tion will exceed p-meson production. But by comparing the s-phase shifts with p-phase shifts at low energy it is clear that when the kinetic energy available to each of the final mesons exceeds 50 Mev the mesons will be predominantly in p states.

The experiment of Blevins, Block, and Harth[12] shows that for 470-Mev π^+ energy the $\pi^+ + p$ cross section is 10% inelastic. This would indicate an inelastic cross section of 2 mb. If the total kinetic energy available in the center-of-mass system is interpreted to be available to the mesons, the calculated $\pi^+ + p \rightarrow n + \pi^+ + \pi^+$ cross section is 1.4 mb and the $\pi^+ + p \rightarrow p + \pi^+ + \pi^0$ cross section is 0.68 mb. Although the sum of these cross sections agrees with experiment quite well, the Born-approximation result gives 1.48 mb and 0.60 mb, respectively, for the above cross sections. Thus agreement with experiment at this energy is not decisive for the determination of the static one-meson approximation. Nevertheless, for 400–550 Mev incident pion energy the final mesons are above the range of s-meson production and below the energy where the approximation of high-energy (3,3) phase shifts with π and the neglect of the other p-phase shifts has an appreciable effect; thus agreement with experiments in this range should be good. Since the one-meson approximation differs from the Born approximation by as much as a factor of three in this energy range, further experiments at these energies will be enlightening.

[12] Blevins, Block, and Harth, Bull. Am. Phys. Soc. Ser. II, 1, 174 (1956).

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The author wants to thank Professor M. L. Goldberger for suggesting this problem and for invaluable help and guidance. He also wishes to thank his wife for assistance in most of the computational work.

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Generalized Effective Range Theory for Nucleon-Nucleon Scattering*

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A method is proposed for obtaining limitations on the shape and possible energy dependence of the force in a given scattering state of the two-nucleon system, from a knowledge of the phase shift in that state over the nonrelativistic domain, and is illustrated for ^1S waves.

I. INTRODUCTION

THE purpose of this paper is to give a method for translating the results of a partial-wave analysis of nucleon-nucleon scattering data into equivalent information on the nuclear force. Our concern is not with the general mathematical problem of deducing a potential from the complete two-body S-matrix.[1] Rather, we seek to obtain only those properties of the nuclear force which are determined by experiment. Just as all potential models must imply the correct effective range[2] in order to fit the low-energy data, so they must all contain the properties we seek in order to fit the higher energy data. Thus, by setting an upper limit to the energies under consideration, we limit the detail with which the incident nucleon is able to observe the force by which it is scattered. All potentials of a class yielding the experimental phase shifts over this energy region may then be considered equivalent to an "effective force," obtained by neglecting fluctuations in members of this class over interparticle distances much smaller than the wavelength of the incident nucleon. For example, at sufficiently low energies the nature of this equivalence is well known as the shape-independent approximation.[2] What limitations are imposed on the "effective force" by data at higher energies?

In Sec. II a method is developed, within the context of S-waves, for deducing this information from the given phase shift on the assumption that the force is static. In essence, the dependence of the phase shift on the force is schematized by replacing the latter by its value at a discrete set of radial points, whose position ratios are so chosen that the representation of the matrix elements of the force is an "optimal" one in the sense of a Gauss-Jacobi quadrature approximation. It is then possible to employ these points as probes of the force by allowing their positions and associated amplitudes to be fixed by the experimental phase shift in its dependence upon energy. The latter is assumed, for the sake of clarity, to be developable in a power series which converges at least asymptotically in the domain of interest. The order of quadrature theorem to be

* A preliminary report of this work can be found in the *Proceedings of the Sixth Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1956).
[1] In this connection see, for example, R. Jost and W. Kohn, Phys. Rev. 87, 977 (1952); 88, 382 (1952).
[2] J. M. Blatt and J. D. Jackson, Phys. Rev. 76, 18 (1949); H. A. Bethe, Phys. Rev. 76, 38 (1949).

used, and hence the number of points representing the force, depends on the number of terms used in a power-series expansion and thus carries with it a corresponding energy range of validity. In this manner, successively higher orders of quadrature become progressively more sensitive to the details of the nuclear force, thus reflecting the physical situation in which the nucleon also becomes more sensitive to the details of the force by which it is scattered as its energy increases. In any order of quadrature, force ratios thus specified may be regarded as constituting an operational definition of the "effective force" alluded to earlier.

In Sec. III the quadrature method is applied in the lowest three orders. In each order, a system of algebraic equations is set up whose solution gives the dependence of the force ratios upon the coefficients of the power-series expansion representing the phase shift. The gross features of the force may be obtained analytically as limiting cases. Thus the lowest order of quadrature, valid in the energy interval 0–15 Mev, establishes a mean interaction distance and hence corresponds to conventional effective-range analysis. The two succeeding orders, valid in the energy intervals 0–40 Mev, 0–120 Mev, respectively, establish narrow bounds on the shape parameters P_1 and P_2 (respectively, the coefficients of E^2 and E^3 in an effective-range expansion) corresponding to an infinite short-range repulsion. It is observed that the Feshbach-Lomon (FL) semi-empirical analysis[3] of nucleon-nucleon scattering accurately confirms these bounds. More detailed information is obtained by resorting to a complete numerical solution. Thus, values of the core radius are obtained which agree well with the results of calculations based on core-type potentials. In the second order of quadrature a single force ratio is derived as a function of P_1. As a further illustration of the method, interaction wave functions are constructed directly from P_1 and compared with wave functions calculated from a corresponding potential model. Turning to the third order of quadrature, a pair of force ratios are obtained as functions of P_1 and P_2. It is shown that the effective force strongly correlates the scattering at different energies. Thus, the bare requirement of a static force is sufficient to strongly restrict the range of values that P_2 can assume for a given value of P_1.

It is entirely possible that a consistent description of nucleon-nucleon scattering cannot be achieved with the aid of static forces, even at quite low energies. In Sec. IV, the quadrature method is generalized to provide a useful technique for the analysis of nonstatic forces. The qualitative success of the quadrature method in ^1S states, its operationally well-defined character, and the simplicity of the numerical analysis involved, all encourage a more extensive application. With the advent of more accurate data, application of the method both in higher orders of approximation and to additional scattering states will become feasible and, it is hoped, prove a fruitful source of information about the two-nucleon force.

[3] H. Feshbach and E. L. Lomon, Phys. Rev. 102, 891 (1956).

