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Meson Production in Meson-Nucleon Collision*†

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By using the Chew-Low theory, meson production cross sections in meson-nucleon collisions have been calculated. The various production amplitudes, in the one-meson approximation, yield integral equations whose kernels involve scattering phase shifts. For these, experimental or theoretical phase shifts were used. Calculations with pseudoscalar mesons show that, for coupling constants deduced from scattering, the Born approximation differs greatly from the results of the one-meson approximation. Agreement with experiments at 470 Mev is good but not decisive.

1. INTRODUCTION

CHEW and Low[1] have recently shown that low-energy p-wave pion-nucleon scattering is adequately described by pseudoscalar mesons with pseudovector coupling to an extended nucleus. In spite of ignoring recoil and pair terms, they reproduced the experimentally measured (3,3) phase shifts including the resonance. Their calculation leads to a set of nonlinear coupled integral equations which in the one-meson approximation can be solved approximately. They retained some of the features of relativistic field theories by satisfying the crossing theorem of Gell-Mann and Goldberger[2] and the requirement of unitarity. The only adjustable parameters in this calculation are the coupling constant and the cutoff.

We calculate meson production cross sections in meson-nucleon collisions in the pseudovector meson theory using the one-meson static approximation. That is, in virtual scattering processes states containing more than one meson and one physical nucleon have been ignored. Although the nucleon is represented by the lowest eigenstate of the complete Hamiltonian and is given enough degrees of freedom to represent its charge and spin states, it is not given translational motion. These two approximations are clearly unjustified for high-energy processes, but no attempt was made to determine the limit of their validity. Configurationt containing two mesons will probably be importans near a resonance for double meson production. Our solution is consistent with the restriction arising from the unitarity of the S matrix in the one-meson approximation, whereas crossing is not exactly satisfied. To facilitate numerical integrations all the phase shifts except the (3,3) are taken to be zero. Experimentally measured phase shifts are used when available.

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1 G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

2 M. Gell-Mann and M. L. Goldberger, *Proceedings of the Fourth Annual Rochester Conference on High-Energy Nuclear Physics* (University of Rochester Press, Rochester, 1954.)

2. INTEGRAL EQUATIONS FOR THE PRODUCTION AMPLITUDE

We shall use the notation of Chew and Low throughout, with the exception of using mesons in definite charge states; hence

$$\tau_n=\begin{cases}\frac{1}{2}(\tau_1\pm i\tau_2), & n=\pm 1\\ \tau_3, & n=0\end{cases}$$

where τ_1, τ_2, τ_3 are the Pauli matrices.

The production matrix for a transition from an initial state containing a nucleon $\Psi_0=\Psi_0(m)$ and a meson q to a final state containing a nucleon $\Psi_0'=\Psi_0(m')$ and two mesons p_1, p_2 is[3]

$$T_{qm}(p_1p_2m)=\langle\Psi_0'|a_{p1}V_{p2}^{(0)\dagger}+a_{p2}V_{p1}^{(0)\dagger}|\Psi_q^{(+)}\rangle.$$

This expression for the production matrix is equal to

$$\langle\Psi'_{p1p2}{}^{(-)}|V_q^{(0)}|\Psi_0\rangle$$

on the energy shell. The latter is very complicated off the energy shell and is not used in obtaining the following integral equations.

Let

$$O\equiv a_{p1}V_{p2}^{(0)\dagger}+a_{p2}V_{p1}^{(0)\dagger}. \tag{2.1}$$

As a consequence of the equations of motion,

$$\begin{aligned}\langle\Psi_0(m')|O|\Psi_q^{(+)}\rangle=&\,\delta_{qp1}\langle\Psi_0'|V_{p2}^{(0)\dagger}|\Psi_0\rangle\\&+\delta_{qp2}\langle\Psi_0'|V_{p1}^{(0)\dagger}|\Psi_0\rangle-\left\langle\Psi_0'\left|V_q^{(0)}\frac{1}{H+\omega_q}O\right|\Psi_0\right\rangle\\&+\left\langle\Psi_0'\left|O\frac{1}{\omega_q-H+i\epsilon}V_q^{(0)}\right|\Psi_0\right\rangle.\end{aligned}\tag{2.2}$$

