addition, Fig. 6 exhibits values of ν_i/p corrected for the nonuniformity of the electric field. The correction is obtained as follows.

In Eq. (7), the coefficient ν_i is interpreted as an effective ionization frequency ν which takes into account the variation of the field with the radius of the cavity and the loss of electrons by radial diffusion. The value of ν lies between the value of ν_i at the axis of the cavity where E_e/p is determined, and the value of ν associated only with the lowest radial diffusion mode. The value of ν associated with the lowest diffusion mode is obtained by a variational calculation.[11] The maximum correction is the difference between ν_i at the axis of the cavity and ν for the lowest diffusion mode. The corrected values are obtained by applying the maximum correction.

The experimental values of ν_i/p do not agree with dc measurements of the Townsend ionization coefficient α that was determined by Rose[12] when the two coefficients are compared by the use of the relation[1] $\nu_i/p=\alpha\mu E/p$. The values of mobility were obtained from reference 8. The discrepancy is greatest at small values of E/p. The values of ν_i/p that are plotted in Fig. 6 are not affected by taking into account the fluctuations of the average electron energy with the time variation of the microwave field.

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[11] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Chap. 6; S. J. Buchsbaum, Quarterly Progress Report, Research Laboratory of Electronics, Massachusetts Institute of Technology, January 15, 1957 (unpublished), p. 10.

[12] D. J. Rose, Phys. Rev. **104**, 273 (1956).

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Statistical Mechanical Theory of Transport Phenomena in a Fully Ionized Gas

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Nonequilibrium statistical mechanics, as developed by Kirkwood, Irving, and Zwanzig, is applied to a system of charged particles interacting via the electromagnetic field. Particles and field are treated statistically both from the classical and quantal points of view. It is shown that Maxwell's equations are valid for the quantum statistical ensemble-averaged fields. An exact form for the hydromagnetic equations is derived, and it is shown that these equations differ from those which are customarily considered to be exact.

I. CLASSICAL THEORY

We consider a system composed of N charged particles subjected to external forces and interacting via the electromagnetic field. Let r: $\mathbf{r}_1, \cdots \mathbf{r}_N$ be the position vectors of the particles and $e_1, \cdots e_N$ their electric charges. It is assumed that apart from the masses $m_1, \cdots m_N$ the particles have no further electrical or mechanical structure. The electric field $\mathbf{E}$ is decomposed into a transverse internal part $\mathbf{E}^T$, an instantaneous Coulomb part $\mathbf{E}^C$, and an external part $\mathbf{E}^{\text{ext}}$:

$$\mathbf{E}=\mathbf{E}^T+\mathbf{E}^C+\mathbf{E}^{\text{ext}}, \tag{1.1}$$

where

$$\mathbf{E}^C=\sum_{k=1}^{N} e_k(\mathbf{r}-\mathbf{r}_k)/|\mathbf{r}-\mathbf{r}_k|^3, \tag{1.2}$$

and

$$\nabla\cdot\mathbf{E}^T=0. \tag{1.3}$$

Similarly the magnetic field $\mathbf{B}$ is expressed as the sum of an internal and an external part:

$$\mathbf{B}=\mathbf{B}^{\text{int}}+\mathbf{B}^{\text{ext}}. \tag{1.4}$$

The system is enclosed in a large volume $V=L^3$ so that the electromagnetic field may be described by a denumerable set of coordinates. Following Heitler,[1] a set of real vector functions $\mathbf{A}_\lambda(\mathbf{r})$ is introduced, complete with respect to transverse vector fields and with the following properties:

$$\int_V \mathbf{A}_\lambda\cdot\mathbf{A}_\mu d\mathbf{r}=4\pi c^2\delta_{\mu\nu}, \tag{1.5}$$

$$\nabla^2\mathbf{A}_\lambda+k_\lambda^2\mathbf{A}_\lambda=0, \tag{1.6}$$

$$\nabla\cdot\mathbf{A}_\lambda=0, \tag{1.7}$$

where $k_\lambda^2\equiv\omega_\lambda^2/c^2$, $\mathbf{k}_\lambda=2\pi\mathbf{n}/L$, and $\mathbf{n}$ is a vector having non-negative integral components. The fields $\mathbf{E}^T$ and $\mathbf{B}^{\text{int}}$ are expressed in terms of the field coordinates q_λ by the equations

$$\mathbf{E}^T=-(1/c)\textstyle\sum_\lambda \dot{q}_\lambda\mathbf{A}_\lambda, \tag{1.8}$$

$$\mathbf{B}^{\text{int}}=\textstyle\sum_\lambda q_\lambda\nabla\times\mathbf{A}_\lambda. \tag{1.9}$$

