E<(m^2+\frac{1}{4}\tau^2)^{\frac{1}{2}}$. We shall discuss these results in more detail in a later paper.

II. SCATTERING AMPLITUDE IN TERMS OF THE FREDHOLM SERIES

In this section we shall write down the Fredholm solution of the scattering integral equation, and from it express the the scattering amplitude in terms of the Fredholm series. The amplitude will be written as an an explicit function of the magnitude of the momentum and that of the momentum transfer. We follow the notation of J.P.

The Schrödinger equation,

$$[\nabla^2+k^2-\lambda V(r)]\psi(\mathbf{r})=0 \tag{1}$$

is written in the usual dimensionless form. The potential strength λ is defined by normalization of $V(r)$ at small distances.

We look for the solution of (1) which for large r behaves like a plane wave plus an outgoing spherical wave.

$$\psi(\mathbf{x})=\exp(i\mathbf{k}\cdot\mathbf{x})+\lambda\int K(\mathbf{x},\mathbf{y})\psi(\mathbf{y})d\mathbf{y}, \tag{2}$$

$$K(\mathbf{x},\mathbf{y})=-V(y)\exp(ik|\mathbf{x}-\mathbf{y}|)/4\pi|\mathbf{x}-\mathbf{y}|. \tag{3}$$

The solution of (2) is a solution of (1), and the Green's function which we have chosen guarantees that our solution will satisfy the required boundary conditions. The asymptotic form of the solution of (2) is

$$\psi(\mathbf{x})\sim\exp(i\mathbf{k}\cdot\mathbf{x})+|\mathbf{x}|^{-1}\exp(ik|\mathbf{x}|)f(k,\tau), \tag{4}$$

where $f(k,\tau)$ is the scattering amplitude, and

$$\tau=k[2(1-\cos\theta)]^{\frac{1}{2}}$$

is the magnitude of the momentum transfer.

Since $K(\mathbf{x},\mathbf{y})$ is singular for $\mathbf{x}=\mathbf{y}$, we cannot directly write the Fredholm solution of (2). Instead, we iterate (2) once and get

$$\psi(\mathbf{x})=F(\mathbf{k},\mathbf{x})+\lambda^2\int K_2(\mathbf{x},\mathbf{y})\psi(\mathbf{y})d\mathbf{y}, \tag{5}$$

where

$$F(\mathbf{k},\mathbf{x})=\exp(i\mathbf{k}\cdot\mathbf{x})+\lambda\int K(\mathbf{x},\mathbf{y})\exp(i\mathbf{k}\cdot\mathbf{y})d\mathbf{y}, \tag{6}$$

and

$$K_2(\mathbf{x},\mathbf{y})=\int K(\mathbf{x},\mathbf{z})K(\mathbf{z},\mathbf{y})d\mathbf{z}. \tag{7}$$

The solution of (5) is given formally by

$$\psi(\mathbf{k},\mathbf{x})=F(\mathbf{k},\mathbf{x})+\lambda^2\int\frac{\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})}{\Delta(\lambda^2,k)}F(\mathbf{k},\mathbf{y})d\mathbf{y}, \tag{8}$$

where

$$\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})=K_2(\mathbf{x},\mathbf{y})+\sum_{n=1}^{\infty}\frac{(-\lambda^2)^n}{n!}\int d\mathbf{x}_1\cdots\int d\mathbf{x}_n \times B^{(n)}(k;\mathbf{x},\mathbf{y},\mathbf{x}_1,\cdots,\mathbf{x}_n), \tag{9}$$

and

$$\Delta(\lambda^2,k)=1+\sum_{n=1}^{\infty}\frac{(-\lambda^2)^n}{n!}\int d\mathbf{x}_1\cdots\int d\mathbf{x}_n \times D^{(n)}(k;\mathbf{x}_1,\cdots,\mathbf{x}_n). \tag{10}$$

The Fredholm determinants $B^{(n)}$ and $D^{(n)}$ are given by

$$B^{(n)}(\mathbf{k};\mathbf{x},\mathbf{y},\mathbf{x}_1,\cdots,\mathbf{x}_n) = \begin{vmatrix} K_2(\mathbf{x},\mathbf{y}) & K_2(\mathbf{x},\mathbf{x}_1) & \cdots & K_2(\mathbf{x},\mathbf{x}_n) \\ K_2(\mathbf{x}_1,\mathbf{y}) & K_2(\mathbf{x}_1,\mathbf{x}_1) & \cdots & K_2(\mathbf{x}_1,\mathbf{x}_n) \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ K_2(\mathbf{x}_n,\mathbf{y}) & K_2(\mathbf{x}_n,\mathbf{x}_1) & \cdots & K_2(\mathbf{x}_n,\mathbf{x}_n) \end{vmatrix}, \tag{11}$$

and

$$D^{(n)}(k;\mathbf{x}_1,\cdots,\mathbf{x}_n) = \begin{vmatrix} K_2(\mathbf{x}_1,\mathbf{x}_1) & \cdots & K_2(\mathbf{x}_1,\mathbf{x}_n) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ K_2(\mathbf{x}_n,\mathbf{x}_1) & \cdots & K_2(\mathbf{x}_n,\mathbf{x}_n) \end{vmatrix}. \tag{12}$$

Under very general assumptions about $V(r)$, namely that $V(r)\leqslant M'/r^2$, and $\int_0^\infty r|V(r)|dr\leqslant M<\infty$, Jost and Pais have shown that for any finite $|\lambda|$ the series implied in (8), (9), and (10) converges uniformly and absolutely, and that if λ is real $\psi(\mathbf{k},\mathbf{x})$ will have no singularities for k real except possibly at $k=0$.