The Kronecker delta functions are simultaneously diagonal in charge and momentum. It also follows that

$$\begin{aligned}\langle\Psi_\nu^{(+)}(n)|O|\Psi_0\rangle=&-\left\langle\Psi_0(n)\left|O\frac{1}{H+\omega_\nu}V_\nu^{(0)\dagger}\right|\Psi_0\right\rangle\\&+\left\langle\Psi_0(n)\left|V_\nu^{(0)\dagger}\frac{1}{\omega_\nu-H-i\epsilon}O\right|\Psi_0\right\rangle.\end{aligned}\tag{2.3}$$

3 N. Fukuda and J. S. Kovacs, Phys. Rev. **104**, 1784 (1956).

Delta functions do not appear in this expression, and a_ν and O commute.

Let us define a new quantity

$$\langle\Psi_0'|O|\Psi_q^{(+)}(m)\rangle'\equiv\langle\Psi_0'|O|\Psi_q^{(+)}(m)\rangle -\delta_{qp1}\langle\Psi_0'|V_{p2}^{(0)\dagger}|\Psi_0\rangle-\delta_{qp2}\langle\Psi_0'|V_{p1}^{(0)\dagger}|\Psi_0\rangle. \quad (2.4)$$

On the energy shell this is equal to

$$\langle\Psi_0'|O|\Psi_q^{(+)}(m)\rangle.$$

Using (2.4) in (2.2) and (2.3) and summing over intermediate states, we get

$$-\langle\Psi_0'|O|\Psi_q^{(+)}(m)\rangle'=\sum_n\left\{\frac{\langle\Psi_0'|V_{p2}^{(0)\dagger}|\Psi_0(n)\rangle\langle\Psi_{p1}^{(+)}(n)|V_q^{(0)}|\Psi_0\rangle}{\omega_{p1}-\omega_q-i\epsilon}+\frac{\langle\Psi_0'|V_{p1}^{(0)\dagger}|\Psi_0(n)\rangle\langle\Psi_{p2}^{(+)}(n)|V_q^{(0)}|\Psi_0\rangle}{\omega_{p2}-\omega_q-i\epsilon}\right.$$
$$\left.+\frac{\langle\Psi_0'|V_q^{(0)}|\Psi_0(n)\rangle\langle\Psi_0(n)|O|\Psi_0\rangle}{\omega_q}-\frac{\langle\Psi_0'|O|\Psi_0(n)\rangle\langle\Psi_0(n)|V_q^{(0)}|\Psi_0\rangle}{\omega_q}\right\}$$
$$+\sum_r\frac{\langle\Psi_0'|O|\Psi_r^{(+)}\rangle'\langle\Psi_r^{(+)}|V_q^{(0)}|\Psi_0\rangle}{\omega_r-\omega_q-i\epsilon}+\sum_\nu\frac{\langle\Psi_0'|V_q^{(0)}|\Psi_\nu^{(+)}\rangle\langle\Psi_\nu^{(+)}|O|\Psi_0\rangle}{\omega_\nu+\omega_q}, \quad (2.5)$$