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[1] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), p. 39.

The Hamiltonian for the system, particles and field, is

$$H=\sum_k \frac{1}{2m_k}\left(\mathbf{p}_k-\frac{e_k}{c}\mathbf{A}_k\right)^2+U+\tfrac{1}{2}\sum_\lambda(p_\lambda^2+\omega_\lambda^2 q_\lambda^2), \quad (1.10)$$

where

$$\mathbf{A}_k=\mathbf{A}_k^{\text{ext}}+\sum_\lambda q_\lambda \mathbf{A}_\lambda(\mathbf{r}_k) \quad (1.11)$$

is the vector potential evaluated at the point $\mathbf{r}_k$, and

$$U=U^{\text{ext}}+\sum_{i<k} e_i e_k/|\mathbf{r}_i-\mathbf{r}_k| \quad (1.12)$$

is the total potential energy. The Hamiltonian, (1.10), is nonrelativistic for the particles, but completely general for the electromagnetic field. The Hamiltonian equations of motion for the system are

$$\partial\mathbf{r}_k/\partial t\equiv\mathbf{u}_k=[\mathbf{p}_k-(e_k/c)\mathbf{A}_k]/m_k, \quad (1.13)$$

$$\partial\mathbf{p}_k/\partial t=(e_k/c)\mathbf{u}_k\cdot(\mathbf{A}_k\overleftarrow{\nabla}_{rk})-\nabla_{rk}U, \quad (1.14)$$

$$\partial q_\lambda/\partial t=p_\lambda, \quad (1.15)$$

and

$$\partial p_\lambda/\partial t=-\omega_\lambda^2 q_\lambda+\textstyle\sum_k(e_k/c)\mathbf{u}_k\cdot\mathbf{A}_\lambda(\mathbf{r}_k). \quad (1.16)$$

The symbol $\overleftarrow{\nabla}$ signifies the usual gradient operator acting on the quantity to the left.

Let $f=f(r,p,q_\lambda,p_\lambda,t)\equiv f(q,p,t)$ be the distribution function (d.f.) in the phase space of the complete system. Then, the Hamiltonian form of the equations of motion implies that the d.f. satisfies Liouville's equation

$$(\partial f/\partial t)+\Lambda f=0, \quad (1.17)$$

where the operator Λ is defined by

$$\begin{aligned}\Lambda f\equiv&\sum_k \mathbf{u}_k\cdot\nabla_{rk}f+\sum_\lambda p_\lambda(\partial f/\partial q_\lambda)\\&+\sum_k\{(e_k/c)\mathbf{u}_k\cdot\mathbf{A}_k-U\}\overleftarrow{\nabla}_{rk}\cdot\nabla_{pk}f\\&+\sum_\lambda\{\sum_k(e_k/c)\mathbf{u}_k\cdot\mathbf{A}_\lambda(\mathbf{r}_k)-\omega_\lambda^2q_\lambda\}(\partial f/\partial p_\lambda), \quad (1.18)\end{aligned}$$

and where it is understood that $\overleftarrow{\nabla}_{rk}$ does not act upon $\mathbf{u}_k$. A formal difficulty is that the d.f. is a function of an infinite number of variables. However, all results of practical interest will be expressed in terms of reduced d.f.'s; that is, in terms of functions obtained by integrating f over all but a small number of variables. Further, if any genuine progress is to be made along these lines, a method to obtain reasonable approximations for the reduced d.f.'s must be found.