Starting from (8), we shall now derive a similar expression for the scattering amplitude. The scattered wave, $\psi_s(\mathbf{k},\mathbf{x})$ is given by

$$\psi_s(\mathbf{k},\mathbf{x})=\lambda\int K(\mathbf{x},\mathbf{y})\exp(i\mathbf{k}\cdot\mathbf{y})d\mathbf{y}+\lambda^2\int\frac{\Delta(\lambda^2,k;\,\mathbf{x},\mathbf{y})}{\Delta(\lambda^2,k)}F(\mathbf{k},\mathbf{y})d\mathbf{y},\quad(13)$$

and the scattering amplitude is

$$f(k,\tau)=\lim_{|\mathbf{x}|\to\infty}[|\mathbf{x}|\exp(-ik|\mathbf{x}|)\psi_s(\mathbf{k},\mathbf{x})].\quad(14)$$

After taking the limit in (14), expanding the determinants of the numerator in (13) in terms of co-factors of their first rows, and rearranging terms, we get

$$f(k,\tau)=-\frac{\lambda}{4\pi}\int\exp(-i\mathbf{k}\cdot\mathbf{y})V(y)\exp(i\mathbf{k}\cdot\mathbf{y})d\mathbf{y}-\frac{\lambda^2}{4\pi}\int\exp(-i\mathbf{k}'\cdot\mathbf{z})V(z)K(\mathbf{z},\mathbf{y})F(\mathbf{k},\mathbf{y})d\mathbf{z}d\mathbf{y}-\frac{\lambda^4}{4\pi}\int\exp(-i\mathbf{k}'\cdot\mathbf{z})V(z)K(\mathbf{z},\mathbf{x}_1)\times\frac{\Delta(\lambda^2,k;\,\mathbf{x}_1,\mathbf{y})}{\Delta(\lambda^2,k)}F(\mathbf{k},\mathbf{y})d\mathbf{z}d\mathbf{x}_1d\mathbf{y},\quad(15)$$

where $\mathbf{k}'=k\mathbf{x}/|\mathbf{x}|$.

Substituting (6) for $F(\mathbf{k},\mathbf{y})$, we get

$$f(k,\tau)=-\frac{\lambda}{4\pi}\int\exp(-i\mathbf{k}'\cdot\mathbf{y})V(y)\exp(i\mathbf{k}\cdot\mathbf{y})d\mathbf{y}-\frac{\lambda^2}{4\pi}\int\exp(-i\mathbf{k}'\cdot\mathbf{z})N_2(\mathbf{z},\mathbf{y})\exp(i\mathbf{k}\cdot\mathbf{y})d\mathbf{z}d\mathbf{y}-\frac{\lambda^3}{4\pi}\int\exp(-i\mathbf{k}'\cdot\mathbf{z})N_3(\mathbf{z},\mathbf{y})\exp(i\mathbf{k}\cdot\mathbf{y})d\mathbf{z}d\mathbf{y}-\frac{\lambda^4}{4\pi}\int\exp(-i\mathbf{k}'\cdot\mathbf{z})\frac{N_4(\mathbf{z},\mathbf{y})}{\Delta(\lambda^2,k)}\exp(i\mathbf{k}\cdot\mathbf{y})d\mathbf{z}d\mathbf{y}-\frac{\lambda^5}{4\pi}\int\exp(-i\mathbf{k}'\cdot\mathbf{z})\frac{N_5(\mathbf{z},\mathbf{y})}{\Delta(\lambda^2,k)}\exp(i\mathbf{k}\cdot\mathbf{y})d\mathbf{z}d\mathbf{y},\quad(16)$$

where

$$N_2(\mathbf{z},\mathbf{y})=V(z)K(\mathbf{z},\mathbf{y}),$$

$$N_3(\mathbf{z},\mathbf{y})=V(z)K_2(\mathbf{z},\mathbf{y}),$$

$$N_4(\mathbf{z},\mathbf{y})=V(z)\int K(\mathbf{z},\mathbf{x}_1)\Delta(\lambda^2,k;\,\mathbf{x}_1,\mathbf{y})d\mathbf{x}_1,\quad(17)$$

$$N_5(\mathbf{z},\mathbf{y})=V(z)\int K(\mathbf{z},\mathbf{x}_1)\Delta(\lambda^2,k;\,\mathbf{x}_1,\mathbf{x}_2)K(\mathbf{x}_2,\mathbf{y})d\mathbf{x}_1d\mathbf{x}_2.$$

Now we introduce a set of new variables corresponding to those used in field-theoretic dispersion relations. We let

$$\boldsymbol{\pi}=\tfrac{1}{2}(\mathbf{k}+\mathbf{k}');\quad \mathbf{r}=\mathbf{y}-\mathbf{z},\qquad \boldsymbol{\tau}=\mathbf{k}-\mathbf{k}';\quad \mathbf{R}=\tfrac{1}{2}(\mathbf{y}+\mathbf{z}).\quad(18)$$

From these definitions it easily follows that $\boldsymbol{\pi}\cdot\boldsymbol{\tau}=0$; $\pi^2=k^2-\frac{1}{4}\tau^2$, and $\cos\theta=(1-\tau^2/2k^2)$. We can now write for (16)

$$f(k,\tau)=-\frac{\lambda}{4\pi}\tilde{V}(\tau)+G_2(k,\tau)+G_3(k,\tau)+\frac{G_4(k,\tau)}{\Delta(\lambda^2,k)}+\frac{G_5(k,\tau)}{\Delta(\lambda^2,k)},\quad(19)$$

where

$$G_j(k,\tau)=-\frac{\lambda^j}{4\pi}\int\exp[i(k^2-\tfrac{1}{4}\tau^2)^{\frac{1}{2}}\mathbf{n}\cdot\mathbf{r}]\times N_j(\mathbf{R}-\tfrac{1}{2}\mathbf{r},\,\mathbf{R}+\tfrac{1}{2}\mathbf{r})\exp(i\boldsymbol{\tau}\cdot\mathbf{R})d\mathbf{r}d\mathbf{R},\quad(20)$$

$$j=2,3,4,5;\quad \mathbf{n}=\boldsymbol{\pi}/|\boldsymbol{\pi}|,$$

and

$$\tilde{V}(\tau)=\int\exp(i\boldsymbol{\tau}\cdot\mathbf{y})V(y)d\mathbf{y}.\quad(21)$$

We note that $G_j(k,\tau)$ does not depend on the direction of $\boldsymbol{\tau}$ nor on the direction of $\mathbf{n}$. Furthermore, it is easy to see from (17) that $N_j(k)=N_j^*(-k)$. Hence we have

$$G_j(k,\tau)=G_j^*(-k,\boldsymbol{\tau}).\quad(22)$$

For any physical scattering, i.e., $|k|\geqslant\frac{1}{2}\tau$, all the series and integrals in (20) converge uniformly and are well defined for a large class of potentials, namely those for which the integrals $\int_0^\infty r|V(r)|dr$ and $\int_0^\infty r^2|V(r)|dr$ are finite. This is already clear in J.P. However, in the next section we are going to let k be complex, and let it vary in the upper-half plane. In that case we have to define the scattering amplitude in the unphysical region on the real axis, $|k|\leqslant\frac{1}{2}\tau$, and the integrals in (20) will not be convergent there for $\tau\neq0$ unless the potential falls off fast enough.