and

$$-\langle\Psi_\nu^{(+)}(n)|O|\Psi_0\rangle=\sum_s\left\{\frac{\langle\Psi_0(n)|V_{p2}^{(0)\dagger}|\Psi_0(s)\rangle\langle\Psi_{p1}^{(+)}(s)|V_\nu^{(0)\dagger}|\Psi_0\rangle}{\omega_\nu+\omega_{p1}}+\frac{\langle\Psi_0(n)|V_{p1}^{(0)\dagger}|\Psi_0(s)\rangle\langle\Psi_{p2}^{(+)}(s)|V_\nu^{(0)\dagger}|\Psi_0\rangle}{\omega_\nu+\omega_{p2}}\right.$$
$$\left.+\frac{\langle\Psi_0(n)|O|\Psi_0(s)\rangle\langle\Psi_0(s)|V_\nu^{(0)\dagger}|\Psi_0\rangle}{\omega_\nu}-\frac{\langle\Psi_0(n)|V_\nu^{(0)\dagger}|\Psi_0(s)\rangle\langle\Psi_0(s)|O|\Psi_0\rangle}{\omega_\nu}\right\}$$
$$+\sum_\rho\frac{\langle\Psi_0(n)|V_\nu^{(0)\dagger}|\Psi_\rho^{(+)}\rangle\langle\Psi_\rho^{(+)}|O|\Psi_0\rangle}{\omega_\rho-\omega_\nu+i\epsilon}+\sum_t\frac{\langle\Psi_0(n)|O|\Psi_t^{(+)}\rangle'\langle\Psi_t^{(+)}|V_\nu^{(0)\dagger}|\Psi_0\rangle}{\omega_\nu+\omega_t}. \quad (2.6)$$

Equations (2.5) and (2.6) are the equations that will be solved to obtain the production matrix when the intermediate states are limited to one nucleon and one-meson–one-nucleon states.

To exhibit the crossing symmetry of the production amplitude it is convenient to rewrite (2.2) on the energy shell after eliminating O

$$T_{qm'}(p_1p_2m)=\left\langle\Psi_0'\right|V_q^{(0)}\frac{1}{H+\omega_q}\left(V_{p2}^{(0)\dagger}\frac{1}{H+\omega_{p1}}V_{p1}^{(0)\dagger}+V_{p1}^{(0)\dagger}\frac{1}{H+\omega_{p2}}V_{p2}^{(0)\dagger}\right)$$
$$+V_{p2}^{(0)\dagger}\frac{1}{\omega_q-\omega_{p1}-H-i\epsilon}\left(V_{p1}^{(0)\dagger}\frac{1}{\omega_q-H+i\epsilon}V_q^{(0)}-V_q^{(0)}\frac{1}{\omega_{p1}+H}V_{p1}^{(0)\dagger}\right)$$
$$+V_{p1}^{(0)\dagger}\frac{1}{\omega_q-\omega_{p2}-H+i\epsilon}\left(V_{p2}^{(0)\dagger}\frac{1}{\omega_q-H+i\epsilon}V_q^{(0)}-V_q^{(0)}\frac{1}{\omega_{p2}+H}V_{p2}^{(0)\dagger}\right)\left|\Psi_0\right\rangle. \quad (2.7)$$

The crossing symmetry is equivalent to requiring the production matrix to be invariant to the substitution

$$V_q^{(0)}\rightleftarrows V_{p1}^{(0)\dagger},$$
$$\omega_q+i\epsilon\rightleftarrows-\omega_{p1},$$

or

$$V_q^{(0)}\rightleftarrows V_{p2}^{(0)\dagger},$$
$$\omega_q+i\epsilon\rightleftarrows-\omega_{p2}.$$

Equation (2.7) is seen to be unchanged by this substitution. This substitution exchanges elements in the third term of (2.2) with those in the fourth term of the same equation. Thus both these terms must be calculated to the same accuracy to satisfy the crossing theorem. In the pseudovector theory, in order to obtain an analytic solution, additional approximations beyond those mentioned above are necessary. These approximations are made in terms which always have a positive energy denominator and which thus are presumably smaller than those in the fourth term of (2.2). We shall return to this point.

In distinction to the behavior of our solutions under crossing, unitarity as consistent with the one-meson approximation is strictly satisfied in our solution. The unitarity of the S matrix requires that

$$\langle i|T|f\rangle^*-\langle f|T|i\rangle =2\pi i\sum_n\delta(E_i-E_n)\langle n|T|f\rangle^*\langle n|T|i\rangle, \quad (2.8)$$

where $E_i=E_f$.

