Hfs of the 5 $^2P_{\frac{3}{2}}$ State of In^{115} and In^{113}: Octupole Interactions in the Stable Isotopes of Indium*†

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The zero-field hfs intervals in the $5\,^2P_{\frac{3}{2}}$ state of In^{115} and In^{113} have been measured by the use of conventional atomic-beam techniques. The magnetic octupole interaction constants for these atomic systems are determined from the measured intervals and corrected for perturbations of the electronic states. They are

$$c'(In^{115})=[0.001702\pm(35)]\times10^6\ \text{sec}^{-1},$$
$$c'(In^{113})=[0.001728\pm(45)]\times10^6\ \text{sec}^{-1}.$$

The estimated values of the nuclear magnetic octupole moments of In^{115} and In^{113} are compared to the predictions of the single-particle and collective models of the nucleus.

INTRODUCTION

RECENT measurements of atomic hfs have established the existence of the nuclear magnetic octupole moments of I^{127}, Ga^{69}, Ga^{71}, and In^{115}. That portion of the hyperfine splittings attributable to a nuclear octupole moment is only about one part in 10^5 of the observed hyperfine splittings. The relevant theory of atomic hfs must, therefore, be valid to at least one part in 10^6 for the unambiguous identification of an octupole interaction. The recent work of Clendenin[1] and of Schwartz[2] has developed certain parts of the theory to this order of precision. However, to interpret all observations of the present work, a further extension of the theory is required and will be developed in a later section of this paper. In this and the preceding paper is described a detailed study of the hfs of the $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ components of the ground state doublet of In^{115} and In^{113}. The work was undertaken to investigate the octupole interaction in these atomic systems, to establish and interpret the differential hfs anomaly between the $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ states, and to test the validity of the present theory of atomic hfs for coupled states such as $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$.

ELEMENTARY THEORY

The ground state configuration of indium is principally that of a single $5p$ electron outside of the filled $5s$ subshell. This configuration gives rise to a normal $P_{\frac{1}{2},\frac{3}{2}}$ doublet. We shall first sketch the theory of the hfs of these states pertinent to the present experiment under the assumption that perturbations of one component of the doublet by the other, configuration interaction, core excitations and other possible perturbing effects are negligible. In the present work we shall evaluate from experimental data the interaction constants under the stated assumptions and then apply corrections to the constants which arise from the perturbations.

Even in the absence of a detailed theory of the nucleus and of the atomic wave function, it is possible to express the hyperfine contribution to the atomic Hamiltonian as a sum, each term of which is the product of an interaction constant and a function of the angular momentum quantum numbers. The expression for the energy of a hyperfine level referred to its energy in the absence of a hyperfine interaction is[2]

$$\frac{W_F}{h}=\frac{Ka}{2}+\frac{3b}{8IJ(2I-1)(2J-1)}\times[K(K+1)-\tfrac{4}{3}(I)(I+1)J(J+1)]+\frac{(5/4)c}{I(I-1)(2I-1)J(J-1)(2J-1)}\{K^3+4K^2+\tfrac{4}{5}K[-3I(I+1)J(J+1)+I(I+1)+J(J+1)+3]-4I(I+1)J(J+1)\},\quad(1)$$

where $K=F(F+1)-I(I+1)-J(J+1)$. I, J, and F are the nuclear, electronic and total angular momentum quantum numbers respectively and a, b, and c are the magnetic dipole, electric quadrupole, and magnetic octupole interaction constants in sec^{-1}. Higher order interactions are possible only when I and J are both greater than $\frac{3}{2}$. For I or J equal to $\frac{1}{2}$, both b and c are zero. For both In^{115} and In^{113}, $I=9/2$. We have made a direct measurement in the $P_{\frac{3}{2}}$ state of

$$f_6=(W_6-W_5)/h,$$
$$f_5=(W_5-W_4)/h,$$
$$f_4=(W_4-W_3)/h.$$

These three frequencies are sufficient to determine the constants a, b, and c appropriate to the state in question.

When an external magnetic field, H, is sufficiently weak, its effect occurs as a small perturbation. The expression for the energies of the magnetic levels to

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[1] W. W. Clendenin, Phys. Rev. 94, 1590 (1954).

[2] C. Schwartz, Phys. Rev. 97, 380 (1955).

terms quadratic in H has previously been given.[3] We use the expression to find the zero-field corresponding to a transition between two F levels from observations of line frequencies at very weak fields.

For $J=\frac{1}{2}$, the expression for the energy for arbitrary I and H has a relatively simple form given by the Breit-Rabi[4] formula. The measurement of the $\Delta\nu$ of the indium isotopes at very weak magnetic fields has been described in the preceding paper. In this paper we describe certain measurements of the hfs of the $P_{\frac{1}{2}}$ state of In^{115} at intermediate magnetic fields. The point of interest is that under the assumption of the applicability of the Breit-Rabi formula, a value of $\Delta\nu$ and of g_I/g_J is obtained in disagreement with the zero field measurements of $\Delta\nu$ and with the value of g_I obtained from nuclear resonance data. To indicate that these quantities are subject to correction through the mutual perturbation of the two components of the fine structure doublet, we shall designate them with double primes in the following discussion of certain properties of the hfs of the $P_{\frac{1}{2}}$ state.