In field theory one is able to define the scattering amplitude in the unphysical region by going back to the Fourier transform of the corresponding four-fold vacuum expectation value. Here, if we want to stay within the bounds of ordinary quantum mechanics, we have no such underlying structure, and to the best of our knowledge restrictions on the potentials are necessary.

III. ANALITICITY AND BOUNDEDNESS OF THE $G_j(k,\boldsymbol{\tau})$

In this section we shall extend the domain of definition of $G_j(k,\tau)$, in (20), into the upper-half complex k-plane, while keeping τ real and fixed. We shall prove the following lemma.

Lemma I.—If the potential, $V(r)$, satisfies the three conditions,

$$|V(r)| \leqslant M'/r^2, \tag{A}$$

$$\int_0^\infty r|V(r)|dr \leqslant M < \infty, \tag{B}$$

$$\int_0^\infty \exp(\alpha r) r^2 |V(r)| dr \leqslant L < \infty, \quad \alpha \geqslant 0, \tag{C}$$

then for real τ, $\frac{1}{2}\tau \leqslant \alpha$, the $G_j(k,\tau)$ are analytic functions of k regular in the region $\mathrm{Im}\, k = \kappa > 0$ and uniformly bounded in the region $\kappa \geqslant 0$. On the real axis they are continuous boundary values of the above analytic functions, with branch points at $k = \pm\frac{1}{2}\tau$.

For later use we note there that the conditions (B) and (C) imply that

$$\int_0^\infty \exp(\alpha r) r |V(r)| dr \leqslant M'' < \infty, \quad \alpha \geqslant 0. \tag{B$'$}$$

The proof of Lemma I depends mainly on a theorem concerning the integrals of analytic functions over a parameter. We shall state this theorem at this point, as it will clarify our approach.

Theorem I.[5] If $\Phi(k,x)$ is analytic in k, regular in a certain region Γ of the complex k-plane, and if $\Phi(k,x)$ is continuous in the closed region composed of Γ and its boundary B, then if

$$f(k) = \int_\Lambda \Phi(k,x) dx,$$

$f(k)$ is analytic in k, regular and uniformly bounded and in the region Γ, provided that there exist $\Psi(x)$ such that $|\Phi(k,x)| \leqslant \Psi(x)$ for all k on B, and provided that

$$\int_\Lambda \Psi(x) dx < \infty.$$

Of course, we can also say that for k on the boundary B, $f(k)$ is continuous in k. Furthermore, recalling the fact that an analytic function takes its maximum value on the boundary of a region of analyticity, we can conclude that if $|\Phi(k,x)| \leqslant \Psi(x)$ for all k on B, this inequality will also hold for all k in Γ.

Our method of attack will follow the conditions of theorem I. The closed curve B in our case will be the real axis from $-k_R$ to $+k_R$, and a large semicircle in the upper half-plane with radius k_R. We shall show that in the integrals in (20) defining the $G_j(k,\tau)$ the integrands are analytic in k, for $\kappa > 0$, and that for k on B they are bounded by a function of $\mathbf{r}$, $\mathbf{R}$, and τ which is integrable if $\frac{1}{2}\tau \leqslant \alpha$.

The proof of our lemma is trivial for the case of $G_2(k,\tau)$. One sees by inspection that the integrand is analytic and regular for $\kappa > 0$. From (20) and (17), one easily gets for $k \geqslant 0$

$$|G_2(k,\tau)| \leqslant \frac{|\lambda|^2}{16\pi^2} \int \exp(\tfrac{1}{2}\tau r) |V(|\mathbf{R} - \tfrac{1}{2}\mathbf{r}|)| \cdot |V(|\mathbf{R} + \tfrac{1}{2}\mathbf{r}|)| (1/r) d\mathbf{R} d\mathbf{r}. \tag{23}$$

Changing back to the variables $\mathbf{z}$ and $\mathbf{y}$ we get

$$|G_2(k,\tau)| \leqslant \frac{|\lambda|^2}{16\pi^2} \int \exp[\tfrac{1}{2}\tau(z+y)] \times \frac{|V(z)| \cdot |V(y)|}{|\mathbf{z} - \mathbf{y}|} d\mathbf{z} d\mathbf{y}. \tag{24}$$

This last integral is finite, by (B$'$) and (C), as long as $\frac{1}{2}\tau \leqslant \alpha$. Thus the conditions of theorem I are satisfied. On the real axis of k, the integrand in (20) has branch points at $k = \pm\frac{1}{2}\tau$, and hence $G_2(k,\tau)$ will have branch points there too.

For the remaining three $G_j(k,\tau)$, $j = 3, 4, 5$, the proof of our lemma depends on finding an upper bound for the iterated kernel $K_2(\mathbf{x},\mathbf{y})$ and for the numerator of the Fredholm resolvent, $\Delta(\lambda^2, k; \mathbf{x},\mathbf{y})$ in the region $\kappa \geqslant 0$. We shall write

$$K_2(\mathbf{x},\mathbf{y}) = A(\mathbf{x},\mathbf{y}) V(y)/4\pi y, \tag{25}$$

where from (3) and (7) we have

$$A(\mathbf{x},\mathbf{y}) = \frac{|\mathbf{y}|}{4\pi} \int \frac{\exp[ik|\mathbf{x} - \mathbf{x}'|] \times V(x') \exp[ik|\mathbf{x}' - \mathbf{y}|] d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'| \cdot |\mathbf{x}' - \mathbf{y}|}. \tag{26}$$

For $\kappa \geqslant 0$, we have

$$|A(\mathbf{x},\mathbf{y})| \leqslant \exp[-\kappa|\mathbf{x} - \mathbf{y}|] \int \frac{|\mathbf{y}|}{4\pi} \frac{|V(x')|}{|\mathbf{x} - \mathbf{x}'| \cdot |\mathbf{x}' - \mathbf{y}|} d\mathbf{x}'.$$

In J.P. it is shown that for any $V(r)$ satisfying (A) and (B) the integral on the right hand side of the above inequality is always less than a constant. Therefore, for $\kappa \geqslant 0$,

$$\begin{aligned} |A(\mathbf{x},\mathbf{y})| &\leqslant N \exp[-\kappa|\mathbf{x} - \mathbf{y}|], \\ |K_2(\mathbf{x},\mathbf{y})| &\leqslant N \exp[-\kappa|\mathbf{x} - \mathbf{y}|] |V(y)|/(4\pi y). \end{aligned} \tag{27}$$

For large k in the upper half-plane one can see from (26) that $|A(\mathbf{x},\mathbf{y})|$ will become very small since it will either oscillate to zero for large $\mathrm{Re}\, k$ or be damped to zero for large κ. In Appendix I we shall prove that for any ϵ no matter how small, one can find k, with $|k|$ large enough, so that

$$|A(\mathbf{x},\mathbf{y})| \leqslant \epsilon \exp[-\kappa|\mathbf{x} - \mathbf{y}|], \quad \kappa \geqslant 0. \tag{28}$$

It is clear from (26), (27), and theorem I that $K_2(\mathbf{x},\mathbf{y})$ is analytic in k, regular for $\kappa > 0$. Furthermore, using (27) we shall show at the end of this section that

[5] See for example E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, New York, 1939), second edition, pp. 99–100.