If $g(q,p,t)$ is a dynamical variable of the system, the statistical ensemble average $\langle g\rangle$ of g is expressed (using essentially the notation of Irving and Kirkwood[2]) by:

$$\langle g\rangle=\langle f,g\rangle=\int fgdqdp, \quad (1.19)$$

f being normalized to unity $\langle f,1\rangle=1$. The notation $\langle f,g\rangle$ expresses the fact that $\langle f,g\rangle$ is a symmetric scalar product of f and g defined over the entire phase space of the system. If the functions f and g are sufficiently well-behaved, the operator Λ is skew-symmetric:

$$\langle\Lambda f,g\rangle=-\langle f,\Lambda g\rangle. \quad (1.20)$$

Equation (1.20) may be used to obtain an expression for the rate of change of $\langle g\rangle$. In fact,

$$\begin{aligned}\partial\langle g\rangle/\partial t&=\langle f,\partial g/\partial t\rangle+\langle\partial f/\partial t,g\rangle\\&=\langle f,\partial g/\partial t\rangle-\langle\Lambda f,g\rangle\\&=\langle f,\partial g/\partial t\rangle+\langle f,\Lambda g\rangle.\end{aligned} \quad (1.21)$$

Equation (1.21) expresses the basic statistical mechanical law of transport.

It is convenient to introduce a new d.f. f^* which is a function of r, u, q_λ, p_λ. Since the Jacobian of the transformation, r, p, q_λ, $p_\lambda\rightarrow r$, u, q_λ, p_λ, is a constant,

$$f(r,p,q_\lambda,p_\lambda,t)=\text{const}\, f^*(r,u,q_\lambda,p_\lambda,t). \quad (1.22)$$

The d.f. f^* also satisfies a Liouville equation,

$$\partial f^*/\partial t+\Lambda^* f^*=0, \quad (1.23)$$

with the operator Λ^* defined by the equation

$$\begin{aligned}\Lambda^*f^*\equiv&\sum_k\mathbf{u}_k\cdot\nabla_{rk}f^*+\sum_\lambda p_\lambda(\partial f^*/\partial q_\lambda)\\&+\sum_k(1/m_k)\{e_k\mathbf{E}_k+(e_k/c)\mathbf{u}_k\times\mathbf{B}_k+\mathbf{X}_k\}\cdot\nabla_{uk}f^*\\&+\sum_\lambda\{\sum_k(e_k/c)\mathbf{u}_k\cdot\mathbf{A}_\lambda(\mathbf{r}_k)-\omega_\lambda^2q_\lambda\}(\partial f^*/\partial p_\lambda), \quad (1.24)\end{aligned}$$

$\mathbf{X}_k$ being the nonelectromagnetic part of the external force acting upon the kth particle.

We now define a symmetric scalar product (f,g) by means of the following expression:

$$(f,g)\equiv\int fgdrdu\prod_\lambda dq_\lambda dp_\lambda, \quad (1.25)$$

so that with f^* normalized to unity, $(f^*,1)=1$, the average value $\langle g\rangle$ of any function g of the particle coordinates, particle velocities, field coordinates, and field momenta is given by

$$\langle g\rangle=(f^*,g). \quad (1.26)$$

The operator Λ^* is skew-symmetric with respect to the scalar product (1.25), so $-(\Lambda^*f,g)=(f,\Lambda^*g)$. Therefore, the basic law of change may be expressed by the relation

$$\partial\langle g\rangle/\partial t=(f^*,\partial g/\partial t)+(f^*,\Lambda^*g). \quad (1.27)$$

The derivation of expressions for the law of change for specific dynamical variables may be accomplished by the use of Eq. (1.27). These derivations are postponed until the quantum mechanical theory is developed, since, for the class of dynamical variables of immediate interest, the quantal and classical expressions are

[2] J. H. Irving and J. G. Kirkwood, J. Chem. Phys. 18, 817 (1950).

formally the same. The quantum expressions, however, make use of quantum (Wigner) d.f.'s f_W, f_W^* which replace f, f^* in Eqs. (1.2)–(1.27).