II. METHOD OF QUADRATURES

We begin with the Schrödinger equation for S-states,

$$[d^2/dx^2+K^2+\lambda f(x)]u(x)=0, \tag{1}$$

where $f(x)$ is the radial dependence of the potential in this state expressed in units of its range R; $\lambda=(MV_0R^2)/\hbar^2$, where V_0 is the strength of the force (positive for attractive potentials); and $K^2=(MER^2)/\hbar^2$ —a notation we shall adhere to throughout this paper. It is convenient to re-express Eq. (1) as an integral equation:

$$u(x)=xj_0(Kx)\lambda(K\cot\delta)\int_0^\infty dx'x'j_0(Kx')f(x')u(x')+\lambda\int_0^\infty dx'G(x,x')f(x')u(x'), \tag{2}$$

in which

$$G(x,x')=-(1/K)(Kx_<)j_0(Kx_<)(Kx_>)n_0(Kx_>). \tag{3}$$

Here j_0 and n_0 are spherical Bessel functions, δ is the ^1S phase shift, and $x_<$ ($x_>$) denotes the lesser (greater) of x and x'. For each energy the integrands in Eq. (2), which shall be denoted by I, are the product of an oscillatory function and another function $w(x)$ having limited spatial extension corresponding to the finite range of the force. Let us schematize this behavior by making the assumption

$$I\sim w(x)\pi_{2n-1}(x,E), \tag{4}$$

where $\pi_{2n-1}(x,E)$ is a polynomial in x of degree $2n-1$, the coefficients of which depend upon the energy. As the energy increases, the rate at which the integrand oscillates within the range of the force becomes more rapid. The value of n used in Eq. (4) must therefore depend upon the energy region in which the scattering is considered.

The usefulness of the assumption Eq. (4) is due to the following theorem. Let $p_n(x)$ be a set of polynomials orthogonal in $(0,\infty)$ with weight function $w(x)$. Then

$$\int_0^\infty \pi_{2n-1}(x)w(x)dx=\sum_{r=1}^{\infty}\lambda_{nr}\pi_{2n-1}(x_{nr}), \tag{5}$$

where x_{nr} is the rth zero of $p_n(x)$, and λ_{nr} are the Christoffel coefficients, which are related to the moments of $w(x)$. These coefficients are given by

$$\lambda_{nr}=\frac{1}{p_n'(x_{nr})}\int_0^\infty dx\frac{w(x)p_n(x)}{x-x_{nr}}. \tag{6}$$

The usefulness of the quadrature theorem Eq. (5) in approximating any definite integral $\int_0^\infty dxw(x)g(x)$ is obvious, and requires only that the integral exist in the Stieltjes sense in order for the sequence of quadrature approximations $\{Q_n\}$ to converge. Such an approximation scheme is "optimal" in the sense that, if $g(x)$ is represented accurately by a $\pi_{2n-1}(x)$, then $\int_0^\infty dxw(x)g(x)$ requires the specification of $g(x)$ at only the n zeros of $p_n(x)$ in order to be represented with equal accuracy.[4]

When the integrals of Eq. (2) are approximated in this manner, one obtains

$$u(x)=xj_0(Kx)\lambda(K\cot\delta_n)\sum_{r=1}^{n}\Lambda_{nr}x_{nr}j_0(Kx_{nr})u(x_{nr}) +\lambda\sum_{r=1}^{n}\Lambda_{nr}G(x,x_{nr})u(x_{nr});\quad(7)$$

where

$$\Lambda_{nr}=\lambda_{nr}f(x_{nr})w^{-1}(x_{nr}),\quad(8)$$

and δ_n is the phase shift corresponding to the nth order of quadrature. At this point, it may be observed that Eq. (7) is the formal solution of Eq. (1) corresponding to the potential

$$\lambda f_n(x)=\lambda\sum_{r=1}^{n}\Lambda_{nr}\delta(x-x_{nr}).\quad(9)$$

That is to say, the effect of the assumption Eq. (4) is to replace the continuous function $f(x)$ by an equivalent set of δ-function shells, whose strengths are related to the values of $f(x)$ at the zeros of an appropriately chosen set of orthogonal polynomials by Eq. (8).

The phase shift δ_n is determined from Eq. (7) by the vanishing of the determinant of the simultaneous algebraic equations for $u(x_{nr})$ formed by setting x successively equal to the zeros of $p_n(x)$. It will in general depend upon the n quadrature strengths Λ_{nr} and a characteristic range R_{nn}. The essence of the method lies in considering these as unknown quantities to be fixed by the experimental phase shift in its dependence on energy. For this purpose it is useful, though not necessary, to express the experimental phase shift as a power series in the energy,

$$K\cot\delta=\bar{a}\bar{x}^{-1}+\tfrac{1}{2}(2\bar{x})K^2+8P_1\bar{x}^3K^4+32P_2\bar{x}^5K^6 +\cdots+(2\bar{x})^{2n-1}P_{n-1}K^{2n},\quad(10)$$

which is assumed to be an accurate transcription of the experimental data at energies $E\leq E_n$. Here $\bar{a}=a^{-1}\bar{r}$, where a is the scattering length and $\bar{r}\simeq1.23\times10^{-13}$ cm is half the effective range. The P_i's are the "shape-dependent coefficients" of the scattering.[2] The $n+1$ unknowns entering into δ_n are now to be fixed by the $n+1$ terms of Eq. (10). The significance of this is that the P_i's contain the characteristics of the force operative for $E\leq E_n$; that is, they represent the "effective force" referred to in Sec. I—modified, of course, by whatever experimental uncertainty is present. If Eq. (4) is indeed valid, then one may expect the shape information contained in Eq. (10) to be equivalently expressed in terms of the ratios of the quadrature strengths Λ_{nr} through the formula

$$\frac{f_n(x_{n,r-1})}{f_n(x_{n,r})}=\left[\frac{\lambda_{n,r}}{\lambda_{n,r-1}}\frac{w(x_{n,r-1})}{w(x_{n,r})}\right]\left(\frac{\Lambda_{n,r-1}}{\Lambda_{n,r}}\right).\quad(11)$$

Once the orthogonal polynomials to be used in the quadrature analysis have been specified, the prescription for obtaining the force ratios of Eq. (11) from the phase-shift data is unambiguous and may be regarded as their operational definition. It is true that this characteristically intimate relationship to the data makes the method peculiarly difficult to justify in a mathematically adequate way. However, it may be noted qualitatively that there are two mechanisms operating to enforce a rapid convergence to the correct set of force ratios for some initially specified potential $f(x)$. First, the "optimal" mathematical convergence afforded by the quadrature theorem assures a good representation of the integrals in Eq. (2) even if Eq. (4) is only a rough approximation. Second, there is the strong condition that the quadrature strengths Λ_{nr} must produce the phase shift corresponding to $f(x)$. If the quadrature theorem were exactly applicable at a given energy, this condition would be automatically satisfied. It may be hoped that, even when such is not the case, the above condition will force a rapid convergence to the correct force ratios.