All the lines $F, m \leftrightarrow F, m-1$, for $I+\frac{1}{2}>m>-(I-\frac{1}{2})$, constitute doublets for the two values of F of frequency separation $2g_I''\mu_0 H/h$. If $m\leqslant 0$, the mean frequency of the doublet has a maximum at a characteristic value of H. The value of $\Delta\nu''$ can readily be determined from the maximum frequency. The details of the analysis have previously been discussed.[5] In the present work the doublet $F, -3\leftrightarrow F, -4$ was chosen for observation because its maximum mean frequency occurs at the readily attainable field of 17 000 gauss. All other doublets with the specified property attain a maximum mean frequency at much higher fields. Since, in the present case, both components have separately a maximum frequency, the observation of both frequencies permits a determination of both $\Delta\nu''$ and of g_I''/g_J. If f_m is the mean frequency of the doublet and if

$$f_\pm = f_m \pm g_I''\mu_0 H/h,$$

then

$$f_{-\,\max} - f_{+\,\max} = -\tfrac{1}{5}\Delta\nu''[14k+1344k^3]$$

and

$$f_{-\,\max} + f_{+\,\max} = \tfrac{1}{5}\sqrt{2}\Delta\nu''[1+48k^2],$$

where $k=[(g_J/g_I'')-1]^{-1}$ is a small negative quantity for In^{115} since g_I'' is negative. Terms containing higher powers of k are negligible.

PROCEDURE AND RESULTS

$P_{\frac{1}{2}}$ State

The apparatus used for the determination of $\Delta\nu''$ and g_I''/g_J of In^{115} has previously been described.[6] It is characterized by long deflecting fields with a large ratio of gradient to field, so that a large deflection of atoms may be obtained at fields of the order of one or two hundred gauss. The atoms are thus in a low field and the transitions $\Delta F=0$, $\Delta m=\pm 1$ may be detected since these lead to a significant moment change at weak fields. The transitions are, however, observed in a field at which their frequency is a maximum, about 17 000 gauss. For H in the vicinity of this value, the frequencies of both components of the doublet are almost entirely field-independent. Small inhomogeneities in the magnetic field have a negligible effect on the widths and shapes of the lines and their frequencies can be determined to a very high precision.

A measurement of the line frequency was made at 20 to 30 settings of the current, I_C, in the coils of the C magnet in the neighborhood of the current at which the line frequency was a maximum. The field is a linear function of the current in the magnet coils over a small range and the frequency varies with the square of the deviation of the field from its value at the maximum frequency. By least-squares fitting of an appropriate curve to the data, a value of the maximum frequency may be determined. It was found that

$$f_{-\,\max} = [1629.5099 \pm (10)] \times 10^6 \text{ sec}^{-1},$$
$$f_{+\,\max} = [1597.7575 \pm (5)] \times 10^6 \text{ sec}^{-1},$$

where the uncertainty in the last-stated figure is here, as elsewhere in the paper, given in parentheses. The uncertainties are approximately three times the probable errors obtained from the analysis of the data and are assigned on the basis of estimates of the departure from linearity of H *vs* I_C and of the limits of applicability of the assumed relationship between f and H. From the values of $f_{\pm\,\max}$, we obtain

$$[g_J(^2P_{\frac{1}{2}})/g_I'']_{In^{115}} = -1005.218 \pm (35),$$
$$\Delta\nu''(In^{115}) = [11\,409.5721 \pm (39)] \times 10^6 \text{ sec}^{-1}.$$

These results are in agreement with those of Taub and Kusch.[7] The value of $\Delta\nu''$ differs from the value of the hfs splitting, $\Delta\nu$, measured at weak magnetic fields by about thirty times the sum of the uncertainties in the two determinations.

$P_{\frac{3}{2}}$ State

Only about 0.02% of the beam consists of atoms of In^{113} in a given magnetic level of the hyperfine structure in the $P_{\frac{3}{2}}$ state. For the observation of the effect of a transition between a pair of magnetic levels of In^{113}, it was thus necessary to use a procedure in which only those atoms that had undergone transition between the two deflecting fields were refocused at the detector. For the $P_{\frac{3}{2}}$ state of indium, high-field conditions set in, very approximately, at 3000 gauss. At higher fields the field-dependent term in the energies of the levels is principally $g_J m_J \mu_0 H$, and the corresponding magnetic moment is $-g_J m_J \mu_0$. The observable transitions have

[3] A. K. Mann and P. Kusch, Phys. Rev. **77**, 427 (1950).
[4] S. Millman *et al.*, Phys. Rev. **53**, 384 (1938).
[5] P. Kusch and H. Taub, Phys. Rev. **75**, 1477 (1949).
[6] Logan, Coté, and Kusch, Phys. Rev. **86**, 280 (1952).
[7] H. Taub and P. Kusch, Phys. Rev. **75**, 1481 (1949).

the high field equivalent $\Delta m_J=\pm 1$ and are of two distinct types; the transitions $m_J=\frac{1}{2}\leftrightarrow-\frac{1}{2}$, observed to determine f_5, involve a change in the sign of the effective moment without a change in its magnitude, while the transitions $m_J=\frac{3}{2}\leftrightarrow\frac{1}{2}$ and $m_J=-\frac{1}{2}\leftrightarrow-\frac{3}{2}$, observed to determine f_6 and f_4, respectively, involve a change in the magnitude of the moment by a factor of 3 without a change in sign.