II. QUANTUM THEORY

A pure state of the system is represented by a wave function which satisfies the Schrödinger equation

$$i\hbar\partial\psi(q,t)/\partial t=\mathcal{H}\psi(q,t), \quad (2.1)$$

where $\mathcal{H}$, the Hamiltonian operator, is given by the expression

$$\mathcal{H}=\sum_k \frac{1}{2m_k}\left(\frac{\hbar}{i}\nabla_{r_k}-\frac{e_k}{c}\mathbf{A}_k\right)^2+U +\tfrac{1}{2}\sum_\lambda\left\{\left(\frac{\hbar}{i}\frac{\partial}{\partial q_\lambda}\right)^2+\omega_\lambda^2 q_\lambda^2\right\}. \quad (2.2)$$

A mixed state having probability a_j for the pure state $\psi_j(qt)$ may be represented by the density matrix $\rho(q,q',t)$ defined by

$$\rho(q;q',t)\equiv\textstyle\sum_j a_j\psi_j(qt)\bar{\psi}_j(q't), \quad (2.3)$$

in which $\bar{\psi}$ denotes the complex conjugate of ψ. The equation of motion for the density matrix follows from (2.1) and is expressed by the equation

$$i\hbar\partial\rho(q;q',t)/\partial t=\{\mathcal{H}(q)-\bar{\mathcal{H}}(q')\}\rho(q;q',t), \quad (2.4)$$

in which $\mathcal{H}(q)$ operates only upon the q variables of ρ. The quantum mechanical phase space d.f. which is the closest analog of the classical d.f. is the Wigner d.f. f_W which is defined by[3,4]

$$f_W(q,p,t)\equiv\int e^{2\pi ipy}\rho(q-\pi\hbar y;q+\pi\hbar y)dy, \quad (2.5)$$

where $py=\sum_a p_a y_a$ and $dy=\prod_a dy_a$. The equation of motion for f_W is obtained by the use of Eqs. (2.4), (2.5), and the inverse,

$$\rho(q-\pi\hbar y;q+\pi\hbar y)=\int e^{-2\pi iyp'}f_W(q,p')dp', \quad (2.6)$$

of Eq. (2.5). This equation of motion, which is the quantum analog of the Liouville equation (1.17), is expressed by

$$\partial f_W/\partial t+\Lambda_W f_W=0, \quad (2.7)$$

where Λ_W is an operator which may be represented several ways, one of which is by the differential form,

$$\Lambda_W f_W=\sum_k\frac{1}{m_k}\left\{\mathbf{p}_k-\frac{e_k}{c}\mathbf{A}_k\cos\left(\frac{\hbar}{2}\overleftarrow{\nabla}_q\cdot\nabla_p\right)\right\}\cdot\nabla_{r_k}f_W +\textstyle\sum_\lambda \mathbf{p}_\lambda(\partial f_W/\partial q_\lambda)+\left\{\sum_k\frac{e_k}{m_k c}\mathbf{A}_k\cdot\mathbf{p}_k-\frac{e_k^2}{2m_kc^2}\mathbf{A}_k^2-U -\tfrac{1}{2}\sum_\lambda\omega_\lambda^2q_\lambda^2\right\}\frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_q\cdot\nabla_p\right)f_W. \quad (2.8)$$

The quantum mechanical ensemble average $\langle g\rangle$ of a function g is given by

$$\langle g\rangle=\langle f_W,g\rangle, \quad (2.9)$$

with $\langle f_W,1\rangle=1$, and

$$\langle f_W,g\rangle\equiv\int f_W g\,dq\,dp. \quad (2.10)$$

For suitably well-behaved f, g the operator Λ_W is again skew-symmetric, $\langle\Lambda_W f,g\rangle=-\langle f,\Lambda_W g\rangle$, so the quantum law of change is given by the following equation

$$\partial\langle g\rangle/\partial t=\langle f_W,\partial g/\partial t\rangle+\langle f_W,\Lambda g\rangle. \quad (2.11)$$

Equation (2.11) expresses the general quantum law of change for any function g.