$\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})$ is analytic in k, regular for $\kappa>0$ and that it has the following bound:

$$|\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})|\leqslant C_2|V(y)|/(4\pi y),\quad \kappa\geqslant 0, \qquad (29)$$

where C_2 is a constant.

On the other hand, using (28), we shall show that for $|k|$ large enough,

$$|\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})| \leqslant \epsilon C_3 \exp[-\kappa|\mathbf{x}-\mathbf{y}|]|V(y)|/(4\pi y),\quad \kappa\geqslant 0, \qquad (30)$$

where C_3 is a constant. This last inequality gives an upper bound for Δ for k on any large semicircle in the upper half-plane.

We remark here that both $K_2(\mathbf{x},\mathbf{y})$ and $\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})$ are continuous in k for $\kappa\geqslant 0$. For $j=4$, 5 we use (29) to estimate the integrands in (20) for k real, and (30) for k on the large semicircle. For the case $j=3$ we use (27) to get a bound on the integrand. Hence we get from (20) and (17) that for k on B,

$$|G_j(k,\tau)|\leqslant|\lambda|^jL_j\int \exp[\tfrac{1}{2}\tau|\mathbf{z}-\mathbf{y}|] \times\frac{|V(z)|\cdot|V(y)|}{4\pi y}d\mathbf{z}d\mathbf{y}, \qquad (31)$$

where the L_j's are constants. Now since the integrand is (20) was analytic in the region inside B, then (31) holds for any k inside B also. Consequently, since we can choose k_R as large as we please, (31) holds for all k such that $\kappa\geqslant 0$. From (B′) and (C) one easily sees that for $\tau/2\leqslant\alpha$ the integral on the right hand side of (31) is finite. On the real axis one sees from (20) that the $G_j(k,\tau)$, for $j=3$, 4, 5, also have branch points at $k=\pm\tau/2$. Hence, except for the assertions we made about $\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})$ the proof of our lemma is completed.

We now prove that $\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})$ is analytic in k regular in the region $\kappa>0$, and we also derive the bounds (29) and (30). From (11) and (25) we see that the Fredholm determinants $B^{(n)}(k;\mathbf{x},\mathbf{y},\mathbf{x}_1,\cdots,\mathbf{x}_n)$ can be written as follows:

$$B^{(n)}(k;\mathbf{x},\mathbf{y},\mathbf{x}_1,\cdots,\mathbf{x}_n) =A^{(n)}(k;\mathbf{x},\mathbf{y},\mathbf{x}_1,\cdots,\mathbf{x}_n)\left(\frac{V(y)}{4\pi y}\right)\prod_{j=1}^{n}\left(\frac{V(x_j)}{4\pi x_j}\right),$$

where $A^{(n)}(k;\mathbf{x},\mathbf{y},\mathbf{x}_1,\cdots,\mathbf{x}_n)$ are the determinants with $A(\mathbf{x}_i,\mathbf{x}_j)$ substituted for $K_2(\mathbf{x}_i,\mathbf{x}_j)$. Using Hadamard's lemma[6] and (27), we get

$$|B^{(n)}(k;\mathbf{x},\mathbf{y},\mathbf{x}_1,\cdots,\mathbf{x}_n)|\leqslant(n+1)^{\frac{1}{2}(n+1)}N^{n+1} \times\left(\frac{|V(y)|}{4\pi y}\right)\prod_{j=1}^{n}\left(\frac{|V(x_j)|}{4\pi x_j}\right),\quad \kappa\geqslant 0. \qquad (32)$$

[6] See reference 4 for detailed references on this point.

From (V) it follows that

$$\left|\int B^{(n)}(k;\mathbf{x},\mathbf{y},\mathbf{x}_1,\cdots,\mathbf{x}_n)d\mathbf{x}_1\cdots d\mathbf{x}_n\right| \leqslant(n+1)^{\frac{1}{2}(n+1)}N^{n+1}M^n\left(\frac{|V(y)|}{4\pi y}\right),\quad \kappa\geqslant 0. \qquad (33)$$

Now the determinants $B^{(n)}$ are analytic in k, and hence from (32), (33), and Theorem I we see that the series (9) defining $\Delta(\lambda^2,k;x,y)$ is a series of analytic functions which are regular for $\kappa>0$ and furthermore

$$|\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})|\leqslant N\frac{|V(y)|}{4\pi y}+\sum_{n=1}^{\infty}\frac{|\lambda^2|^n}{n!}(n+1)^{\frac{1}{2}(n+1)} \times N^{n+1}M^n\frac{|V(y)|}{4\pi y},\quad \kappa\geqslant 0. \qquad (34)$$

The series on the right is convergent for finite $|\lambda|$ and we thus get (29). Also $\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})$ is analytic in k regular for $\kappa>0$, for it is defined by a uniformly convergent series of analytic functions.

To get (30), we use (28), and we have, for large $|k|$,

$$|B^{(n)}(k;\mathbf{x},\mathbf{y},\mathbf{x}_1,\cdots,\mathbf{x}_n)| \leqslant(n+1)!\epsilon^{n+1}\exp[-\kappa|\mathbf{x}-\mathbf{y}|] \times\left(\frac{|V(y)|}{4\pi y}\right)\prod_{j=1}^{n}\left(\frac{|V(x_j)|}{4\pi x_j}\right),\quad \kappa\geqslant 0, \qquad (35)$$

where we have intentionally not used the Hadamard lemma so as to enable us with the help of the triangle inequality to get the exponential on the right hand side. We can now easily get an analog to (34) with ϵ substituted for N:

$$|\Delta(\lambda^2,k,\mathbf{x},\mathbf{y})|\leqslant\epsilon\frac{|V(y)|}{4\pi y}\exp[-\kappa|\mathbf{x}-\mathbf{y}|] \times\{1+\sum_{n=1}^{\infty}|\lambda^2|^n(n+1)\epsilon^nM^n\},\quad \kappa\geqslant 0. \qquad (36)$$

For $|k|$ large enough, ϵ could have been chosen small enough to make the series in (36) convergent. This proves (30). It is also clear from (36) that in the limit as $|k|\to\infty$, $\kappa\geqslant 0$, $\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})$ vanishes.