The field in the A, B, and C magnets was always in the same direction. For observation of lines leading to a determination of f_5 the gradients in the A and B fields were in the same direction and of approximately the same magnitude. For the observation of lines leading to a determination of f_6 and f_4 the gradients in the A and B fields were in opposite directions and that in the A magnet was approximately three times that in the B magnet. With the first of these arrangements the observed intensity of transition was that arising from atoms in either of the two initial states, $m_J=\pm\frac{1}{2}$. With the second arrangement only those transitions were observable for which $|m_J|$ of the initial state was $\frac{1}{2}$. The intensities of the lines observed to determine f_6 and f_4 were thus about $\frac{1}{2}$ of those of the lines entering into the determination of f_5.

The apparatus used in the study of the $P_{\frac{3}{2}}$ state was the same as that used for the low-field measurements in the $P_{\frac{1}{2}}$ state. Two different hairpins were used. Most of the measurements were made using a hairpin similar in design to the one shown in Fig. 2(b) of reference 7. It was possible with this hairpin to observe the π transitions $\Delta F=\pm 1$, $\Delta m=\pm 1$, but not the σ transitions $\Delta F=\pm 1$, $\Delta m=0$. Since this circuit was unshielded there existed the possibility of a shift in the frequency of a resonance due to a phase shift in the rf along the length of the transition region. To test for the presence of such an effect, measurements were also made with the hairpin constructed[8] for the determination of $\Delta\nu$ of gallium. Both π and σ transitions were observed using this circuit. No significant discrepancy was found between the results obtained with the two different hairpins.

All observations of the hyperfine transitions within the $P_{\frac{3}{2}}$ state were taken in fields of from 2 to 10 gauss. The π lines chosen for measurement were pairs of lines with the following properties: the linear dependence of frequency on H for one member of the pair was equal in magnitude and opposite in sign to that of the other member; the quadratic dependence on H was the same for both members of the pair; the change in m_J of the levels at high field allowed the observation of both members of the pair for a single arrangement of the apparatus as previously described. From observation of a pair of lines it was then possible to find the zero field interval between the two F levels. The σ lines $6,0\leftrightarrow 5,0$ and $5,0\leftrightarrow 4,0$ were also observed. Their frequencies do not involve terms linearly dependent on the field. To determine the value of a small term, quadratically dependent on the field, a pair of π lines was observed at the same field as that in which a σ line was observed.

Of all possible pairs of π lines, those were chosen for measurement whose frequencies are least sensitive to the strength of the magnetic field for weak fields. Thus broadening and distortion of the resonance peak due to small inhomogeneities in the field are minimal and symmetrical for members of the pair. The width at half maximum of the observed resonance peaks was approximately 20×10^3 sec^{-1} in good agreement with the value predicted in the absence of magnetic field inhomogeneities for the 2.54-cm-long transition region.

A single determination of a hyperfine interval consisted of about four determinations of the frequency corresponding to the resonance maximum for each of a pair of π lines or four determinations of the resonance maximum of a σ line. The results of the measurements are tabulated in Table I. The uncertainties given in Table I are considerably larger than the statistical probable error obtained from the scatter of the individual measurements. They were obtained from estimates of the uncertainties arising from a number of sources, such as the transmission error in WWV, the comparison of our local standard with WWV, the comparison of the oscillator frequency with that of the local standard, and the possible bias of the observer in making observations. We feel that the stated uncertainties are large enough to encompass the effects of all sources of error.

A previous measurement,[3]

$$f_5(\mathrm{In}^{113})=[1115.807\pm(22)]\times 10^6\ \mathrm{sec}^{-1},$$

agrees within the experimental uncertainties with the value we obtain. The values of $f_6(\mathrm{In}^{115})$ and $f_5(\mathrm{In}^{115})$ differ by appreciably more than the stated uncertainties from the previously published[9] preliminary results. These were obtained on the basis of limited data taken under experimental conditions that gave a comparatively large noise background. A review of the data indicates that the estimates of error placed on the preliminary results were overly optimistic.

From the data of Table I and Eq. (1), we find, in

TABLE I. Results of measurement of zero-field hfs intervals in the $^2P_{\frac{3}{2}}$ state in In.