A class of dynamical variables sufficiently general for many applications consists of functions g_c of the form

$$g_c=g_0(q)+\sum_a g_a(q)p_a+\sum_{k,l}\sum a_{kl}\mathbf{u}_k\mathbf{u}_l, \quad (2.12)$$

with a_{kl} independent of q, p. For such functions

$$\Lambda_W g_c=\Lambda g_c, \quad (2.13)$$

and the classical and quantal expressions, (1.21) and (2.11), are formally identical. For functions of the type g_c, it is again convenient to transform from the r,p,q_λ,p_λ variables to the r, u, q_λ, p_λ variables, with the result that

$$\partial\langle g_c\rangle/\partial t=(f_W^*,\partial g_c/\partial t)+(f_W^*,\Lambda^* g_c). \quad (2.14)$$

which is formally identical with Eq. (1.27).

The equation of motion for the various reduced d.f.'s may be derived with the aid of Eq. (2.11). For example, the 1-particle d.f. $f_\nu^{(1)}(\mathbf{r},\mathbf{p},t)$ for particles of type ν is defined by

$$f_\nu^{(1)}\equiv\langle f_W,\delta(\mathbf{r}-\mathbf{r}_\nu)\delta(\mathbf{p}-\mathbf{p}_\nu)\rangle, \quad (2.15)$$

so the equation of motion is

$$\partial f_\nu^{(1)}/\partial t=\langle f_W,\Lambda_W\delta(\mathbf{r}-\mathbf{r}_\nu)\delta(\mathbf{p}-\mathbf{p}_\nu)\rangle. \quad (2.16)$$

Equation (2.16) may be expressed as

$$\frac{\partial f_\nu^{(1)}}{\partial t}+\frac{\mathbf{p}}{m_\nu}\cdot\nabla f_\nu^{(1)}-U^{\text{ext}}\frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_r\cdot\nabla_p\right)f_\nu^{(1)}=\Omega_\nu, \quad (2.17)$$

where

$$\Omega_\nu=\left\langle f_W,\left\{-\frac{e_\nu}{c}\mathbf{A}_\nu\cos\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{r_\nu}\cdot\nabla_{p_\nu}\right)\cdot\nabla_{r_\nu} +\left(\frac{e_\nu}{m_\nu c}\mathbf{A}_\nu\cdot\mathbf{p}_\nu-\frac{e_\nu^2}{2m_\nu c^2}A_\nu^2-\tfrac{1}{2}\sum_{\mu\neq\nu}\frac{e_\nu e_\mu}{|\mathbf{r}_\nu-\mathbf{r}_\mu|}\right)\frac{2}{\hbar} \times\sin\left(\frac{\hbar}{2}\overleftarrow{\nabla}_{r_\nu}\cdot\nabla_{p_\nu}\right)\right\}\delta(\mathbf{r}-\mathbf{r}_\nu)\delta(\mathbf{p}-\mathbf{p}_\nu)\right\rangle.$$

[3] E. Wigner, Phys. Rev. **40**, 749 (1932).
[4] J. H. Irving and R. W. Zwanzig, J. Chem. Phys. **19**, 1173 (1951).

The quantity Ω_ν may be expressed in terms of various particle-particle and particle-field pair d.f.'s.

III. MAXWELL'S EQUATIONS

The charge density at a point $\mathbf{r}$ is defined by

$$g_c=\sum_k e_k\delta(\mathbf{r}-\mathbf{r}_k). \tag{3.1}$$

If the average value of g_c is designated by ρ_e, then according to Eqs. (2.14), (1.24),

$$\begin{aligned}\partial\rho_e/\partial t&=(f_W^*, \Lambda^*\sum_k e_k\delta(\mathbf{r}-\mathbf{r}_k))\\&=-\nabla\cdot(f_W^*, \sum_k e_{\mathbf{u}_k}\delta(\mathbf{r}-\mathbf{r}_k)).\end{aligned} \tag{3.2}$$

However, $g_c=\sum_k e_{\mathbf{u}_k}\delta(\mathbf{r}-\mathbf{r}_k)$ is the electric current density at $\mathbf{r}$, and if $\mathbf{J}_e$ denotes the average value of this electric current density, Eq. (3.2) may be written

$$\partial\rho_e/\partial t+\nabla\cdot\mathbf{J}_e=0, \tag{3.3}$$

which is the law of conservation of charge for the ensemble averaged charge and current densities.