III. ^1S EFFECTIVE FORCE

$n=1$

The lowest order of quadrature is equivalent to replacing $f(x)$ by

$$f_1(x)=\Lambda_{11}\delta(x-1)\quad(12)$$

in the integrals of Eq. (2). Here the characteristic range R has been set equal to R_{11}, the radial position of the zero of $p_1(x)$. The corresponding quadrature phase shift is easily obtained by the method described in II:

$$K\cot\delta_1=\Lambda_{11}^{-1}j_0^{-2}(K)[1-\Lambda_{11}j_0(2K)].\quad(13)$$

The over-all potential strength λ occurring in Eq. (2) has been absorbed into Λ_{11}. Comparison of the first two terms of a power-series expansion of Eq. (13) with the first two terms of Eq. (10) leads, in the limit of zero binding[5] (infinite scattering length), to the identifications

$$\Lambda_{11}=1;\quad R=\tfrac{3}{2}\bar{r}\simeq2.0\times10^{-13}\text{ cm}.\quad(14)$$

The location of the δ-function shell which reproduces the observed S-phase shift for $E\leq E_1$ is thus fixed.

[4] A thorough discussion of the Gauss-Jacobi quadrature theorem may be found in G. Szegö, *Orthogonal Polynomials* (American Mathematical Society Colloquium Publications, New York, 1939), Vol. 23, pp. 46–8, 340–54.

[5] For the purposes of this paper, binding corrections are unimportant.

However, no shape information can be obtained even if Eq. (4) should be satisfied exactly in this energy range. Thus the lowest order of quadrature corresponds precisely to the shape-independent approximation. The energy region in which Eq. (13) is valid is that for which the scattering length and effective range represent the phase shift accurately, i.e., $0 \leq E \leq E_1 \sim 15$ Mev.

$n=2$

Turning now to the second order of quadrature, use of the potential

$$f_2(x)=\Lambda_{21}\delta(x-x_1)+\Lambda_{22}\delta(x-1) \quad (15)$$

in the integrals of Eq. (2) produces, in the same manner as above, the second-order quadrature phase shift:

$$K\cot\delta_2=\frac{[1-\Lambda_{21}x_1j_0(2Kx_1)][1-\Lambda_{22}j_0(2K)]-\Lambda_{21}\Lambda_{22}x_1^2j_0^2(Kx_1)\cos^2K}{\Lambda_{21}x_1^2j_0^2(Kx_1)[1+\Lambda_{22}j_0(2K)]+\Lambda_{22}j_0^2(K)[1-\Lambda_{21}x_1j_0(2Kx_1)]}. \quad (16)$$

In Eqs. (15) and (16), λ has been absorbed into the quadrature strengths Λ_{2i}, $R=R_{22}$, and $x_1=R_{21}/R_{22}\leq 1$. Consider the basic quadrature polynomials to have been chosen, so that x_1 is fixed. There are thus three free parameters in Eq. (16)—R, Λ_{21}, Λ_{22}—which are to be fixed by the conditions that the phase shift shall correspond to the observed scattering length, effective range, and shape-dependent parameter P_1. Assuming zero binding, equating the first three terms of a power-series expansion of Eq. (16) with the corresponding terms of Eq. (10), one obtains the conditions

$$K^0\colon\ \Lambda_{21}x_1=(1-\Lambda_{22})/(1-\mu), \quad (17)$$

$$K^2\colon\ 3\bar{x}=-\mu^2+(3-x_1)\mu+2x_1, \quad (18)$$

$$K^4\colon\ \sum_{n=0}^{6}(a_n+P_1b_n)\mu^n=0, \quad (19)$$

in which $\mu=\Lambda_{22}(1-x_1)$ and a_n, b_n are functions of x_1 alone.

Numerical methods are needed to obtain a complete solution of Eqs. (17)–(19). However, some general features may be seen analytically. Assume that the amplitude Λ_{22} of the outer shell is positive (attractive). Then Eq. (17) shows that the inner shell is repulsive for $1-x_1\leq\mu\leq 1$—indeed becoming infinitely so at $\mu=1$. From Eq. (19) one easily determines $P_1^{(\infty)}$, the value of P_1 corresponding to an infinitely repulsive inner shell:

$$P_1^{(\infty)}=-\sum a_n/\sum b_n = -(3/40)(x_1+2)^{-3}(x_1^3+2x_1^2+8x_1+4). \quad (20)$$

By definition, the ratio of zeros x_1 must lie in the interval (0,1); and from Eq. (20) $P_1^{(\infty)}$ is observed to be monotonic in this region. Equation (20) thus provides bounds on $P_1^{(\infty)}$:

$$-0.0416\leq P_1^{(\infty)}\leq -0.0375. \quad (21)$$

It is then sufficient that these bounds be obeyed experimentally in order for the data to be consistent with a force having a short-range repulsion. By using the first three terms of the power-series expansion Eq. (10), it is seen that Eq. (21) implies a maximum in the ^1S phase shift in the range 40–60 Mev. The method is thus shown to be consistent since, if the force be static and well-behaved, the core feature follows directly from the existence of such a maximum.

In order to understand the effect of the shape parameter P_1 on the force, one must choose a physically appropriate set of polynomials orthogonal in $(0,\infty)$, i.e., a set whose weight function reflects the spatially concentrated nature of the force. Choose for example the Laguerre polynomials $L_n^{(\alpha)}(x)$, with weight function

$$w(x)=x^\alpha e^{-c(\alpha)x}, \quad (22)$$

in which the scale factor $c(\alpha)=\alpha+2+(\alpha+2)^{\frac{1}{2}}$ is such that $x_{22}=1$, corresponding to Eq. (15). As α becomes very large, it is easily shown that x_1 approaches unity as a limit. One may use Eq. (18) to define a "core" radius as the radius of the inner shell corresponding to infinite repulsion:

$$r_c\equiv R_{21}^{(\infty)}=\left(\frac{3x_1}{x_1+2}\right)\bar{r}. \quad (23)$$

As the core region grows in size, the attractive region grows narrower and deeper by virtue of the imposed binding. In the limit $\alpha=\infty$ one obtains an infinitely repulsive region bounded by a δ-function attractive shell of infinite amplitude having radius $\bar{r}$. In view of the physical unreasonableness of such a narrowly confined attractive region, further detailed examination will be limited to the choices $\alpha=0$, 1, and 2. This does not affect the limits of Eq. (21) on $P_1^{(\infty)}$ seriously. In fact it can be shown that Eq. (21) remains correct for these values of α, provided one allows the inner shell to have a finite amplitude. The bounds on the core radius are

$$0.26\times10^{-13}\text{ cm}\leq r_c\leq 0.50\times10^{-13}\text{ cm}, \quad (24)$$

in which the lower limit corresponds to $\alpha=0$ and the upper limit to $\alpha=2$. These bounds are consistent with core radii obtained from potential models[6] for which P_1 lies in the range of Eq. (21).