Hfs Interval	Isotope	Number of determinations	Frequency in 10^6 sec^{-1}
f_6	115	5	$1752.6865\pm(2)$
f_6	113	3	$1745.4575\pm(5)$
f_5	115	10	$1117.1676\pm(2)$
f_5	113	6	$1115.8253\pm(5)$
f_4	115	8	$668.9631\pm(2)$
f_4	113	4	$670.9552\pm(5)$

[8] A. Lurio and A. G. Prodell, Phys. Rev. **101**, 79 (1956).

[9] P. Kusch and T. G. Eck, Phys. Rev. **94**, 1799 (1954).

units of 10^6 sec^{-1},

$$a_{\frac{3}{2}}(\mathrm{In}^{115}) = 242.164807 \pm (23),$$
$$a_{\frac{3}{2}}(\mathrm{In}^{113}) = 241.641040 \pm (58),$$
$$b(\mathrm{In}^{115}) = 449.54568 \pm (21),$$
$$b(\mathrm{In}^{113}) = 443.41568 \pm (52),$$
$$c(\mathrm{In}^{115}) = 0.000100 \pm (13),$$
$$c(\mathrm{In}^{113}) = 0.000151 \pm (32).$$

It is to be noted that these are not the true interaction constants but only those calculated under the assumptions implicit in Eq. (1).

DETAILED THEORY

We have given values of the interaction constants in the $P_{\frac{3}{2}}$ state derived from experimental data on the assumption that the state is pure. Further, we have found a value of $\Delta\nu''$ for the $P_{\frac{3}{2}}$ state from intermediate field measurements under the assumption, again, that the state is pure. This value is in conspicuous disagreement with a value of $\Delta\nu$ obtained from low-field measurements under the same assumption. We now examine the theory in more detail to obtain better values of the interaction constants, particularly c, in the interest of obtaining a good value of the octupole moments of the indium nuclei. Further, we reconcile the discrepancy in the two values of $\Delta\nu$ for the $P_{\frac{1}{2}}$ state of In^{115}.

$H=0$

The theory of atomic hfs in the absence of external fields has been systematically treated by Schwartz.[2] As concerns the derivation of the interaction constants from the measured hfs intervals, he finds that the elementary theory previously outlined is significantly deficient in that it does not take into account the mutual perturbations of the $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ state and the perturbations of these states by other electronic states. These perturbations give rise to second-order corrections to the interaction constants. In calculating these second-order terms only the matrix elements of the dipole and quadrupole interactions need be considered, since the magnitude of the corrections arising from the octupole interaction are negligible compared to the uncertainties of our results. The corrections to $a_{\frac{1}{2}}$, $a_{\frac{3}{2}}$, and b are small compared to the values of these quantities, though many times the experimental uncertainties. However, in the expression for the corrected octupole constant, $c'=c-c^{(2)}$, the absolute value of $c^{(2)}$ is much larger for indium than that of c.

Schwartz[2] has derived an expression for $c^{(2)}$, considering only the matrix elements connecting the two states of the doublet. The contribution from all other electronic levels is difficult to calculate. Estimates of its magnitude, however, indicate that it is quite small,[2,10] and we shall neglect it. The radial integrals of the off-diagonal matrix elements can be related to those of the diagonal matrix elements of the $P_{\frac{3}{2}}$ state and therefore to the interaction constants in this state. Equation (61) of reference 2 gives

$$c^{(2)} = -\left(\frac{3}{10}\right)\left(\frac{I-1}{I\delta}\right)\left[\frac{5}{16}I\eta\xi\zeta a_{\frac{3}{2}}b - \frac{\eta^2 b^2}{2(2I-1)}\right], \quad (2)$$

where ξ and η are dimensionless constants of the order of one derived from normalization constants and relativistic correction factors and δ is the fine structure splitting. ζ is a correction factor which takes account of configuration mixing corresponding to the possibility of one of the $5s$ electrons of indium being raised to a higher s state, s' ($s \rightarrow s'$). While this type of configuration mixing has virtually no effect on the octupole and quadrupole matrix elements, it does change the value of the dipole matrix elements by a sizeable amount. Schwartz has derived an expression for ζ from the experimental value of the ratio $a_{\frac{3}{2}}/a_{\frac{1}{2}}$. However, the form of the expression he obtains is correct only if the value of $a_{\frac{3}{2}}$ used in Eq. (2) is the theoretical value, i.e., the value that would be obtained experimentally if there were no configuration interaction. Since the theoretical value of $a_{\frac{3}{2}}$ is difficult to obtain, we have derived an expression for ζ that can properly be used in conjunction with the experimental value. The details of this derivation are given in Appendix I. We obtain for a $P_{\frac{1}{2},\frac{3}{2}}$ doublet

$$\zeta = 1 + \left(\frac{(a_{\frac{3}{2}}/a_{\frac{1}{2}}) - 5\theta}{1+5\theta}\right)\left(1+\frac{16}{5\xi}\right),$$

where θ is a normalization and relativistic correction factor. For indium,[11]

$$\xi = 1.0499, \quad \delta = 6.633\times 10^{13}\ \mathrm{sec}^{-1}, \quad \eta = 1.1085,$$
$$a_{\frac{3}{2}}/a_{\frac{1}{2}} = 9.423, \quad \theta = 1.3026, \quad \zeta = 2.568.$$