The magnetic field intensity is expressed by

$$g_c=\mathbf{B}(\mathbf{r},t)=\sum_\lambda q_\lambda\nabla\times\mathbf{A}_\lambda(\mathbf{r})+\mathbf{B}^{\text{ext}}, \tag{3.4}$$

and we find that

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}\langle\mathbf{B}\rangle&=\frac{1}{c}(f_W^*, \sum p_\lambda\nabla\times\mathbf{A}_\lambda)+\frac{1}{c}\frac{\partial\mathbf{B}^{\text{ext}}}{\partial t},\\&=-\nabla\times\left(f_W^*, -\frac{1}{c}\sum p_\lambda\mathbf{A}_\lambda\right)-\nabla\times\mathbf{E}^{\text{ext}},\\&=-\nabla\times\langle\mathbf{E}\rangle.\end{aligned} \tag{3.5}$$

The last expression follows because $\nabla\times\langle\mathbf{E}^C\rangle=0$.

Let g_c be the transverse electric field $\mathbf{E}^T$,

$$g_c=-(1/c)\sum_\lambda p_\lambda\mathbf{A}_\lambda. \tag{3.6}$$

Then it follows that

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}\langle\mathbf{E}^T\rangle&=-\frac{1}{c^2}\left(f_W^*, \sum_\lambda\left\{\sum_k\frac{e_k}{c}\mathbf{A}_\lambda\cdot\mathbf{u}_k-\omega_\lambda^2q_\lambda\right\}\mathbf{A}_\lambda(\mathbf{r})\right)\\&=-\frac{4\pi}{c}\left(f_W^*, \frac{1}{4\pi c^2}\sum_\lambda\sum_k e_k\mathbf{u}_k\cdot\mathbf{A}_\lambda(\mathbf{r}_k)\mathbf{A}_\lambda(\mathbf{r})\right)\\&\quad+(f_W^*, \sum_\lambda\nabla\times[\nabla\times\mathbf{A}_\lambda(\mathbf{r})]q_\lambda)\\&=-(4\pi/c)\mathbf{J}_e^T+\nabla\times\langle\mathbf{B}^{\text{int}}\rangle,\end{aligned} \tag{3.7}$$

in which $\mathbf{J}_e^T$ designates the ensemble average of the transverse current density.

If g_c is the instantaneous Coulomb field,

$$g_c=\sum_k e_k(\mathbf{r}-\mathbf{r}_k)/|\mathbf{r}-\mathbf{r}_k|^3,$$

it follows that

$$\begin{aligned}\frac{1}{c}\frac{\partial}{\partial t}\langle\mathbf{E}^c\rangle&=\left(f_W^*, \frac{1}{c}\sum_k e_k\mathbf{u}_k\cdot\nabla\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_k|}\right)\\&=\left(f_W^*, \frac{1}{c}\sum_k e_k\left\{\mathbf{u}_k\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}_k|}+\nabla\times\left(\nabla\times\frac{\mathbf{u}_k}{|\mathbf{r}-\mathbf{r}_k|}\right)\right\}\right)\\&=-\frac{4\pi}{c}\left(f_W^*, \sum_k e_k\left\{\mathbf{u}_k\delta(\mathbf{r}-\mathbf{r}_k)-\frac{1}{4\pi}\nabla\times\left(\nabla\times\frac{\mathbf{u}_k}{|\mathbf{r}-\mathbf{r}_k|}\right)\right\}\right)\\&=-(4\pi/c)\mathbf{J}_e^L,\end{aligned} \tag{3.8}$$

where $\mathbf{J}_e^L$ is the average value of the longitudinal current density. By combining Eqs. (3.7), (3.8), and the corresponding relation for the external fields, the second Maxwell equation

$$\nabla\times\langle\mathbf{B}\rangle=\frac{4\pi}{c}\mathbf{J}_e+\frac{1}{c}\frac{\partial}{\partial t}\langle\mathbf{E}\rangle \tag{3.9}$$

is obtained. Similarly, we find

$$\nabla\cdot\langle\mathbf{B}\rangle=0,\quad \nabla\cdot\langle\mathbf{E}\rangle=4\pi\rho_e. \tag{3.10}$$