It remains to investigate the dependence of the second-order force ratio upon P_1 for the three choices of α. Solutions of Eq. (19) were obtained as a function of P_1 by use of the computing facilities at the University of California's Livermore laboratories. The outer shell, located at R_{22} slightly greater than R_{11}, always has amplitude $\Lambda_{22}\sim 1$; while the inner shell, located at $x_1R_{22}\sim 0.2-0.6\times10^{-13}$ cm, has an amplitude Λ_{21} depending sensitively on P_1. The force ratio is given by

$$\frac{f(x_{21})}{f(x_{22})}=\frac{\Lambda_{21}(P_1)}{\Lambda_{22}(P_1)}\left[\frac{\alpha+2-\sqrt{(\alpha+2)}}{\alpha+2+\sqrt{(\alpha+2)}}\right]^{\alpha+1}e^{\sqrt{(\alpha+2)}}. \quad (25)$$

[6] R. B. Raphael, Phys. Rev. 102, 905 (1956).

Equation (25) is plotted in Fig. 1 as a function of P_1 for $\alpha=0, 1, 2$. All three curves exhibit the same general behavior, which is also in qualitative agreement with what is expected on the basis of potential-model calculations. As P_1 decreases from large positive values, the force becomes progressively less attractive at small distances until, at $P_1=-0.0375$, it becomes repulsive at the inner-shell radius R_{21}. Note that this value of P_1 is only weakly dependent upon x_1, as is reasonable from Eq. (17). Below the asymptotic value $P_1^{(\infty)}$, the force ratio decreases from values corresponding to infinite attraction at R_{21}. This behavior cannot be readily interpreted in terms of a smoothly varying "effective force" having binding appropriate to ${}^1S$ states even though the shells themselves maintain this binding. Accordingly, $P_1^{(\infty)}$ will be taken as the lower limit in P_1 for which the quadrature method in this order gives meaningful results. In Fig. 2 the radius R_{21} of the inner shell, in units of $\bar{r}$, has been plotted as a function of P_1 for the three choices of α. Again the curves for different α exhibit a similar behavior. It is of interest that in each case R_{21} attains its minimum value near the "core" region, and that this minimum value equals r_c to a good approximation.

An energy range of validity must now be established for results in the second order of quadrature. Most simply, it is the energy region over which the first three terms of a power-series expansion of $K \cot\delta_2$ represent this function within a few percent. However, due to the rapid variation of $K \cot\delta_2$ with energy above $\sim$15 Mev, it is not the most suitable function for representation by a power series. It is much more appropriate to make

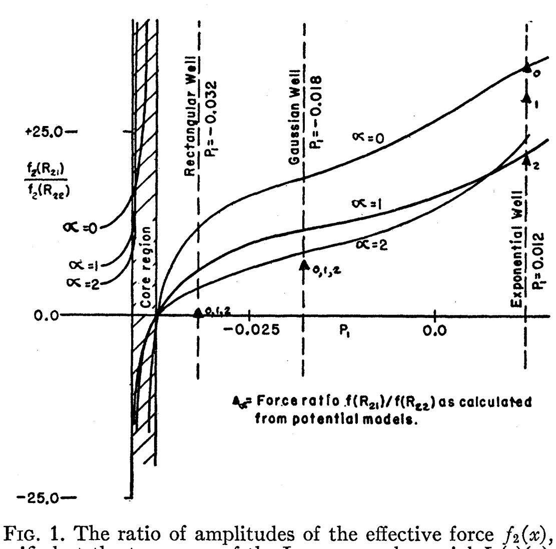

FIG. 1. The ratio of amplitudes of the effective force $f_2(x)$, as specified at the two zeros of the Laguerre polynomial $L_2^{(\alpha)}(x)$ by Eq. (25), in its dependence upon the shape parameter P_1 for $\alpha=0, 1, 2$. The cross-hatched area, corresponding to Eq. (21), shows the bounds on P_1 sufficient for a short-range repulsion in the ${}^1S$ force. Also shown are force ratios calculated from typical potential functions. Observe the excellent agreement in the case of the exponential potential which is the weight function for the polynomials $L_n^{(\alpha)}(x)$.

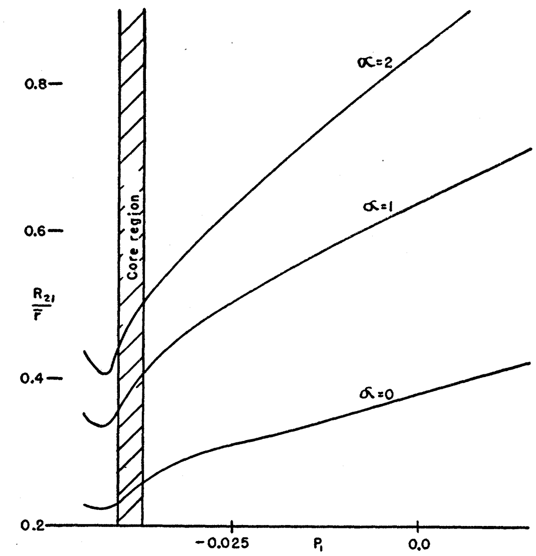

FIG. 2. The radial position R_{21} of the inner shell as a function of P_1 for the three quadrature polynomials $L_2^{(\alpha)}(x)$, $\alpha=0, 1, 2$. R_{21} may be viewed as a core radius for P_1 lying within the core region specified by Eq. (21).

use of the logarithmic derivative

$$\Gamma_2=K \cot(K\bar{x}+\delta_2), \tag{26}$$

in which $\bar{x}$ is so chosen that Γ_2 satisfies the condition

$$\left[\frac{\partial\Gamma_2}{\partial K^2}\right]_{x=\bar{x}}=0, \quad E=0. \tag{27}$$

The parameter $\bar{x}$ thus determined is, for zero binding, identical[6] with that occurring in the effective-range expansion Eq. (10). Because Γ_2 has a much weaker dependence on energy than $K \cot\delta_2$, its power-series representation is valid over a much broader energy interval. This is particularly evident in the case of core configurations, for which δ_2 passes through zero in the vicinity of 150 Mev. In contrast, $K \cot(K\bar{x}+\delta_2)$ remains well behaved over the entire nonrelativistic domain. In the case of zero binding, the approximation to Γ_2 which corresponds to a knowledge of P_1 is

$$\Gamma_2'=-(8P_1+\tfrac{1}{2})\bar{x}^3K^4. \tag{28}$$

In Fig. 3 the quantity

$$\epsilon_2=(\Gamma_2-\Gamma_2')\Gamma_2^{-1} \tag{29}$$

is plotted as a function of energy for $\alpha=0, 1, 2$ and for values of P_1 representative of various potential functions. With few exceptions, Γ_2' agrees with Γ_2 within 15% below 40 Mev. The interval $0\leq E\leq 40$ Mev is therefore attributed to results obtained in the second order of quadrature.

