quantitative comparison with the reduced-width ratios of 0.36 and 0.12 calculated by Elliott for the $J=0$ and $J=2$ excited states. Qualitatively the second value is in agreement with the size of the differential cross section which was observed, but the first value would appear to be too high, since there was no clear indication of an excited state in this region of excitation at a laboratory deuteron angle of zero degrees.

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Nuclear Spin-Orbit Energy for Oscillator Wave Functions*

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Matrix elements of the two-body spin-orbit interaction between two-nucleon states in LS coupling are expressed in a convenient form which is particularly suitable for oscillator wave functions. The interaction between an almost-closed shell and an external inequivalent nucleon is also considered.

1. INTRODUCTION

THE tensor operator methods of Racah[1] have been used to evaluate the matrix elements of two-body noncentral nuclear forces between antisymmetric LS-coupling states arising from two inequivalent nucleons[2–6] and for configurations involving almost-closed shells.[7] The results obtained expressed the elements of the energy matrix in terms of radial integrals and their coefficients, the radial dependent terms of the operator being first separated from the remainder.

In this paper we obtain expressions for the elements of the radial-orbit amplitude matrix of the spin-orbit operator in which the radial terms are not entirely separated from the orbital coefficients. It is shown that this approach leads to a gain in simplicity of form, and that the elements are not unduly difficult to evaluate. The true two-nucleon case is considered first, and some results for configurations involving almost-closed shells are derived later.

In the course of this work we evaluate single-nucleon amplitude elements for the tensor operators R^1 and P^1 with oscillator wave functions.

2. SPIN-ORBIT OPERATOR

We begin by recalling some results in II.

Apart from the intrinsic spin[8] and isotopic spin[9] dependences, the two-nucleon spin-orbit operator is of the form $J(r_{12})L^1$, in which $J(r_{12})$ is a distance function and L^1 is a two-nucleon tensor operator.

It is convenient, following the methods used in nuclear central force calculations, to expand the distance function

$$J(r_{12})=\sum_{k=0}^{\infty} J_k(r_1,r_2)(C_{(1)}{}^k\cdot C_{(2)}{}^k) \tag{1}$$

in terms of the radius vectors of the individual nucleons and scalar products of the single-nucleon tensor operators C^k. In this the radial and orbital components of the function are separated.

The operator L_1 can be expressed

$$L^1=-i\sqrt{2}\sum_{s,t=1}^{2}(R_{(s)}{}^1\odot{}^1P_{(t)}{}^1), \tag{2}$$

in which the single-nucleon tensor operators R^1 and P^1 are constructed from the Cartesian components of the individual nucleon position and momentum vectors. Each of the operators R^1 and P^1 contain both radial and orbital components, so that the radial component of the operator L^1 is not separate. The tensor product $\odot^k$ is defined in I.

* Much of this work was carried out at the University of Southampton, Southampton, England, and forms part of the writer's Ph.D. thesis.

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[2] J. P. Elliott, Proc. Roy. Soc. (London) **A218**, 345 (1953).

[3] L. W. Longdon, Phys. Rev. **90**, 1125 (1953).

[4] J. Hope, Phys. Rev. **89**, 884 (1953).

[5] J. Hope and L. W. Longdon, Phys. Rev. **101**, 710 (1956), referred to as I.

[6] J. Hope and L. W. Longdon, Phys. Rev. **102**, 1124 (1956), referred to as II.

[7] The term "almost-closed shell" as used in this paper implies a shell closed except for a single vacancy.

[8] An expression for the amplitude elements of the spin-orbit intrinsic spin operator is quoted in II.

[9] The elements of the isotopic spin operator for neutral, symmetric, and charged dependences are given in I.

3. TWO-NUCLEON ENERGY

Hitherto the operator $J(r_{12})L^1$ has been expressed in a form in which its radial component has been separated, as in II, Eq. (7). But it is suggested here that advantage is to be gained by not taking this approach.