In the one-meson approximation the solution will be required to satisfy (2.8) when the states $\langle n|$ are limited to one physical nucleon and one nucleon-one meson scattering states. It is readily shown that if

$$\langle f|T|i\rangle \equiv T_{qm}(p_1p_2m') = \langle \Psi_0'|O|\Psi_q^{(+)}(m)\rangle,$$

then

$$\langle i|T|f\rangle = \langle \Psi_q^{(-)}(m)|O^\dagger|\Psi_0'\rangle. \qquad (2.9)$$

Upon comparing (2.9) with (2.2), it is seen that $\langle i|T|f\rangle^*$ differs from (2.2) by the replacement of ϵ by $-\epsilon$. Thus

$$\langle i|T|f\rangle^* - \langle f|T|i\rangle = 2\pi i \sum_n \langle \Psi_0(m')|O|n\rangle\langle n|V_q^{(0)}|\Psi_0(m)\rangle\delta(E_n-\omega_q),$$

and unitary is automatically satisfied by the equations of motion. The following is clear: in (2.2) the term with always positive energy denominator has nothing to do with unitarity, whereas the last term of (2.2) is responsible for unitarity and must be treated to a corresponding degree of accuracy as the requirement of unitarity demands. In the pseudovector theory the one-meson approximation requires the retention of one-nucleon–one-meson scattering states in the last term of (2.2).

The solution of the coupled integral equations (2.2) and (2.3) in the one-meson approximation is further simplified by using an angular and isotopic spin eigenstate decomposition of $\langle \Psi_0(m')|O|\Psi_q^{(+)}\rangle$ and $\langle \Psi_\nu^{(+)}(n)|O|\Psi_0\rangle$.[1,4–6] The coefficients of the former quantity in this decomposition will be labelled by $T_{(J,T),(l,t)}$ corresponding to final mesons of total orbital angular momentum l and a total angular momentum of the whole state J. The isotopic-spin subscripts T, t have similar meanings.

Our procedure may now be compared to that of Barshay,[7] Franklin,[8] and Rodberg.[6] Barshay starts with the relativistic pseudoscalar meson theory in the Heisenberg representation and obtains the meson production amplitude which he later reduces to the static limit. After establishing Eq. (2.7) he sum over both intermediate states, keeping terms containing one nucleon in both, and one nucleon in one and a nucleon and a meson in the other. On the other hand, Franklin observes that the meson production matrix off the energy shell has a singularity of the type seen in (2.2). After identifying only one of the δ_{qp_1}, δ_{qp_2} singularities, he obtains an integral equation of the type (2.5). In this integral equation he omits sums over intermediate states containing one-meson–one-nucleon states, i.e., the fifth and sixth terms in Eq. (2.5). The result is the inhomogeneous part of his integral equation. But because only one of the δ-function singularities was isolated the resultant expression is not symmetric with respect to the final mesons, at the end he symmetrizes this expression. Although the inhomogeneous part of Eq. (2.5) contains some of the terms in Eq. (27) of Franklin, it does not contain all terms whereas our inhomogeneous terms involve a further sum over one-meson intermediate states and presumably higher order effects. Franklin gives a further comparison of Barshay's approximation to his. The result of Barshay's and Franklin's approximations is that their result does not obey unitarity in the one-meson approximation. Since unitarity implies bounds for partial cross sections of given angular momentum, our resultant production cross sections are smaller than theirs. In distinction to the pseudovector theory, in the charged scalar theory we[9] have calculated the meson production amplitude in the one-meson approximation without any further assumptions; that is, a solution obeying unitarity and crossing symmetry has been obtained. This may afford an interesting comparison as to the influence of violating the crossing symmetry in an approximate solution. Rodberg[6] uses equations of the type used by Franklin which also treats the two final mesons differently but retains one-meson states in terms with singular denominators. Thus he obtains a single linear integral equation.