Let us now discuss what value of P_1 best fits the scattering. For this purpose, it is most convenient to refer to the Feshbach-Lomon (FL) analysis,[3] which

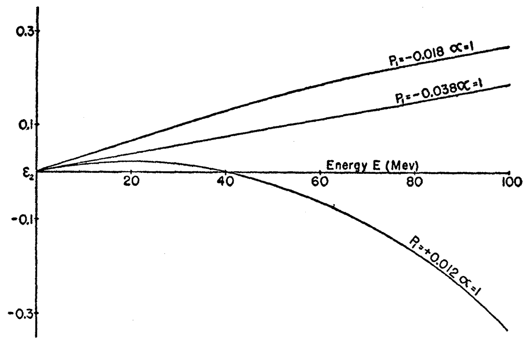

FIG. 3. The accuracy ϵ_2 [Eq. (29)] with which the logarithmic derivative $\Gamma_2=K\cot(K\bar{x}+\delta_2)$ is represented by the first non-vanishing term (of order E^2) of its power-series expansion, is shown as a function of energy; and for a number of values of shape parameter P_1 representative of both monotonic and core-type potentials. The illustration is for the typical case $\alpha=1$.

provides a charge-independent fit of nucleon-nucleon bound state and scattering data below 250 Mev. FL use a specific model which confines the nuclear force in each state to a sphere of radius r_0, within which the interaction is assumed sufficiently strong to be independent of the relative kinetic energy of the colliding particles. This permits a description of the scattering in terms of energy-independent boundary conditions applied to the logarithmic derivative of the interaction wave function evaluated at $r=r_0$. Now it has been shown[6] that the FL parameter r_0 is equal to $\bar{r}$ within experimental error. The success of the FL fit suggests additional restrictions on the energy variation of Γ_2:

$$\left[\frac{\partial^2\Gamma_2}{\partial(K^2)^2}\right]_{x=\bar{x}}=0, \quad E=0. \tag{30}$$

Hence one obtains the condition

$$P_1=-0.0416. \tag{31}$$

Thus if the logarithmic derivative were rigorously constant with energy P_1 would be equal to the lower limit of $P_1^{(\infty)}$ in Eq. (22). However, the logarithmic derivative may change somewhat with energy and still preserve the FL fit. Uncertainties in the extent to which higher partial waves contribute to P–P scattering in the 30-Mev region cause a corresponding uncertainty in P_1. Bounds on P_1 which generously allow for this effect are[6]

$$-0.0416\leq P_1\leq -0.0367, \tag{32}$$

which corresponds closely to Eq. (22). A more direct empirical confirmation of these bounds is highly desirable.

It is of interest to express the dynamical information obtained by the quadrature method directly in terms of interaction wave functions. The wave function $u_n(x)$ corresponding to the nth order of quadrature clearly consists of $n+1$ sine-wave segments joined at the positions of the n zeros of the quadrature polynomial $p_n(x)$. The discontinuity in slope at these points is governed by the force ratios, which are themselves fixed from the phase shift. The quadrature wave functions $u_2^{(\alpha)}(x)$ (normalized to unit incident particle density) corresponding to $P_1=-0.032$, $E=100$ Mev, is plotted in Fig. 4 for $\alpha=0, 1$; and for comparison, the exact wave function $u_R(x)$ for the rectangular well which has this value of P_1. In this typical case, the changes in slope at the zeros of the Laguerre polynomial $L_2^{(\alpha)}(x)$ conspire, for each α, to produce a phase shift in good agreement with that for the rectangular well. The behavior of the quadrature wave function within the range of the force has considerable latitude, as is indicated by comparing curves having different α-values. This arbitrariness reflects the insensitivity of the scattering to the details of the force at small distances. Nevertheless certain qualitative features of the interaction wave function are specified once P_1 is given. In Fig. 5 we plot the quadrature wave functions $u_2^{(0)}(x)$ at $E=50$ Mev for a range of P_1 values. When P_1 lies in the range given by Eq. (21), the core-like behavior of the force is clearly indicated. The qualitative dependence of the $u_2^{(0)}(x)$ on P_1 is to be expected from a consideration of usual potential shapes.

$n=3$

To what extent do these results remain valid as the energy is increased? In particular, what conditions

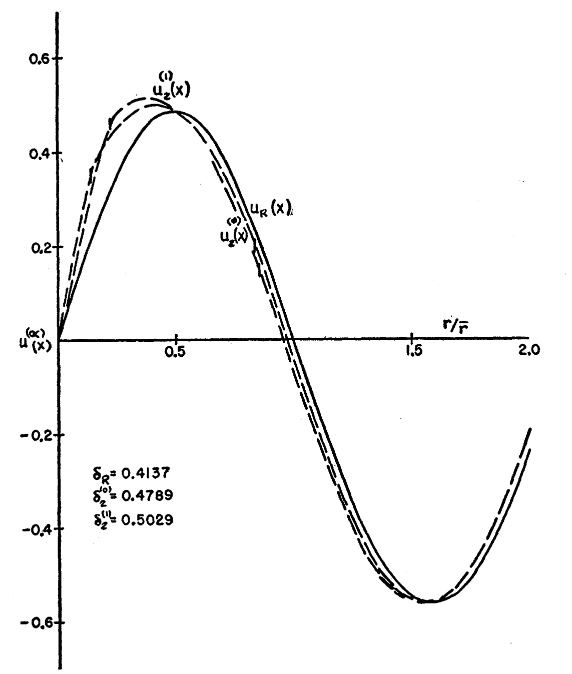

FIG. 4. The wave function $u_2^{(\alpha)}(r)$ as constructed from the shape parameter $P_1=0.032$ (corresponding to a rectangular well) is illustrated at 100 Mev for $\alpha=0$, 1. The wave function $u_R(x)$ (dashed curve), constructed from the rectangular well, is shown for comparison. Also listed for comparison are the corresponding phase shifts $\delta_2^{(\alpha)}$ and δ_R. All wave functions are normalized to unit incident-particle density.