Equation (2) yields

$$c^{(2)}(\mathrm{In}^{115}) = -0.001555\times 10^6\ \mathrm{sec}^{-1},$$
$$c^{(2)}(\mathrm{In}^{113}) = -0.001531\times 10^6\ \mathrm{sec}^{-1}.$$

There remains the question of the importance of configuration interactions other than the one we have just considered. Calculations by Schwartz[10] give a small amount of configuration mixing due to excitation of a p-electron from a closed shell into the valence shell ($p \rightarrow p'$, P_0). Because of this mixing, the magnitude of $c^{(2)}$ must be increased in the case of indium by about 3%[12] over the value obtained from Eq. (2). The estimated uncertainty in $c^{(2)}$ is then about 2%.

The values of the second-order corrections to the dipole and quadrupole constants were obtained from

[10] C. Schwartz, Phys. Rev. **105**, 173 (1957).

[11] The values of ξ, η, and θ were kindly made available to us by C. Schwartz.

[12] C. Schwartz (private communication).

expressions similar to Eq. (2). For these constants we have not calculated the effects of the $(p\rightarrow p', P_0)$ mode of excitation. The corrected values of the $P_{\frac{3}{2}}$ state interaction constants, which we have denoted by primes, are, in units of 10^6 sec^{-1},

$$a_{\frac{3}{2}}'(\mathrm{In}^{115})=242.165057\pm(23),$$
$$b'(\mathrm{In}^{115})=449.59656\ \pm(21),$$
$$c'(\mathrm{In}^{115})=\ \ 0.001702\pm(35),$$
$$a_{\frac{3}{2}}'(\mathrm{In}^{113})=241.641293\pm(58),$$
$$b'(\mathrm{In}^{113})=443.46626\ \pm(52),$$
$$c'(\mathrm{In}^{113})=\ \ 0.001728\pm(45).$$

The uncertainties given for $a_{\frac{3}{2}}'$ and b' are experimental uncertainties only, while for c' the 2% uncertainty in $c^{(2)}$ is included.

$H\neq 0$

An expression for the energy levels of the $P_{\frac{1}{2}}$ state of a $P_{\frac{1}{2},\frac{3}{2}}$ doublet in the presence of an external magnetic field has been derived by Clendenin[1] who used single-particle, relativistic electronic wave functions. The matrix elements of the Hamiltonian were computed in the F, m representation and the resultant secular determinant solved to order $1/\delta$, where δ is the fine structure splitting. The only matrix elements off-diagonal in J that were considered were those of the dipole hyperfine interaction and of the magnetic field. In the case of indium, where the magnitude of the quadrupole interaction is approximately $\frac{1}{5}$ that of the dipole interaction, the effect of off-diagonal quadrupole matrix elements on the energy levels of the $P_{\frac{1}{2}}$ state is by no means negligible. Clendenin's calculation has been extended[13] to include the off-diagonal quadrupole matrix elements and the $(s\rightarrow s')$ configuration interaction. In the notation of the preceding section, the expression for the energy levels of the $P_{\frac{1}{2}}$ state is

$$\frac{W_{I\pm\frac{1}{2},m}}{h}=\frac{-\Delta\nu}{2(2I+1)}+\frac{g_I'm\mu_0H}{h}-\frac{[\frac{5}{16}\xi\zeta a_{\frac{3}{2}}]^2}{\delta}\frac{2I(I+1)}{}$$
$$-\frac{\eta^2b^2(I+1)(2I+3)}{8\delta I(2I-1)}-\left(\frac{2}{9}\right)\frac{\mu_0^2N^2H^2}{\delta h^2}$$
$$\pm\frac{\Delta\nu}{2}\Bigg[1+\frac{4mx}{(2I+1)}+x^2$$
$$+\frac{\eta bN}{3I(2I-1)\delta(g_J-g_I')}\Bigg\{\frac{16I(I+1)mx}{(2I+1)}$$
$$+12m^2x^2-(4I^2+4I+3)x^2\Bigg\}\Bigg]^{\frac{1}{2}},\quad(3)$$

[13] A. Lurio, Ph.D. thesis, Columbia University (unpublished).

where

$$\Delta\nu=a_{\frac{1}{2}}'(I+\tfrac{1}{2})-\frac{(2I+1)}{\delta}\Bigg[\{\tfrac{5}{16}\xi\zeta a_{\frac{3}{2}}\}^2$$
$$+\tfrac{5}{16}\xi\zeta a_{\frac{3}{2}}\eta b\frac{(2I+3)}{2I}-\frac{3}{16}\frac{\eta^2b^2(2I+3)}{I^2(2I-1)}\Bigg]$$
$$x=(g_J-g_I')\mu_0H/(h\Delta\nu),$$
$$g_I'=g_I\Bigg[1+\left(\frac{5}{12}\right)\frac{\xi\zeta Na_{\frac{3}{2}}}{\delta g_I}-\frac{\eta bN}{I(2I-1)\delta g_I}\Bigg].$$