IV. ANALYTIC PROPERTIES OF $\Delta(\lambda^2,k)$ AND THEIR RELATION TO THE BOUND STATES

In this section we shall show that $\Delta(\lambda^2,k)$ is an analytic function of k, regular for $\kappa>0$, and uniformly bounded in the region $\kappa\geqslant 0$. Furthermore, we shall show that the zeros of $\Delta(\lambda^2,k)$ are related to the bound states of (1).

From (10) and (11) we have

$$\Delta(\lambda^2,k)=1+\sum_{n=1}^{\infty}\frac{(-\lambda^2)^n}{n!}\int d\mathbf{x}_1\cdots\int d\mathbf{x}_n \times D^{(n)}(k;\mathbf{x}_1,\cdots,\mathbf{x}_n), \quad (10)$$

and

$$D^{(n)}(k;\mathbf{x}_1,\cdots,\mathbf{x}_n)=\begin{vmatrix} K_2(\mathbf{x}_1,\mathbf{x}_1) & \cdots & K_2(\mathbf{x}_1,\mathbf{x}_n) \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ K_2(\mathbf{x}_n,\mathbf{x}_1) & \cdots & K_2(\mathbf{x}_n,\mathbf{x}_n)\end{vmatrix}. \quad (11)$$

Since K_2 is analytic in k regular for $\kappa>0$, then $D^{(n)}(k;\mathbf{x}_1,\cdots,\mathbf{x}_n)$ is also analytic in k. Now from (25) we have

$$D^{(n)}(k;\mathbf{x}_1,\cdots,\mathbf{x}_n)=A^{(n)}(k;\mathbf{x}_1,\cdots,\mathbf{x}_n)\prod_{j=1}^{n}\left(\frac{V(x_j)}{4\pi x_j}\right).$$

Using Hadamard's lemma and (27), we get

$$|D^{(n)}(k;\mathbf{x}_1,\cdots,\mathbf{x}_n)|\leqslant N^n n^{n/2}\prod_{j=1}^{n}\left(\frac{V(x_j)}{4\pi x_j}\right), \quad \kappa\geqslant 0,$$

and hence,

$$\left|\int D^{(n)}(k;\mathbf{x}_1,\cdots,\mathbf{x}_n)d\mathbf{x}_1\cdots d\mathbf{x}_n\right|\leqslant N^n M^n n^{n/2}, \quad \kappa\geqslant 0. \quad (37)$$

We conclude by theorem I that the series (10) is a series of analytic functions. Furthermore, we have

$$|\Delta(\lambda^2,k)|\leqslant 1+\sum_{n=1}^{\infty}\frac{|\lambda^2|^n}{n!}n^{n/2}N^nM^n, \quad \kappa\geqslant 0. \quad (38)$$

The series on the right hand side of the inequality is convergent for any finite $|\lambda|$. Hence, $\Delta(\lambda^2,k)$ is analytic in k, regular for $\kappa>0$ and uniformly bounded for $\kappa\geqslant 0$.

Considered as a function of λ, $\Delta(\lambda^2,k)$ vanishes for $\lambda=\pm\lambda_n$, where λ_n are the eigenvalues of the homogeneous equation,

$$\psi_n(\mathbf{k},\mathbf{x})=\lambda_n\int K(\mathbf{x},\mathbf{y})\psi_n(\mathbf{k},\mathbf{y})d\mathbf{y}. \quad (39)$$

In this case the resolvent kernel $\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})/\Delta(\lambda^2,k)$ is *not* an irreducible fraction. According to Poincaré,[6] one can factor out an entire function of λ from both the numerator and denominator, and get

$$\frac{\Delta(\lambda^2,k;\mathbf{x},\mathbf{y})}{\Delta(\lambda^2,k)}=\frac{\Delta_1(\lambda,k;\mathbf{x},\mathbf{y})}{\Delta_1(\lambda,k)}, \quad (40)$$

where $\Delta_1(\lambda,k;\mathbf{x},\mathbf{y})$ and $\Delta_1(\lambda,k)$ have no zeros in common. This procedure will not change the convergence properties proved earlier, and correspondingly $G_4(k,\tau)/\Delta(\lambda^2,k)$ and $G_5(k,\tau)/\Delta(\lambda^2,k)$ will respectively become $G_4^{(1)}(k,\tau)/\Delta_1(\lambda,k)$ and $G_5^{(1)}(k,\tau)/\Delta_1(\lambda,k)$ with the same analytic properties as before.

Now if we fix λ, then as functions of k $\Delta_1(\lambda,k;\mathbf{x},\mathbf{y})$ and $\Delta_1(\lambda,k)$ will have no zeros in common. Thus if $\Delta_1(\lambda,k)=0$, this means that there exists at least one solution of the homogeneous equation,

$$\psi_n(\mathbf{k}_n,\mathbf{x})=\lambda\int K(\mathbf{x},\mathbf{y})\psi_n(\mathbf{k}_n,\mathbf{y})d\mathbf{y}. \quad (41)$$

For k_n real, $k_n\neq 0$, Jost and Pais have shown that λ must be complex in (41). Hence for real λ, $\Delta_1(\lambda,k)$ will have no zeros on the real k axis, except possibly at $k=0$.