must be satisfied by the phase shift at higher energies in order to retain a "static" interpretation of the ^1S force? In order to answer these questions and to obtain an understanding of how the higher energy data determines shape, a detailed examination of the third order of quadrature has been made. In this order, application of the quadrature theorem is equivalent to solving Eq. (1) with the potential

$$f_3(x)=\Lambda_{31}\delta(x-y)+\Lambda_{32}\delta(x-z)+\Lambda_{33}\delta(x-1). \quad (33)$$

Here $R=R_{33}$, so that $y=R_{31}/R_{33}$, $z=R_{32}/R_{33}$; and the order $y\leq z\leq 1$ has been adopted. Just as before, an expression is derived for $K\cot\delta_3$, expanded as a power series in the energy, and compared with the first four terms of Eq. (10). The four parameters Λ_{3i} and R are thus fixed by the scattering length a (chosen infinite), the mean interaction distance $\bar{r}$, and two shape parameters P_1 and P_2.

The bounds Eq. (21) on P_1 corresponding to a hard core remain unchanged in this order as does Eq. (23) for the core radius. However, the following restriction on the parameter P_2 must hold simultaneously with these in order for the higher energy data to be consistent with a hard core:

$$0.00234\leq P_2^{(\infty)}\leq 0.00416. \quad (34)$$

In order to observe the effect of variations in P_1 and P_2 upon the force, numerical calculations were carried out using the Laguerre polynomials $L_n^{(0)}(x)$, for which

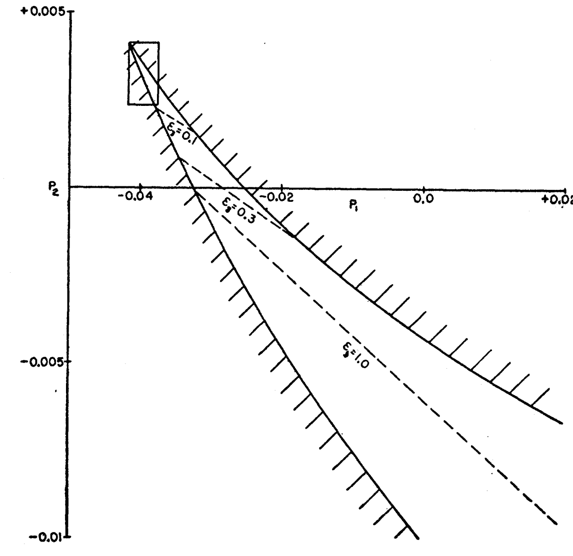

FIG. 6. The range of values which P_2 can assume consistently with a static force (unshaded region) is shown as a function of P_1. Along the dashed lines, the quantity $\epsilon_3=(\Gamma_3-\Gamma_3')\Gamma_3^{-1}$ assumes roughly constant values as indicated. The point $P_1=-1/24$, $P_2=1/240$, where the bounding curves intersect, corresponds to a constant logarithmic derivative Γ, in accord with the FL analysis. The rectangle marks off the values of P_1 and P_2 corresponding to a short-range repulsion, as specified by Eqs. (21) and (34).

$y\simeq 0.0661$, $z\simeq 0.3648$. Consider Λ_{33} and R fixed by means of conditions analogous to those of Eqs. (17) and (18). A given value of P_1 then specifies, by means of a condition analogous to that of Eq. (19), a *restricted range* of values of Λ_{31} and Λ_{32} [just as it had previously given a *unique* value of Λ_{21}, Eq. (19)]. This restriction in turn implies a limitation on P_2 by means of a condition, similar to the preceding ones, on the E^3 term of a power-series expansion of $K\cot\delta_3$. That is, if one is able to find P_1 by means of a good knowledge of the scattering below 40 Mev, it then becomes possible to place restrictions on the scattering at higher energies by requiring that it be caused by the same force. Conversely, conformity to this condition by the phase shifts at the different energies is a sufficient condition that the force responsible for the scattering be static. This limitation is plotted in Fig. 6 as a function of P_1. In Fig. 7, the two force ratios specified by this order of quadrature are plotted as functions of P_2 for several choices of P_1—these choices being consistent with the static limitation of Fig. 6. The overall variation of the force with P_1 at middle distances ($\sim z\bar{r}$) remains substantially unchanged from that observed in second order. However, large variations can be induced by P_2 without being inconsistent with a static force. Thus for $P_1=-0.039$, a small decrease in the strength of repulsion at $\sim 0.4\times 10^{-13}$ cm requires, by virtue of a binding condition analogous to Eq. (17), a very great increase in the strength of repulsion at $\sim 10^{-14}$ cm. Note that when P_1 takes on values corre-

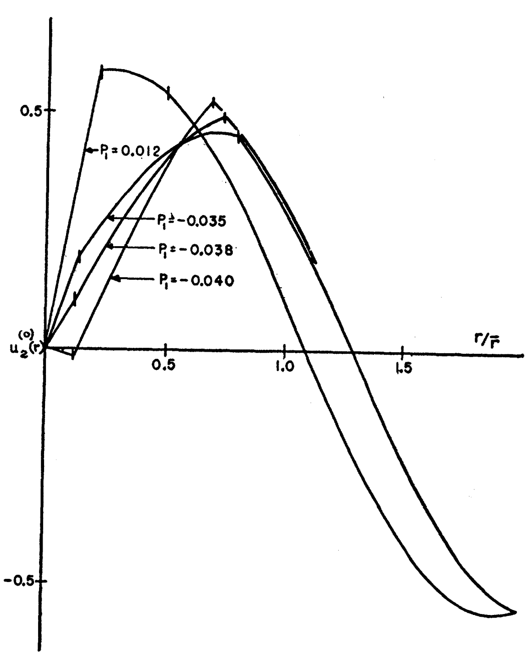

FIG. 5. The interaction wave function $u_2^{(0)}(r)$ is illustrated at 50 Mev for various values of P_1 representative of monotonic and core-type forces. Note especially the suppression of the wave function at small distances as P_1 approaches the "core" region of Eq. (21). All wave functions are normalized to unit incident-particle density.