3. RESULTS AND CONCLUSIONS

Meson production cross sections have been calculated in the one-meson approximation in the pseudovector theory. The solution obeys unitarity as demanded by the one-meson approximation. When the production matrix is expanded in terms of angular momentum and isotopic spin eigenstates, the resultant amplitudes yield integral equations which are coupled to only one other equation if (3,3) phase shifts alone are used. Of these amplitudes $T_{(\frac{3}{2},\frac{3}{2}),(2,2)}$ can be solved exactly. This amplitude is independent of the (1,1), (3,1), and (1,3) phase shifts. At low energy, when the (3,3) phase

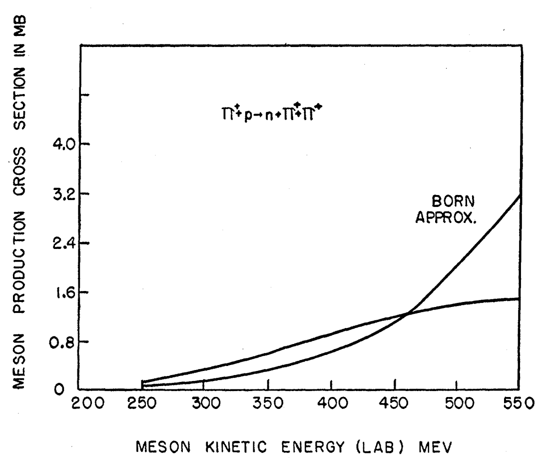

FIG. 1. Calculated total meson production cross section for the reaction $\pi^+ + p \rightarrow n + \pi^+ + \pi^+$ *vs* meson kinetic energy in the laboratory system. For comparison the Born-approximation result is included.

[4] B. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[5] B. d'Espagnat, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **2B**, No. 11 (1945).
[6] L. S. Rodberg, Phys. Rev. **106**, 1090 (1957).
[7] Saul Barshay, Phys. Rev. **103**, 1102 (1956).
[8] Jerrold Franklin, Phys. Rev. **105**, 1101 (1957). We thank Dr. Franklin for communicating his results ahead of publication.

[9] See author's thesis, University of Chicago, 1957 (unpublished).

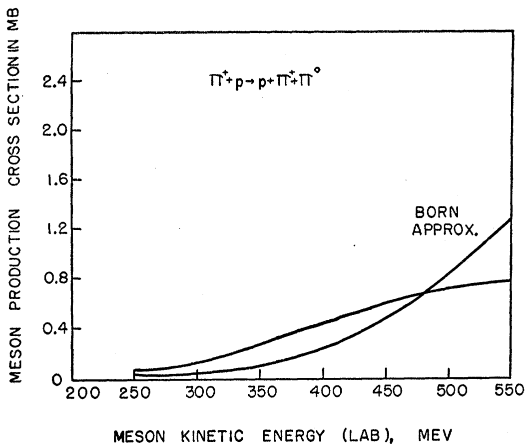

FIG. 2. Calculated total meson production cross section for the reaction $\pi^+ + p \rightarrow p + \pi^+ + \pi^0$ *vs* meson kinetic energy in the laboratory system. For comparison the Born-approximation result is included.

shifts only are used, this turns out to be larger than any other eigenamplitudes. Since the calculation of $T_{(\frac{3}{2},\frac{3}{2}),(2,2)}$ makes no approximations in the crossed terms, the crossing theorem is strictly obeyed for this amplitude. On the other hand, the calculation of $T_{(\frac{3}{2},\frac{3}{2}),(2,1)}$, etc., require the approximation of terms which come from crossed diagrams with lower order expressions; thus crossing symmetry is not exactly satisfied for these amplitudes. But since these amplitudes are smaller than $T_{(\frac{3}{2},\frac{3}{2}),(2,2)}$ at low energy, our solution obeys the crossing theorem to a high accuracy in this region. The calculation was done with $f^2 = 0.08$ and a 6.3 μc^2 meson energy cutoff; the production cross sections are shown in Figs. 1, 2, 3, and 4. The (3,3) phase shifts are taken from Bethe[10] and Margulies[11]