The quantity N is a normalization constant defined by Clendenin[1] and is very nearly equal to one. For $b=0$ and $\zeta=1$, these expressions are in agreement (except for normalization and relativistic factors and the sign convention for g_I) with those of reference 1. $a_{\frac{1}{2}}'$ is the corrected dipole interaction constant for the $P_{\frac{1}{2}}$ state. If the equation for $\Delta\nu$ is solved for $a_{\frac{1}{2}}'$, the terms proportional to $1/\delta$ are the same as those that would be obtained by calculating the second order correction to $a_{\frac{1}{2}}$ by the techniques of the preceding section. The correction terms proportional to b/δ within the square root of the expression for $W_{I\pm\frac{1}{2},m}$ arise from cross-terms involving the off-diagonal matrix elements of the quadrupole interaction and the magnetic field.

It is now possible, by use of Eq. (3), to correct the values of $\Delta\nu''(\mathrm{In}^{115})$ and $[g_J(^2P_{\frac{1}{2}})/g_I'']_{\mathrm{In}^{115}}$. The only correction terms we need consider are the terms proportional to $1/\delta$ within the square root of Eq. (3), since all other correction terms in the expression for $W_{I\pm\frac{1}{2},m}$ merely shift the center of gravity of the hfs multiplet. The simplest procedure is to apply to our measured frequencies a correction for the terms in (3) which do not occur in the Breit-Rabi formula. The corrected frequencies may then be used in the Breit-Rabi formula to obtain $\Delta\nu$ and g_J/g_I' exactly as $\Delta\nu''$ and g_J/g_I'' were obtained. Neglecting the terms in Eq. (3) that are irrelevant to this discussion and expanding the square root to terms linear in $1/\delta$, we have

$$\frac{W_{I\pm\frac{1}{2},m}}{h}=\frac{g_I'm\mu_0H}{h}\pm\frac{\Delta\nu}{2}\Bigg[\Bigg\{1+\frac{4mx}{(2I+1)}+x^2\Bigg\}^{\frac{1}{2}}$$
$$+\frac{\beta}{2\{1+4mx/(2I+1)+x^2\}^{\frac{1}{2}}}\Bigg],$$

where β is the sum of the correction terms within the square root of Eq. (3). Using this expression and the values of x corresponding to the maximum frequencies, we find the following corrections for $f_{\pm\,\max}$:

$$\Delta_+=+0.02416\times10^6\ \mathrm{sec}^{-1},\quad \Delta_-=+0.02545\times10^6\ \mathrm{sec}^{-1}.$$

The shifts, introduced by the correction terms, in the values of x corresponding to the maximum frequencies have a negligible effect on the above calculations. The

corrected frequencies yield

$$\Delta\nu(\mathrm{In}^{115})=[11\,409.7474\pm(39)]\times10^6\ \mathrm{sec}^{-1},$$
$$[g_J(^2P_{\frac{3}{2}})/g_I']_{\mathrm{In}^{115}}=-1005.192\pm(35).$$

The discrepancy between this value of $\Delta\nu(\mathrm{In}^{115})$ and the value determined from measurements taken at weak magnetic fields is well within the experimental uncertainties.

DISCUSSION OF RESULTS

Now that we have obtained corrected values of the interaction constants it is possible to discuss the phenomena which motivated this investigation. The differential hfs anomaly between the $P_{\frac{1}{2}}$ and $P_{\frac{3}{2}}$ states of the stable indium isotopes has been considered in the preceding paper. Here we shall examine the validity of the present theory of atomic hfs for the $P_{\frac{1}{2},\frac{3}{2}}$ states of indium and obtain a rough estimate of the magnitudes of the quadrupole and octupole moments of In^{115} and In^{113}.

g_I Anomaly

We have already seen that the extended theory of the $P_{\frac{1}{2}}$ state of indium gives agreement between the low- and intermediate-field determinations of $\Delta\nu(\mathrm{In}^{115})$. A more stringent test of the theory is a comparison of the theoretical value of the ratio g_I' (atomic hfs)/g_I (nuclear resonance) and the experimental value of this ratio. From Eq. (3) and the values of the constants in the equation, we find:

$$R^{-1}(\mathrm{In}^{115})=g_I'(\mathrm{In}^{115})/g_I(\mathrm{In}^{115})=1-0.00581,$$

where we have set $N=1$ and used Ting and Williams' value[14] of $g_I(=-1.22976/1836.6)$. The discrepancy between this value of R^{-1} and that given by Schwartz[2] arises from our new treatment of configuration interaction (i.e., ζ calculation). From the value we have obtained for $g_J(\mathrm{In}^{115},{}^2P_{\frac{1}{2}})/g_I'(\mathrm{In}^{115})$ and the ratios[5,15,16]