Now for λ real, and $\kappa_n>0$, we claim that all the zeros are on the positive imaginary axis, and that $\psi_n(\mathbf{k}_n,\mathbf{x})$ are the bound states. The possible zero at $k=0$, may or may not correspond to a bound state. For large $|\mathbf{x}|$,

$$\psi_n(\mathbf{k}_n,\mathbf{x})\sim|\mathbf{x}|^{-1}\exp(ik_n|\mathbf{x}|)f_n(k_n,\theta). \quad (42)$$

From the Schrödinger equation, we have

$$i\nabla\cdot(\psi_n^*\nabla\psi_n-\psi_n\nabla\psi_n^*)=i(k_n^{*2}-k_n^2)\psi_n^*\psi_n. \quad (43)$$

Integrating over a large sphere, we get

$$i\oint(\psi_n^*\nabla\psi_n-\psi_n\nabla\psi_n^*)\cdot d\mathbf{S} =i(k_n^{*2}-k_n^2)\int_V|\psi_n|^2d\mathbf{x}, \quad (44)$$

and substituting the asymptotic form on the left, we have

$$-(k_n+k_n^*)\exp[i(k_n-k_n^*)|\mathbf{x}_v|]\int|f_n(k_n,\theta)|^2d\Omega =i(k_n^{*2}-k_n)\int_V|\psi_n|^2d\mathbf{x}. \quad (45)$$

For $\kappa_n>0$ we divide both sides by $(k_n^*+k_n)$, and get

$$-\exp[-2\kappa_n|\mathbf{x}_v|]\int|f_n(k_n,\theta)|^2d\Omega=2\kappa_n\int_V|\psi_n|^2d\mathbf{x}. \quad (46)$$

This is a contradiction, and therefore, we conclude that $k_n^*+k_n=0$, i.e., $k_n=i\kappa_n$.

For $\kappa_n=0$, and $k_n\neq 0$, we also get a contradiction. Hence as in Jost and Pais, if $\kappa_n=0$, then $k_n=0$, if λ is real. The states $\psi_n(k_n,\mathbf{x})$ are all (except for $k_n=0$) normalizable, and since they are solutions of the homogeneous equation, they are the bound states.

It is a known fact that if $\int_0^\infty r|V(r)|dr\leqslant M<\infty$ the number of bound states is finite.[7] Furthermore, the κ_n will have a finite maximum, corresponding to the lowest energy state.

Finally, we claim that for $|k|\to\infty$, $\kappa\geqslant 0$, $\Delta(\lambda^2,k)\to 1$. This follows from (28). Using the same steps we used to get to (38), we can show that for $|k|$ large enough,

$$|\Delta(\lambda^2,k)-1|\leqslant\sum_{n=1}^{\infty}|\lambda^2|^n\frac{n^{n/2}}{n!}\epsilon^nM^n, \ \kappa\geqslant 0. \quad (38)$$

[7] V. Bargmann, Proc. Natl. Acad. Sci. U. S. 38, 961 (1952).

The right hand side of this inequality could be made arbitrarily small.

We note there that all our work up to this section holds even if λ is complex. However, as we have seen above, we have to take λ real to get a relation between the poles and the bound states. As we have mentioned above, it is proved in J.P. that for complex λ, there might be poles on the real k axis.

Finally, we remark here that all the results of this section follow from the conditions (A) and (B) only, and hence are valid for a very large class of potentials.

V. ASYMPTOTIC BEHAVIOR OF THE SCATTERING AMPLITUDE

In this section we shall discuss the behavior of the scattering amplitude for large $|k|$ $(\kappa\geqslant 0)$.

It is already clear from (28) and (36) that the $G_j(k,\tau)$, for $j=3$, 4, 5, vanish in the limit as $|k|$ approaches infinity. Furthermore, by the Riemann-Lebesgue lemma one can easily show that $G_2(k,\tau)$ will vanish[8] for large k_r (where $k_r=\mathrm{Re}\,k$).

In field theory the behavior of the scattering amplitude for large values of the energy is not known exactly. What one knows is that it increases *not* faster than a polynomial in the energy. This behavior is closely related to the singularities on the light cone.

In our case everything is well defined, but a somewhat similar behavior, though rather restricted, is caused by the singularity of the potential at $r=0$. We have succeeded in proving that if we restrict the behavior of the potential at zero more than it already is restricted in (A) and (B), then not only do the $G_j(k,\tau)$ for, $j=3,4,5$, vanish for $|k|$ approaching infinity, but they vanish as $1/|k|$. Namely, one can show that if the potential satisfies the following two conditions,

$$\int_0^\infty r|V'(r)|dr<\infty, \tag{D}$$

$$\int_0^\infty |V(r)|dr<\infty, \tag{E}$$

in addition to (A), (B), and (C), one can get a bound for $A(\mathbf{x},\mathbf{y})$ for large $|k|$ of the form

$$|A(\mathbf{x},\mathbf{y})|\leqslant\frac{S}{|k|}\exp[-\kappa|\mathbf{x}-\mathbf{y}|],\quad \kappa\geqslant 0, \tag{28$'$}$$

instead of (28). It would be interesting to relax the conditions (A) and (B) as far as the singularity at zero is concerned, instead of making them stricter as in (D) and (E), and then see what the behavior of the amplitude at infinity would be.

[8] This is enough to make the Cauchy integral of G_2, on a large enough semicircle in the upper half-plane, arbitrarily small.

VI. DERIVATION OF THE DISPERSION RELATIONS

We rewrite (19) as follows:

$$g(k,\tau)=G_2(k,\tau)+G_3(k,\tau)+\frac{G_4{}^{(1)}(k,\tau)}{\Delta_1(\lambda,k)}+\frac{G_5{}^{(1)}(k,\tau)}{\Delta_1(\lambda,k)}, \tag{47}$$

where

$$g(k,\tau)=f(k,\tau)+\lambda/4\pi\tilde{V}(\tau). \tag{48}$$

We have already shown in the preceding sections that for $\frac{1}{2}\tau\leqslant\alpha$, $g(k,\tau)$ is analytic in k, regular in the region $\kappa>0$, continuous and uniformly bounded for $\kappa\geqslant 0$, except for a finite number of poles at the zeros of $\Delta_1(\lambda,k)$ which for real λ all lie on the positive imaginary axis. Furthermore, on the real axis $g(k,\tau)$ has branch points at $k=\pm\frac{1}{2}\tau$. For $|k|$ approaching infinity $g(k,\tau)$ vanishes.

As usual we shall derive our dispersion relations in terms of the energy variable E, where $E=k^2$ in the proper units.