For the configurations involving two inequivalent nucleons, and for any particular value of k, the elements of the amplitude matrix are of the form

$$(n_1l_1,n_2l_2,L\|J_k(r_1,r_2)(C_{(1)}{}^k\cdot C_{(2)}{}^k)L^1\|n_1'l_1',n_2'l_2',L'),$$

and can be expressed as

$$(n_1l_1,n_2l_2,L\|J_k(r_1,r_2)(C_{(1)}{}^k\cdot C_{(2)}{}^k)\|\nu_1\lambda_1,\nu_2\lambda_2,L)\times(\nu_1\lambda_1,\nu_2\lambda_2,L\|L^1\|n_1'l_1',n_2'l_2',L'),\quad(3)$$

in which we have employed a dummy suffix notation implying summation over all ν, λ.

The first element of this product is of the type encountered in central-force expressions and, following common practice, we separate its radial term and arrive at

$$(n_1l_1,n_2l_2|J_k(r_1,r_2)|\nu_1\lambda_1,\nu_2\lambda_2)(l_1l_2L\|C_{(1)}{}^k\cdot C_{(2)}{}^k\|\lambda_1\lambda_2L)\times(\nu_1\lambda_1,\nu_2\lambda_2,L\|L^1\|n_1'l_1',n_2'l_2',L'),\quad(4)$$

in which the element of the amplitude matrix is expressed as a linear combination of radial integrals of the central-force type, or of the type I^k defined in II, Eq. (16). The coefficients of these radial integrals consist of an orbital term of the central-force type, which is dependent on k, together with the amplitude elements of the operator L^1, which are independent of k but contain a radial component.

The operator L^1 can be expanded as in Eq. (2), and if $s=t$ the tensor product reduces to the single-nucleon angular momentum operator $\mathfrak{L}^1$, whose elements are quoted in II, Eq. (14). For this reason the expansion of the element $(\nu_1\lambda_1,\nu_2\lambda_2,L\|L^1_{(s=t)}\|n_1'l_1',n_2'l_2',L')$ contains a term $\delta(\nu_1,n_1')\delta(\nu_2,n_2')\delta(\lambda_1,l_1')\delta(\lambda_2,l_2')$ and the elements of the amplitude matrix for the operator $J(r_{12})L^1$ for $s=t$ in the expansion of L^1 lead to the coefficient of the radial integral I^k given in II, Eq. (16). Since the single-nucleon elements of $\mathfrak{L}^1$ contain no explicit radial dependence other than the Kronecker deltas quoted above, the approach suggested here is identical with that used in II, and we shall content ourselves with giving the coefficient of I^k as it appears in II.

$$(n_1l_1,n_2l_2,L\|J(r_{12})L^1_{(s=t)}\|n_1'l_1',n_2'l_2',L')=\Delta 2^{-1}I^k\times\Theta(l_1l_2l_1'l_2';LL;kk;k000)\{L(L+1)\}^{\frac{1}{2}}\delta(L,L'),\quad(5)$$

in which the functions Δ and Θ are defined in I, Eq. (10).

The remaining "crossed" terms of the expansion of L^1, for $s\neq t$, lead to two-nucleon tensor products, and their amplitude elements can be expanded using I, Eq. (3). The result is

$$(\nu_1\lambda_1,\nu_2\lambda_2,L\|L^1_{(s\neq t)}\|n_1'l_1',n_2'l_2',L')=3^{-\frac{1}{2}}\{\Psi(PR)-\Psi(RP)\}\chi(\lambda_1\lambda_2l_1'l_2';LL';11;1),\quad(6)$$

in which

$$\{\Psi(PR)-\Psi(RP)\}=-i\sqrt{2}\{(\nu_1\lambda_1\|P^1\|n_1'l_1')(\nu_2\lambda_2\|R^1\|n_2'l_2')-(\nu_1\lambda_1\|R^1\|n_1'l_1')(\nu_2\lambda_2\|P^1\|n_2'l_2')\},$$

and in the next section we shall derive explicit expressions for the single-nucleon amplitude elements of the operators R^1 and P^1.