$$g_J(\mathrm{Na},{}^2S_{\frac{1}{2}})/g_J(\mathrm{In}^{115},{}^2P_{\frac{1}{2}})=3.00729\pm0.01\%,$$
$$g_J(K,{}^2S_{\frac{1}{2}})/g_J(\mathrm{Na},{}^2S_{\frac{1}{2}})=1\pm0.0025\%,$$
$$g_J(K,{}^2S_{\frac{1}{2}})/g_I(\mathrm{proton})=-658.2274\pm0.0004\%,$$

we find

$$g_I'(\mathrm{In}^{115})/g_I(\mathrm{proton})=0.217747\pm0.011\%.$$

Ting and Williams[14] give

$$g_I(\mathrm{In}^{115})/g_I(\mathrm{proton})=0.219101\pm0.006\%.$$

Thus the experimental value of $R^{-1}(\mathrm{In}^{115})$ is $1-[0.00618\pm(13)]$. The diamagnetic correction has been ignored in the ratios, since it cancels out in all comparisons. It is difficult to estimate the uncertainty in the theoretical value of R^{-1} without knowledge of the magnitudes of effects due to perturbations not considered, such as further configuration interaction and the perturbations related to the finite size of the nucleus. In view of this we consider the agreement between the theoretical and experimental values of $R^{-1}(\mathrm{In}^{115})$ to be quite good.

Quadrupole and Octupole Moments

A straightforward calculation of the nuclear moments from the interaction constants requires a knowledge of the electronic wave function. Since this knowledge is available only for the simplest of atomic systems, the usual procedure is to estimate the magnitude of the nuclear moment by comparing the interaction constant and a quantity, such as the fine structure splitting, that is related in a similar way to the electronic wave function. Koster,[17] using such a procedure, has obtained values of the quadrupole moment, Q, in In^{115} and In^{113} that have uncertainties of the order of a few percent. We have calculated Q for these isotopes of indium using values of $a_{\frac{3}{2}}$ corrected for the $(s\rightarrow s')$ mode of configuration interaction. The results agree within 2% with those of Koster. The value of the quadrupole moment ratio can of course be obtained to a much higher precision than can the values of the quadrupole moments. Except for small differential second-order effects, such as those related to the difference in the size of the nuclei,

$$Q(\mathrm{In}^{115})/Q(\mathrm{In}^{113})=b'(\mathrm{In}^{115})/b'(\mathrm{In}^{113})=1.0138236\pm(13).$$

To obtain estimates of the octupole moment, Ω, of the stable isotopes of indium, we shall use Eq. (72) of reference 10,

$$\Omega=\frac{c'\mu F_{\frac{3}{2}}1.96\times10^8}{a_{\frac{3}{2}}'IT(1-\beta_{\frac{3}{2}})Z^2}. \tag{4}$$

In this expression Ω is in units of nuclear magneton barns (nmb) and μ in nuclear magnetons. Z is the atomic number and $T/F_{\frac{3}{2}}$ is a correction factor that is shown graphically in Fig. 3 of reference 10. The expression for $\beta_{\frac{3}{2}}$ is given in the preceding paper. For indium $Z=49$, $T/F_{\frac{3}{2}}=0.90$, $\beta_{\frac{3}{2}}=-0.3873$, $\mu/I(\mathrm{In}^{115})=1.22976$, and $\mu/I(\mathrm{In}^{113})=1.2272$.

$$\Omega(\mathrm{In}^{115})=(0.565\pm0.012)\ \mathrm{nmb},$$
$$\Omega(\mathrm{In}^{113})=(0.574\pm0.015)\ \mathrm{nmb},$$

where the stated uncertainty is that arising from the uncertainty in c'. These values of the octupole moments should probably not be trusted for more than a qualitative comparison with nuclear models because of the uncertainty in the calculation of polarization effects.[10]

The values of Ω predicted by the single-particle model of the nucleus may be obtained from Eq. (56) of reference 2. For indium the predicted value of $\Omega/\mu_N\langle r^2\rangle$ is 4.68. For the strong-coupling case of A. Bohr's asymmetric core model of the nucleus, the single-particle

[14] Y. Ting and D. Williams, Phys. Rev. **89**, 595 (1953).
[15] P. Kusch and H. M. Foley, Phys. Rev. **74**, 250 (1948).
[16] P. Franken and S. Koenig, Phys. Rev. **88**, 199 (1952).
[17] G. F. Koster, Phys. Rev. **86**, 148 (1952).

value of Ω must be reduced by the projection factor given by Eq. (59) of reference 2, yielding for indium a value of $\Omega/\mu_N\langle r^2\rangle$ equal to 1.37. If we use the approximation

$$\langle r^2\rangle=\tfrac{3}{5}(1.2A^{\frac{1}{3}})^2\times10^{-26}\text{ cm}^2,$$

we have for both In^{115} and In^{113} an experimental value of $\Omega/\mu_N\langle r^2\rangle\cong2.81$, which lies approximately halfway between the values predicted by the single-particle and strong-coupling asymmetric core models of the nucleus. However, no real conclusions can be drawn concerning the strength of the coupling between the odd nucleon and the core, because of the large uncertainty in the value of Ω and in the applicability of the approximation for $\langle r^2\rangle$.