Let us write $\phi(E,\tau)\equiv g(k,\tau)$. Then $\phi(E,\tau)$ will be analytic everywhere in the E plane except for a branch cut on the real positive axis and the poles on the negative real axis. We apply the Cauchy formula to $\phi(E,\tau)$ and integrate over the contour C shown in Fig. 1. We get

$$\oint_C\frac{\phi(E',\tau)}{E'-E}dE'=2\pi i\sum_{j=1}^N\frac{R_j(\tau)}{E_j-E};\quad E_j<0, \tag{49}$$

where $R_j(\tau)$ are the residues of $\phi(E',\tau)$ at the bound states E_j. The integral over the large circle vanishes as the radius becomes infinite, and after taking the contributions from the two small circles, and letting the lines approach the real axis, we get†

$$\mathrm{Re}\,\phi(E,\tau)=\frac{1}{\pi}P\int_0^\infty\frac{\mathrm{Im}\,\phi(E',\tau)}{E'-E}dE'+\sum_{j=0}^N\frac{R_j(\tau)}{E-E_j}, \tag{50}$$

where $E_0=0$, P indicates the principal value of the integral, and $R_0(\tau)$ is the residue at $E=0$. If we now let $f(k,\tau)\equiv M(E,\tau)$ we get the desired dispersion relations,

$$\mathrm{Re}\,M(E,\tau)=\frac{1}{\pi}P\int_0^\infty\frac{\mathrm{Im}\,M(E',\tau)}{E'-E}dE'+\sum_{j=0}^N\frac{R_j(\tau)}{E-E_j}-\frac{\lambda}{4\pi}\tilde{V}(\tau). \tag{51}$$

We finally remark that all the residues $R_j(\tau)$ are real. This follows from the fact that for k on the positive imaginary axis both $K(\mathbf{x},\mathbf{y})$ and $K_2(\mathbf{x},\mathbf{y})$ are real. Hence, all the resolvent kernals in (20) are real, and since the $G_j(k,\tau)$ depend only on the magnitude of τ,

† *Note added in proof.*—Here we used the fact that $\phi(E+i\epsilon,\tau)=\phi^*(E-i\epsilon,\tau)$, $E\geqslant 0$. This follows from (22) and the fact that $\Delta(\lambda^2,k)=\Delta^*(\lambda^2,-k)$.

they are real too. Thus for k on the positive imaginary axis $f(k,\tau)$ is real, since $\tilde{V}(\tau)$ is also real for a central potential.

The relations (51) hold as long as $\frac{1}{2}\tau \leqslant \alpha$. Hence for potentials which fall off as a Gaussian or faster, (51) holds for all finite τ. In applying (51), one has to integrate over the unphysical region, $0<E'<\frac{1}{4}\tau^2$, and the experimental data do not give us $M(E',\tau)$ in that region. In the following section we shall show that the partial-wave expansion can be used to determine the scattering amplitude in the unphysical region from the phase shifts.

In the forward direction, $\tau=0$, (51) holds even if $\alpha=0$ in (C). This leaves a very large class of potentials.

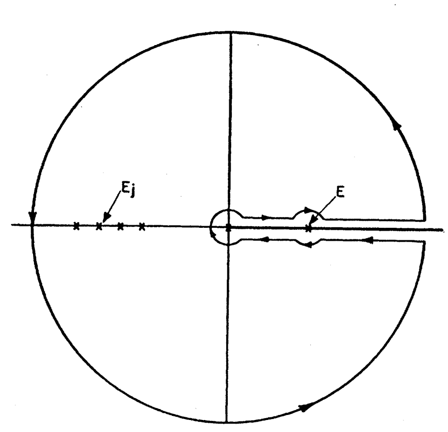

FIG. 1. Contour of integration in E-plane.

VII. CONTINUATION OF THE PARTIAL-WAVE EXPANSION INTO THE UNPHYSICAL REGION

In this section we shall show that the partial-wave expansion for $f(k,\tau)$ is convergent in the unphysical region $0<k<\frac{1}{2}\tau$ if $\tau<\alpha$. Thus one can use it to define $f(k,\tau)$ in the unphysical region. (The variable k is real throughout this section.)

Since $\cos\theta = 1-\tau^2/2k^2$, we can write

$$f(k,\tau)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)[\exp(2i\delta_l(k))-1] \times P_l(1-\tau^2/2k^2). \quad (52)$$

For k real and $k \geqslant \frac{1}{2}\tau$, (52) is convergent for a large class of potentials, which certainly includes the potentials that sayisfy (A) and (B).

As k becomes less than $\frac{1}{2}\tau$, τ being kept fixed, the argument of the Legendre polynomial becomes less than -1; and for $k \to 0$ it approaches $-\infty$.

Carter[9] has shown rigorously that the absolute value of the phase shift for large l is always bounded by the Born-approximation expression for δ_l for large l; i.e.,

$$|\delta_l(k)| \leqslant C' \int_0^\infty r|V(r)| J_{l+\frac{1}{2}}^2(kr)dr. \quad (53)$$

Now if (B′) is satisfied, $\int_0^\infty \exp(\alpha r) r|V(r)|dr \leqslant M''$, then since $r|V(r)|$ is integrable at zero, and $J_{l+\frac{1}{2}}(0^+)$ vanishes very fast for large l, we can write

$$|\delta_l(k)| \leqslant C'' \int_0^\infty \exp(-ar) J_{l+\frac{1}{2}}^2(kr)dr; \quad a<\alpha. \quad (54)$$

The integral in (54) can be evaluated exactly[10]:

$$\int_0^\infty \exp(-ar) J_{l+\frac{1}{2}}^2(kr)dr = \frac{1}{\pi k} Q_l(1+a^2/2k^2), \quad (55)$$

where Q_l is the Legendre function of the second type. From (52) we have

$$|f(k,\tau)| \leqslant \frac{1}{2|k|}\sum_{l=0}^{\infty}(2l+1)|\{\exp(2i\delta_l(k))-1\}| \cdot |P_l(1-\tau^2/2k^2)|. \quad (56)$$

For large l we have

$$|\exp(2i\delta_l)-1| \cong 2|\delta_l| \leqslant \frac{2C''}{\pi|k|}|Q_l(1+a^2/2k^2)|,$$

$$|Q_l(x)| \cong \frac{2^l(l!)^2}{(2l+1)!}|x|^{-l-1}, \quad |x|>1,$$

$$|P_l(x)| \cong \frac{(2l)!}{2^l(l!)^2}|x|^l, \quad |x|>1.$$

Hence for large l each term in (56) is of the form

$$|1-\tau^2/2k^2|^l / |1+a^2/2k^2|^{l+1},$$

and the series will converge, in the region $0<k<\frac{1}{2}\tau$, if

$$|1-\tau^2/2k^2| < |1+a^2/2k^2|, \quad (57)$$

i.e.,

$$\tau < a.$$

Since $a<\alpha$, we have, finally, $\tau<\alpha$.