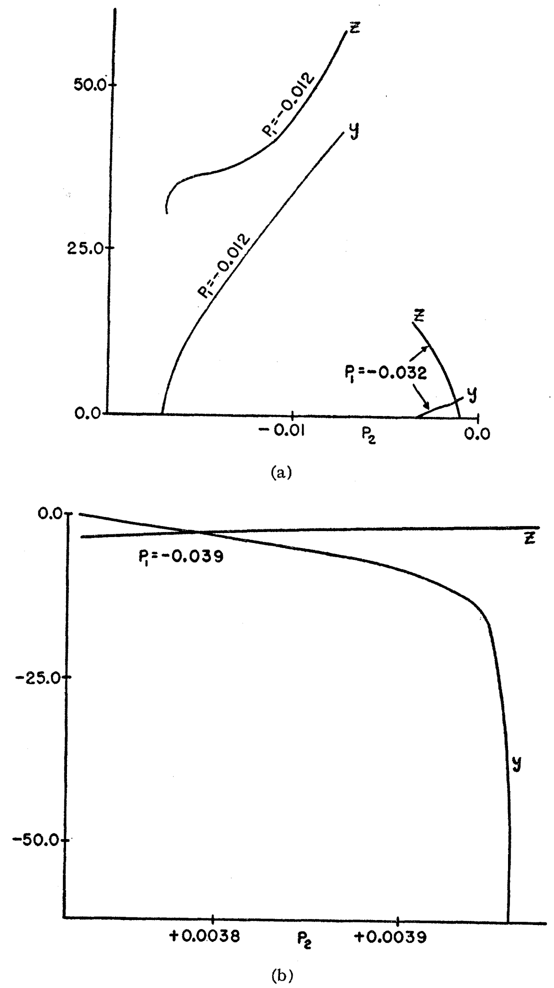

FIG. 7. The two ratios of the effective force $f_3(x)$, as specified in the third order of quadrature, shown in their dependence on P_2 for several values of P_1. Curves labeled "Z" denote the ratio $f_3(z)/f_3(1)$; while curves labeled "Y" denote the ratio $0.1f_3(y)/f_3(1)$. The interaction range R (radial position of the outermost shell) is an increasing function of P_1 and varies from $R\simeq 1.3\bar{r}$ for $P_1=-0.039$ to $R\simeq 2.0\bar{r}$ for $P_1=0.012$. The radial positions of the inner and middle shells are given by $R_{31}=yR$, $R_{32}=zR$ respectively. The outer-shell amplitude $\Lambda_{33}\simeq 1$ for $P_1=-0.039$, and decreases to $\Lambda_{33}\simeq 0.5$ for $P_1\doteq +0.012$. Neither R nor Λ_{33} depend strongly on P_2 as long as the latter is consistent with the restriction of Fig. 6.

sponding to monotonic forces, the effect of P_2 is no longer concentrated at small distances.

The energy range of validity of these results is obtained by comparing the logarithmic derivative $\Gamma_3=K\cot(K\bar{x}+\delta_3)$ with the first few terms of its power-series expansion:

$$\Gamma_3'=-(8P_1+\tfrac{1}{3})\bar{x}^3K^4+(32P_2+8P_1+\tfrac{1}{5})\bar{x}^5K^6. \quad (35)$$

Within the core region, the discrepancy $\epsilon_3=(\Gamma_3-\Gamma_3')\Gamma_3^{-1}$ is less than 10% at 120 Mev. However, as is shown in Fig. 6 the agreement rapidly grows poorer on moving into the monotonic region.

The restriction of Eq. (34) on P_2 seems to be consistent with experiment. Thus the point in Fig. 6 where the pair of curves delimiting the "static" region join, corresponds exactly to the constant logarithmic derivative employed by the FL semiempirical analysis. However, it must be emphasized that the data is not sufficiently accurate to exclude the possibility of a nonstatic or nonlocal force, especially if such a force does not lead to a strongly energy-dependent logarithmic derivative. We turn briefly to consider this alternate interpretation in the light of the quadrature method.

IV. NONSTATIC FORCES[7]

Let us now take the view that the ^1S force is nonstatic; and that it is the energy dependence rather than the detailed shape which is of greatest importance in reproducing the data. There exist very many ways of describing a nonstatic force. All of these should possess in common the property that, for sufficiently low energies, nonstatic effects are negligible. This is in strict analogy to the static situation, in which the many ways of describing static forces must all be equivalent at sufficiently low energies. We may then ask what the energy dependence must be in order to fit the higher energy data, in the same way that we have previously asked for the shape dependence.

As a first orientation, consider a rectangular well the depth of which is energy dependent:

$$\begin{aligned} V_R(x,E)&=\lambda(E), \quad x\leq 1 \\ &=0, \qquad\quad x>1. \end{aligned} \quad (36)$$

The range R and the strength $\lambda(0)$ are fixed by the scattering length and effective range. Let us determine $\lambda(E)$ by requiring that the logarithmic derivative $\Gamma_R=K\cot(K\bar{x}+\delta_R)$ constructed from V_R be constant with energy, in accordance with the FL analysis. Not unexpectedly, we observe that the force must become progressively less attractive as the energy increases, and indeed change sign at $E\sim 130$ Mev. That nonstatic effects are negligible at low energies is shown in Fig. 8 by the least-squares fit[8] $\lambda'(E)=2.2916-14\times 10^{-5}E^2$ which accurately reproduces $\lambda(E)$.

These results may be obtained both more simply and more generally by means of the quadrature method. So long as the force is a local and smoothly varying function of position and energy, Eqs. (13) and (16) remain valid expressions for the first-order and second-order quadrature phase shifts. The quadrature amplitudes Λ_{nr} are now considered functions of energy. Thus in Eq. (13) write[8] $\Lambda_{11}=\Lambda_{11}^{(0)}+\Lambda_{11}^{(2)}E^2$, and allow R, $\Lambda_{11}^{(0)}$, $\Lambda_{11}^{(2)}$ to be fixed by a, $\bar{r}$, and P_1 in the usual way. Choosing $P_1=-0.0416$ in accordance with Eq. (32), one obtains immediately $R=\frac{3}{2}\bar{r}$, $\Lambda_{11}^{(0)}=1$, and $\Lambda_{11}^{(2)}$

[7] A portion of this material may be found in the author's doctoral thesis, Harvard, 1954 (unpublished).

[8] Terms linear in the energy may be ignored, since they amount to a redefinition of the effective range, and may be compensated for by a scale transformation.