Therefore Maxwell's equations are valid for the quantum mechanical ensemble-averaged fields $\langle\mathbf{E}\rangle$, $\langle\mathbf{B}\rangle$, and these equations have the form of the macroscopic equations. However, as has been emphasized by Irving and Kirkwood,[2] the true macroscopic equations must be expressed in terms of suitable space-time averages of the ensemble averaged quantities. The macroscopic quantities so obtained satisfy the same equations.

IV. HYDROMAGNETIC EQUATIONS

Let ρ_{m_ν} be the average value of the mass density $m_\nu\sum_{(\nu)}\delta(\mathbf{r}-\mathbf{r}_k)$,[5] for particles of type ν, and $\rho_{m_\nu}\mathbf{u}_\nu$ the average value of $m_\nu\sum_{(\nu)}\mathbf{u}_k\delta(\mathbf{r}-\mathbf{r}_k)$. Considerations of the same type as those which led to the law of conservation of charge yield the equation of continuity for particles of type ν,

$$\frac{\partial}{\partial t}\rho_{m_\nu}+\nabla\cdot(\rho_{m_\nu}\mathbf{u}_\nu)=0. \tag{4.1}$$

The exact form for the hydromagnetic equations is obtained by application of the law of change (2.14) to the momentum density, $m_\nu\sum_{(\nu)}\mathbf{u}_k\delta(\mathbf{r}-\mathbf{r}_k)$, for par-

[5] $\sum_{(\nu)}f_k$ represents the summation of f_k over all k belonging to type ν.

ticles of type ν. The result is expressed by

$$\frac{\partial}{\partial t}(\rho_{m_\nu}\mathbf{u}_\nu)=\Bigg(f_W{}^*,\, m_\nu\sum_{(\nu)}\mathbf{u}_k\cdot\nabla\mathbf{u}_k\delta(\mathbf{r}-\mathbf{r}_k)+\sum_{(\nu)}\left\{e_k\mathbf{E}_k+\frac{e_k}{c}\mathbf{u}_k\times\mathbf{B}_k\right\}\delta(\mathbf{r}-\mathbf{r}_k)\Bigg)$$
$$=\nabla\cdot\boldsymbol{\sigma}_\nu-\rho_{m_\nu}\mathbf{u}_\nu\mathbf{u}_\nu+n_\nu\mathbf{X}_\nu+\rho_{e_\nu}\mathbf{E}^{\text{ext}}+(\mathbf{J}_{e_\nu}/c)\times\mathbf{B}^{\text{ext}}+(f_W{}^*,\, e_\nu\sum_{(\nu)}\{\mathbf{E}_k{}^{\text{int}}+\mathbf{u}_k\times\mathbf{B}_k{}^{\text{int}}\}\delta(\mathbf{r}-\mathbf{r}_k)), \tag{4.2}$$

in which the stress tensor $\boldsymbol{\sigma}_\nu$ is defined by

$$\boldsymbol{\sigma}_\nu\equiv-m_\nu(f_W{}^*,\sum_{(\nu)}(\mathbf{u}_k-\mathbf{u}_\nu)(\mathbf{u}_k-\mathbf{u}_\nu)\delta(\mathbf{r}-\mathbf{r}_k)), \tag{4.3}$$

and the number density n_ν of particles of type ν by

$$n_\nu\equiv(f_W{}^*,\sum_{(\nu)}\delta(\mathbf{r}-\mathbf{r}_k))=N_\nu f_\nu{}^{(1)}(\mathbf{r}), \tag{4.4}$$