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APPENDIX I

Here we shall consider the effect of $s\rightarrow s'$ configuration mixing on the dipole hyperfine interaction for an s^2l doublet. The formulation of the problem and the notation used are those of reference 2, which should be consulted for the definitions of the symbols.

The electronic reduced dipole matrix elements can be written as

$$(J\|T_e^{(1)}\|J')=(J\|T_l^{(1)}\|J')+\Delta_{JJ'},$$

where the first and second terms on the right arise from the interaction of the nuclear magnetic dipole moment with the valence l electron and the s electrons, respectively. From Eqs. (45a), (45b), (46), and (48) of reference 2, the following expressions are obtained:

$$(J\|T_e^{(1)}\|J)=(J\|T_l^{(1)}\|J)+\Delta_{JJ},\tag{A1a}$$

$$(J-1\|T_e^{(1)}\|J-1)=\left[\frac{(J+1)(2J-1)}{(J-1)(2J+1)}\right]^{\frac{1}{2}}\theta(J\|T_l^{(1)}\|J)-\left[\frac{(J-1)(2J-1)}{(J+1)(2J+1)}\right]^{\frac{1}{2}}\Delta_{JJ},\tag{A1b}$$

$$(J\|T_e^{(1)}\|J-1)=-\frac{[(J+1)(2J-1)]^{\frac{1}{2}}}{(2J+1)(2J-1)}\xi(J\|T_l^{(1)}\|J)+\left[\frac{2J-1}{J+1}\right]^{\frac{1}{2}}\Delta_{JJ}.\tag{A1c}$$

The dipole interaction constants are related to the diagonal reduced matrix elements by

$$a_J=\frac{M_1(J\|T_e^{(1)}\|J)}{I[J(J+1)(2J+1)]^{\frac{1}{2}}}.\tag{A2}$$

Equations (A1a), (A1b), and (A2) give

$$a_J=a_J(l)+a_J(s),\tag{A3a}$$

$$a_{J-1}=\left(\frac{J+1}{J-1}\right)\theta a_J(l)-a_J(s),\tag{A3b}$$

where

$$a_J(l)=\frac{M_1(J\|T_l^{(1)}\|J)}{I[J(J+1)(2J+1)]^{\frac{1}{2}}},\tag{A3c}$$

and

$$a_J(s)=\frac{M_1\Delta_{JJ}}{I[J(J+1)(2J+1)]^{\frac{1}{2}}}.\tag{A3d}$$

The off-diagonal dipole matrix elements are given by

$$\begin{aligned}&\langle IJF|T_e^{(1)}\cdot T_n^{(1)}|IJ-1F\rangle\\&=(-1)^{I+J-F-1}W(IJIJ-1;F1)\\&\quad\times\left[\frac{(I+1)(2I+1)}{I}\right]^{\frac{1}{2}}M_1(J\|T_e^{(1)}\|J-1)\\&=-[(I+J-F)(J-I+F)(I-J+F+1)\\&\quad\times(I+J+F+1)]^{\frac{1}{2}}\frac{(J+1)\xi a_J}{2(2J+1)(2J-1)}\\&\qquad\times\left\{1-\frac{a_J(s)}{a_J}\left[1+\frac{(2J+1)(2J-1)}{(J+1)\xi}\right]\right\},\end{aligned}$$

where the second equality is established by using Eqs. (A1c), (A3c), and (A3d), and the relation

$$\begin{aligned}&(-1)^{I+J-F-1}W(IJIJ-1;F1)\\&=\left[\frac{(I+J-F)(J-I+F)(I-J+F+1)(I+J+F+1)}{(I+1)(2I+1)(2I)(2J+1)(2J)(2J-1)}\right]^{\frac{1}{2}}.\end{aligned}$$

For $a_J(s)=0$, the off-diagonal matrix elements reduce as they should to the form for no configuration mixing. Thus the effect of $s\rightarrow s'$ configuration interaction on $\langle IJF|T_e^{(1)}\cdot T_n^{(1)}|IJ-1F\rangle$ is to replace the factor a_J by $a_J\zeta$, where

$$\zeta=1-\frac{a_J(s)}{a_J}\left[1+\frac{(2J+1)(2J-1)}{(J+1)\xi}\right].$$

By use of Eqs. (A3a) and (A3b), ζ can be written in terms of the measured dipole interaction constants:

$$\zeta=1+\left\{\frac{\dfrac{a_{J-1}}{a_J}-\left(\dfrac{J+1}{J-1}\right)\theta}{1+\left(\dfrac{J+1}{J-1}\right)\theta}\right\}\left[1+\frac{(2J+1)(2J-1)}{(J+1)\xi}\right].$$

For a $P_{\frac{1}{2},\frac{3}{2}}$ doublet ($J=\frac{3}{2}$), we obtain for ζ the expression given in the text.