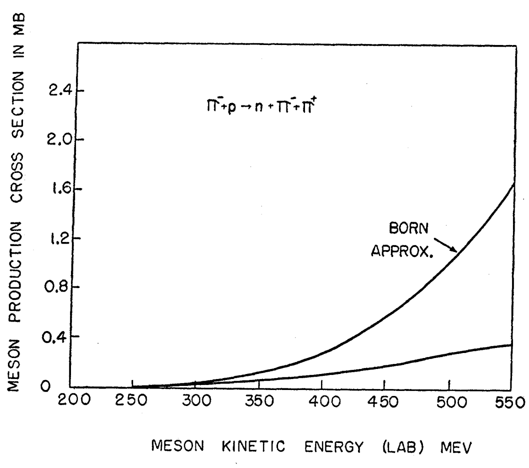

FIG. 3. Calculated total meson production cross section for the reaction $\pi^- + p \rightarrow n + \pi^- + \pi^+$ *vs* meson kinetic energy in the laboratory system. For comparison the Born-approximation result is included.

[10] H. A. Bethe and F. de Hoffmann, *Mesons and Fields* (Row, Peterson, and Company, Evanston, 1955), Vol. 2, pp. 121–126.

[11] R. S. Margulies, Phys Rev. **100**, 1255(A) (1955).

up to 400 Mev. For higher energies these phase shifts are extrapolated. This cannot be done with a great deal of arbitrariness. Nevertheless the low-energy production cross sections will probably be independent of the high-energy phase shifts. At 500 Mev and above, the phase shift is taken to be constant and equal to π. This is undoubtedly wrong but in absence of experimental information the most convenient choice was made. The other p-phase shifts have been ignored. This is permissible for less than 400-Mev meson kinetic energy. As a consequence of these two approximations the production cross sections calculated on this basis will be too low for high energy.

The crossed-diagram contributions which were treated to a lower approximation could have been handled equally well in the presence of all the p-phase shifts. Therefore the use of (3,3) phase shifts alone was not

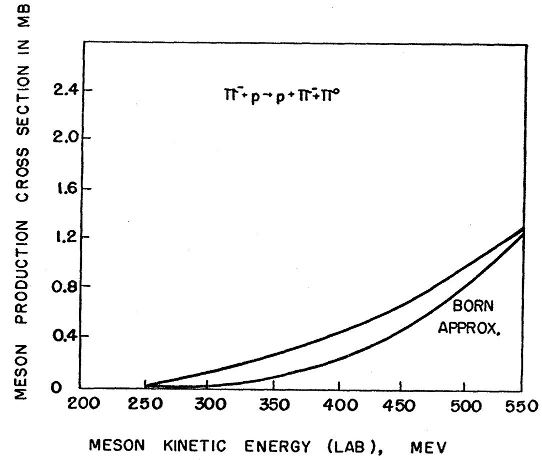

FIG. 4. Calculated total meson production cross section for the reaction $\pi^- + p \rightarrow p + \pi^- + \pi^0$ *vs* meson kinetic energy in the laboratory system. For comparison the Born-approximation result is included.

critical in obtaining solutions for the production amplitudes.

At threshold for the process $\pi^+ + p \rightarrow n + \pi^+ + \pi^+$, only $T_{(\frac{3}{2},\frac{3}{2}),(2,2)}$ contributes if (3,3) phase shifts alone are used. For this process the cross section at threshold is $\sim$13 times the Born approximation result. Again at threshold the cross section for $\pi^+ + p \rightarrow p + \pi^+ + \pi^0$ is $\sim$13 times the Born approximation result. The cross sections for $\pi^- + p \rightarrow n + \pi^- + \pi^+$ and $\pi^- + p \rightarrow p + \pi^- + \pi^0$ are increased by factors of $\sim$6 and $\sim$7, respectively, over their values at threshold in the Born approximation. These conclusions, as discussed in the previous paragraph, could not be very far from the strict use of the one-meson calculation. This increase of all cross sections is a consequence of the resonance in scattering at an energy of 2.1 μc^2 in the center-of-mass system. Since $\pi^+ + p$ is in a pure isotopic spin triplet, whereas $\pi^- + p$ is not, for reactions starting from the former initial state greater enhancement is obtained.