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APPENDIX I

In this Appendix we shall prove the inequality (28). The proof is essentially an analog of the Riemann-

[9] D. S. Carter, thesis, Princeton, 1952 (unpublished).

[10] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, London, 1922), p. 389.

Lebesgue lemma. From (26) we write

$$A(k;\mathbf{x},\mathbf{y})=\frac{|\mathbf{y}|}{4\pi}\int \exp\{(ik|\mathbf{x}-\mathbf{z}|+|\mathbf{z}-\mathbf{y}|)\}\times\frac{V(z)d\mathbf{z}}{|\mathbf{x}-\mathbf{z}|\cdot|\mathbf{z}-\mathbf{y}|}. \quad \text{(A-1)}$$

To simplify the notation we let

$$Q(\mathbf{z})=\frac{|\mathbf{y}|}{4\pi}\left(\frac{V(z)}{|\mathbf{x}-\mathbf{z}|\cdot|\mathbf{z}-\mathbf{y}|}\right), \quad \text{(A-2)}$$

and

$$w(\mathbf{z})=|\mathbf{x}-\mathbf{z}|+|\mathbf{z}-\mathbf{y}|. \quad \text{(A-3)}$$

From J.P. we know that for potentials satisfying (A) and (B) we have

$$\int|Q(\mathbf{z})|d\mathbf{z}\leqslant N<\infty. \quad \text{(A-4)}$$

Hence there exists a $Q_\epsilon(\mathbf{z})$ such that for any ϵ the following inequality holds,

$$\int|Q(\mathbf{z})-Q_\epsilon(\mathbf{z})|d\mathbf{z}\leqslant \epsilon, \quad \text{(A-5)}$$

where $Q_\epsilon(\mathbf{z})$ is continuous, differentiable, and vanishes with its first partial derivatives outside a large cube with center at origin. Furthermore, $|Q_\epsilon(\mathbf{z})|\leqslant C_1$, and $|\partial Q_\epsilon(\mathbf{z})/\partial z_i|\leqslant C_2$, $i=1, 2, 3$, where C_1 and C_2 are constants. If we now write for (A-1)

$$A_Q=\int \exp[ikw(\mathbf{z})]Q(\mathbf{z})d\mathbf{z}, \quad \text{(A-1}')$$

we get

$$A_Q=A_{Q-Q_\epsilon}+A_{Q_\epsilon}. \quad \text{(A-6)}$$

Using (A-5), and the fact that $w(\mathbf{z})\leqslant|\mathbf{x}-\mathbf{y}|$, we get for $\kappa\geqslant 0$

$$|A_{Q-Q_\epsilon}|\leqslant \epsilon\exp[-\kappa|\mathbf{x}-\mathbf{y}|].$$

Hence, we now have

$$|A_Q|\leqslant \epsilon\exp[-\kappa|\mathbf{x}-\mathbf{y}|]+|A_{Q_\epsilon}|. \quad \text{(A-7)}$$

We choose Cartesian coordinates z_1, z_2, z_3 such that $\mathbf{x}$ and $\mathbf{y}$ lie in the [23] plane. In that case we have,

$$w_1(\mathbf{z})=\frac{\partial w(z)}{\partial z_1}=\frac{z_1}{|\mathbf{x}-\mathbf{z}|}+\frac{z_1}{|\mathbf{z}-\mathbf{y}|}. \quad \text{(A-8)}$$

From (A-1′) we have,

$$A_{Q_\epsilon}(\mathbf{z})=\int\int\int \exp[ikw(\mathbf{z})]w_1(\mathbf{z})\frac{Q_\epsilon(\mathbf{z})}{w_1(\mathbf{z})}dz_1dz_2dz_3. \quad \text{(A-9)}$$

We now want to integrate (A-9) by parts over z_1 to get a factor $1/k$. But $w_1(Z)$ is zero at $z_1=0$. Hence we have to exclude the plane $z_1=0$ from our region of integration, and write

$$A_{Q_\epsilon}=\int\int\int_\Lambda \exp[ikw(\mathbf{z})]w_1(\mathbf{z})\frac{Q_\epsilon(\mathbf{z})}{w_1(\mathbf{z})}dz_1dz_2dz_3+\int\int\int_\delta \exp[ikw(\mathbf{z})]Q_\epsilon(\mathbf{z})d\mathbf{z}, \quad \text{(A-10)}$$

where δ is a thin infinite slab with the plane $z_1=0$ as its median plane, and Λ is the rest of the space. If we now do a partial integration on the first term, we get

$$|A_{Q_\epsilon}|\leqslant\frac{1}{|k|}\int\int\int_\Lambda \exp[-\kappa|\mathbf{x}-\mathbf{y}|]\times\left|\frac{\partial}{\partial z_1}\frac{Q_\epsilon(\mathbf{z})}{w_1(\mathbf{z})}\right|dz_1dz_2dz_3+\frac{C_\delta'}{|k|}\exp[-\kappa|\mathbf{x}-\mathbf{y}|]+\left|\int\int\int_\delta \exp[ikw(\mathbf{z})]Q_\epsilon(\mathbf{z})d\mathbf{z}\right|. \quad \text{(A-11)}$$

The first integral is finite, and in the second one the slab can be chosen thin enough to make $\int_\delta|Q_\epsilon|d\mathbf{z}<\epsilon$. We thus get

$$|A_{Q_\epsilon}|\leqslant\frac{C_3}{|k|}\exp[-\kappa|\mathbf{x}-\mathbf{y}|]+\epsilon\exp[-\kappa|\mathbf{x}-\mathbf{y}|], \quad \text{(A-12)}$$

where C_3 is a constant.

Substituting back in (A-7), we get the desired inequality, since we can always choose $|k|$ large eough to make $C_3/|k|<\epsilon$ and $C_\delta'/|k|<\epsilon$.

An estimate of the form $|A(k,\mathbf{x},\mathbf{y})|\leqslant(C/|k|)\times\exp[-\kappa|\mathbf{x}-\mathbf{y}|]$, is not always possible if $V(r)$ satisfies only (A) and (B). One can easily give a counterexample if one sets $\mathbf{x}=0$ in (A-1), and performs the integration. It is clear in this case that one would need condition (E) to give a bound that goes as $1/|k|$. In general, the addition of conditions (D) and (E), given in Sec. V, to (A) and (B) is enough to prove the inequality given at the beginning of this paragraph. This inequality is needed in the Dirac case and we shall give its proof elsewhere.