N_ν being the total number of particles of type ν. Equation (4.2) may be written in the form

$$\rho_{m_\nu}\frac{d^\nu}{dt}\mathbf{u}_\nu=\nabla\cdot\boldsymbol{\sigma}_\nu+n_\nu\mathbf{X}_\nu+\rho_{e_\nu}\mathbf{E}^{\text{ext}}+(\mathbf{J}_{e_\nu}/c)\times\mathbf{B}^{\text{ext}}+\mathbf{G}_\nu \tag{4.5}$$

where $d^\nu/dt=(\partial/\partial t)+\mathbf{u}_\nu\cdot\nabla$ and

$$\mathbf{G}_\nu=N_\nu e_\nu(f_W{}^*,\{\mathbf{E}_\nu{}^{\text{int}}+(\mathbf{u}_\nu/c)\times\mathbf{B}_\nu{}^{\text{int}}\}\delta(\mathbf{r}-\mathbf{r}_\nu)), \tag{4.6}$$

or

$$\mathbf{G}_\nu=N_\nu e_\nu\Bigg(f_W{}^*,\left\{\sum_\mu N_\mu e_\mu\frac{\mathbf{r}-\mathbf{r}_\mu}{|\mathbf{r}-\mathbf{r}_\mu|^2}-\frac{1}{c}\sum_\lambda p_\lambda\mathbf{A}_\lambda(\mathbf{r})+\frac{\mathbf{u}_\nu}{c}\times\sum_\lambda q_\lambda\nabla\times\mathbf{A}_\lambda(\mathbf{r})\right\}\delta(\mathbf{r}-\mathbf{r}_\nu)\Bigg). \tag{4.7}$$

If the following pair distribution functions are introduced:

$$f_{\mu\nu}{}^{(2)}(\mathbf{r}_a,\mathbf{r}_b)=(f_W{}^*,\,\delta(\mathbf{r}_a-r_\mu)\delta(\mathbf{r}_b-\mathbf{r}_\nu)), \tag{4.8}$$

$$f_{\nu\lambda}{}^{(2)}(\mathbf{r},p_\lambda')=(f_W{}^*,\,\delta(\mathbf{r}-\mathbf{r}_\mu)\delta(p_\lambda'-p_\lambda)), \tag{4.9}$$

and

$$f_{\nu\lambda}{}^{(2)}(\mathbf{r},\mathbf{u},q_\lambda')=(f_W{}^*,\,\delta(\mathbf{r}-\mathbf{r}_\nu)\times\delta(\mathbf{u}-\mathbf{u}_\nu)\delta(q_\lambda'-q_\lambda)), \tag{4.10}$$

$\mathbf{G}_\nu$ may be expressed by the relation

$$\mathbf{G}_\nu=N_\nu e_\nu\sum_\mu N_\mu e_\mu\int\frac{\mathbf{r}-\mathbf{r}_\mu}{|\mathbf{r}-\mathbf{r}_\mu|^3}f_{\nu\mu}{}^{(2)}(\mathbf{r},\mathbf{r}_\mu)d\mathbf{r}_\mu-N_\nu\frac{e_\nu}{c}\sum_\lambda\mathbf{A}_\lambda(\mathbf{r})\int P_\lambda f_{\nu\lambda}{}^{(2)}(\mathbf{r},p_\lambda)dp_\lambda-N_\nu\frac{e_\nu}{c}\sum_\lambda[\nabla\times\mathbf{A}_\lambda(\mathbf{r})]\times\int q_\lambda\mathbf{u}\times f_{\nu\lambda}{}^{(2)}(\mathbf{r},\mathbf{u},q_\lambda)d\mathbf{u}dq_\lambda. \tag{4.11}$$

Equations (4.5) bear some resemblance to the equations which are customarily considered to be exact for a gas.[6] The Coulomb term in $\mathbf{G}_\nu$ corresponds to the collision term in the usual formulas. However, even if the field and particles motions are uncorrelated, Eqs. (4.5) do not appear to reduce exactly to the usual expressions.

Equations for the transport of other quantities may be obtained, but until we have in our possession a reasonable approximate procedure for computing the various reduced d.f.'s, it hardly seems worthwhile to derive them.

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[6] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1955), p. 18.