$=-1.8\times10^{-5}$. Higher orders of quadrature may be used when the energy dependence is not uniform over the interaction volume. As an example, suppose that the force appears static if the colliding particles are separated by distances of the order of a meson Compton wavelength, so that in Eq. (16) $\Lambda_{22}=\text{const}$; but assume further that $\Lambda_{21}=\Lambda_{21}^{(2)}E^2$. One then finds $\Lambda_{22}=1$, $R=\frac{3}{2}\bar{r}$ as before, and $\Lambda_{21}^{(2)}=-(1.8\times10^{-5})x_1^{-2}$. Thus a force which at low energies is purely attractive develops a short-range repulsion as the energy increases, this repulsion being the stronger at a given energy the more narrowly it is confined.

The data below 100 Mev is insufficient to distinguish between static and nonstatic forces. However, scattering and polarization data at higher energies have not as yet been reconciled with static forces. The quadrature method should be of use in a further testing of the consequences of an assumed energy dependence. Generalizations to nonlocal forces are also easily made and have been found useful in an area where the standard approximate methods are often not adequate.

V. CONCLUDING COMMENTS

In recent years, nucleon-nucleon scattering techniques and methods of empirical analysis have improved rapidly. It is reasonable to suppose that the lowest partial waves contributing to nucleon-nucleon scattering will soon be known to good accuracy over much of the nonrelativistic domain. In this paper we have inquired how much detailed dynamical information can

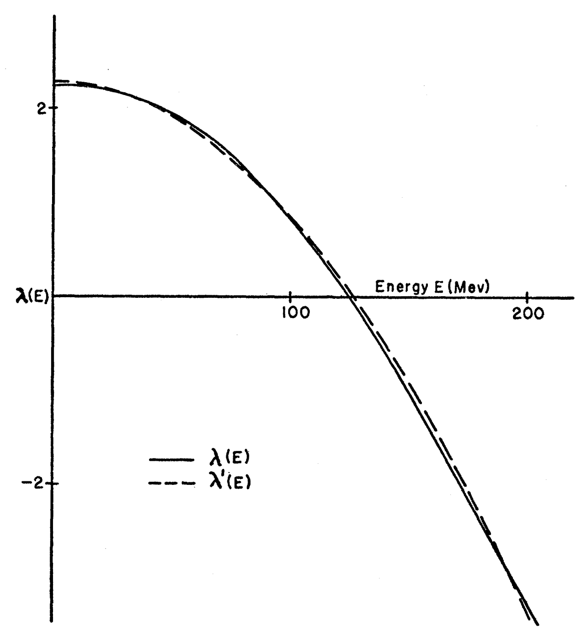

FIG. 8. The variation with energy of the strength λ of a rectangular well necessary to insure constancy of the logarithmic derivative $\Gamma_R=K\cot(K\bar{x}+\delta_R)$ in accordance with the FL analysis. Above 130 Mev, the force becomes strongly repulsive. The least-square fit $\lambda'(E)=2.2916-0.000\,14E^2$ is seen to be an accurate approximation to $\lambda(E)$ below 200 Mev.

be extracted from this knowledge. The limitation on the energy region under consideration implies that one cannot expect to identify a unique functional form for the force. However, it remains meaningful to ask for the properties of an "effective force," in which fluctuations occurring in distances small compared to the Compton wavelength of the colliding nucleons have been averaged out. In this fashion the concept of an "effective range" is generalized to that of an "effective force," to be known in progressively greater detail as the data over wider energy regions is taken into account. The assumed smooth variation of the effective force with position (and perhaps energy) makes possible, by the application of a quadrature theorem, an optimally accurate representation of the scattering by force ratios, the number of which is dependent upon the energy region in question and the accuracy desired.

There exists a number of physically appropriate choices of orthogonal polynomials upon which the quadrature theorem may be based. The results which have been obtained are, for the most part, not strongly dependent upon which of these is chosen. Thus, gross features of the force, such as its range and the existence of a short-range repulsion for suitable values of the shape parameters P_1 and P_2, are quite insensitive to this choice. For the three choices of polynomials examined in order $n=2$, the core radius remains within fairly narrow bounds, while the qualitative behavior of the force ratio with P_1 is the same in all cases.

It does not seem possible to adduce any mathematically binding arguments for the validity of the quadrature method. However, in addition to being physically plausible by construction, it yields results which are qualitatively in agreement with those obtained using the more familiar potential models. This is the case with the $n=2$ force ratio in its dependence upon P_1, and is also well illustrated by the interaction wave functions constructed from the phase shift. One is thus encouraged to extend the region of application of the method, and also to make use of the quadrature amplitudes as an alternate description of the scattering. As compared with potential models, such a description is computationally simpler and more systematic.

In order to conveniently compare the results obtained by quadrature analysis with experiment, use was made of the Feshbach-Lomon (FL) semiempirical analysis. This analysis must be substantiated by more direct methods in order for the comparison to have weight.[9] Careful measurements of p-p scattering in the 40-Mev region in progress at the time of this writing,[10] should provide this check. Another reason for emphasizing accurate intermediate-energy experimentation[11] is the strong correlation which was observed between P_1 and

[9] For a critical discussion of the FL analysis, see A. M. Saperstein and L. Durand, Phys. Rev. **104**, 1102 (1956).
[10] L. H. Johnson (private communication).
[11] In this connection, see also H. P. Noyes, Bull. Am. Phys. Soc. Ser. II, **2**, 72 (1957).

the shape restrictions imposed by higher energy data (P_2).

At present it seems permissible to treat the ^1S force below 120 Mev as static. However, such treatment is by no means mandatory in view of uncertainties in both the data and the analyses thereof. It has been seen that the quadrature method is easily modified to take account of nonstatic forces. For example, it was shown that the FL requirement of a constant logarithmic derivative permits a force which, monotonic at low energies, contains a short-range repulsion at higher energies. Generally speaking, the quadrature method can be useful in the analysis of complicated energy-dependent, and especially nonlocal, interactions for which simple analytic approximation methods are not available.

It has been implied that a power-series representation of the data is by no means necessary in order to apply the quadrature method. For example, the convergence problems associated with the use of power series can be avoided by using the phase shifts themselves as input data. The force ratios in a given order would then express how the force must be restricted as a result of specifying the phase shift at several energies. While inappropriate to the systematic presentation with which we have been mainly concerned, this technique has the advantage of being more easily adapted to experiment.[12]

The extent of application of the quadrature method is limited only by the amount and accuracy of available phase-shift information. Its generalization to higher angular-momentum states and tensor forces is very straightforward and easily amenable to machine computation. It is hoped that as more data becomes available the method will prove useful in its twin capacities as a tool for analyzing phase-shift data and as a simple way of examining the consequences of assumed laws of force.

[12] Such modifications are being investigated by H. P. Noyes and T. Northrup.

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