It is clear that at very low energies s-meson production will exceed p-meson production. But by comparing the s-phase shifts with p-phase shifts at low energy it is clear that when the kinetic energy available to each of the final mesons exceeds 50 Mev the mesons will be predominantly in p states.

The experiment of Blevins, Block, and Harth[12] shows that for 470-Mev π^+ energy the $\pi^+ + p$ cross section is 10% inelastic. This would indicate an inelastic cross section of 2 mb. If the total kinetic energy available in the center-of-mass system is interpreted to be available to the mesons, the calculated $\pi^+ + p \rightarrow n + \pi^+ + \pi^+$ cross section is 1.4 mb and the $\pi^+ + p \rightarrow p + \pi^+ + \pi^0$ cross section is 0.68 mb. Although the sum of these cross sections agrees with experiment quite well, the Born-approximation result gives 1.48 mb and 0.60 mb, respectively, for the above cross sections. Thus agreement with experiment at this energy is not decisive for the determination of the static one-meson approximation. Nevertheless, for 400–550 Mev incident pion energy the final mesons are above the range of s-meson production and below the energy where the approximation of high-energy (3,3) phase shifts with π and the neglect of the other p-phase shifts has an appreciable effect; thus agreement with experiments in this range should be good. Since the one-meson approximation differs from the Born approximation by as much as a factor of three in this energy range, further experiments at these energies will be enlightening.

[12] Blevins, Block, and Harth, Bull. Am. Phys. Soc. Ser. II, 1, 174 (1956).

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Generalized Effective Range Theory for Nucleon-Nucleon Scattering*

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A method is proposed for obtaining limitations on the shape and possible energy dependence of the force in a given scattering state of the two-nucleon system, from a knowledge of the phase shift in that state over the nonrelativistic domain, and is illustrated for ^1S waves.

I. INTRODUCTION

THE purpose of this paper is to give a method for translating the results of a partial-wave analysis of nucleon-nucleon scattering data into equivalent information on the nuclear force. Our concern is not with the general mathematical problem of deducing a potential from the complete two-body S-matrix.[1] Rather, we seek to obtain only those properties of the nuclear force which are determined by experiment. Just as all potential models must imply the correct effective range[2] in order to fit the low-energy data, so they must all contain the properties we seek in order to fit the higher energy data. Thus, by setting an upper limit to the energies under consideration, we limit the detail with which the incident nucleon is able to observe the force by which it is scattered. All potentials of a class yielding the experimental phase shifts over this energy region may then be considered equivalent to an "effective force," obtained by neglecting fluctuations in members of this class over interparticle distances much smaller than the wavelength of the incident nucleon. For example, at sufficiently low energies the nature of this equivalence is well known as the shape-independent approximation.[2] What limitations are imposed on the "effective force" by data at higher energies?

In Sec. II a method is developed, within the context of S-waves, for deducing this information from the given phase shift on the assumption that the force is static. In essence, the dependence of the phase shift on the force is schematized by replacing the latter by its value at a discrete set of radial points, whose position ratios are so chosen that the representation of the matrix elements of the force is an "optimal" one in the sense of a Gauss-Jacobi quadrature approximation. It is then possible to employ these points as probes of the force by allowing their positions and associated amplitudes to be fixed by the experimental phase shift in its dependence upon energy. The latter is assumed, for the sake of clarity, to be developable in a power series which converges at least asymptotically in the domain of interest. The order of quadrature theorem to be

* A preliminary report of this work can be found in the *Proceedings of the Sixth Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1956).

[1] In this connection see, for example, R. Jost and W. Kohn, Phys. Rev. 87, 977 (1952); 88, 382 (1952).

[2] J. M. Blatt and J. D. Jackson, Phys. Rev. 76, 18 (1949); H. A. Bethe, Phys. Rev. 76, 38 (1949).