We can thus express the typical element of the amplitude matrix for two-nucleon configurations, apart from the coefficient of I^k quoted in Eq. (5), using the well-known result for the amplitude elements of a scalar product of tensor operators,[10] and Eqs. (4) and (6) as follows:

$$\begin{aligned}&(n_1l_1,n_2l_2,L\|J(r_{12})L^1_{(s\neq t)}\|n_1'l_1',n_2'l_2',L')\\&=(-1)^{l_1+\lambda_2-L}\cdot 3^{-\frac{1}{2}}W(l_1l_2\lambda_1\lambda_2;Lk)\\&\quad\times\chi(\lambda_1\lambda_2l_1'l_2';LL';11;1)(l_1\|C^k\|\lambda_1)(l_2\|C^k\|\lambda_2)\\&\quad\times\{\Psi(PR)-\Psi(RP)\}\\&\qquad\times(n_1l_1,n_2l_2|J_k(r_1,r_2)|\nu_1\lambda_1,\nu_2\lambda_2).\end{aligned}\quad(7)$$

4. FUNCTION Ψ

Before we can evaluate the function $\{\Psi(PR)-\Psi(RP)\}$, we must find expressions for the single-nucleon amplitude elements of the operators R^1 and P^1. Now, using II, Eqs. (5) and (6), we may write

$$(nl\|R^1\|n'l')=(l\|C^1\|l')(nl|r|n'l'),\quad(8)$$

in which $(l\|C^1\|l')$ can be found by using I, Eq. (8), and in which the single-nucleon radial integral is equivalent to

$$(nl|r|n'l')=\int_\infty U_{nl}(p)rU_{n'l'}(p)p^2dp,$$

where $U_{nl}(p)$ is a single-nucleon wave function, $p=r/a_0$, and a_0 is the well parameter.

For oscillator wave functions, we have

$$U_{nl}(p)=(-1)^{l+1+n}\{2\Gamma(n+\tfrac{1}{2})\Gamma(n-l)\}^{\frac{1}{2}}p^l\times\exp(-\tfrac{1}{2}p^2)T_{l+1}{}^{n-l-1}(p^2),$$

in which $T_{l+1}{}^{n-l-1}(p^2)$ is a Sonine polynomial. It can be shown[11] from this that, for $l'=l+1$,

$$(nl|r|n'l')=a_0\left\{\frac{\Gamma(n'+\frac{1}{2})\Gamma(n-l)}{\Gamma(n+\frac{1}{2})\Gamma(n'-l')}\right\}\{\delta(n',n+1)-\delta(n',n)\}.$$

We then find, from Eq. (8), the result

$$\begin{aligned}&a_0^{-1}(nl\|R^1\|n'l')\\&=-\{(l+1)(n+\tfrac{1}{2})\}^{\frac{1}{2}}\delta(l',l+1)\delta(n',n+1)\\&\quad+\{(l+1)(n-l-1)\}^{\frac{1}{2}}\delta(l',l+1)\delta(n',n)\\&\quad-\{l(n-l)\}^{\frac{1}{2}}\delta(l',l-1)\delta(n',n)\\&\quad+\{l(n-\tfrac{1}{2})\}^{\frac{1}{2}}\delta(l',l-1)\delta(n',n-1),\end{aligned}\quad(9)$$

[10] See reference 1, Eq. (38).

[11] For an account of this derivation see J. Hope, Ph.D. thesis, London University, 1952 (unpublished).

together with a similar result for the operator P^1 which can be derived from the above by use of the relation

$$a_0^2(nl\|P^1\|n'l')=-(i)^{N'-N}(nl\|R^1\|n'l'), \qquad (10)$$

in which $N=2n-l-1$ is the total quantum number of the single-nucleon state. Equation (10) can be demonstrated by an argument similar to that used in the matrix theory of the linear oscillator.

Using Eqs. (9) and (10), we list in Table I the function $\{\Psi(PR)-\Psi(RP)\}$. We find that there are only eight values of the parameters ν, λ for which the function is nonzero.[12] This is not too cumbersome a result.

5. OPERATOR IN SINGLE-NUCLEON FORM

The expansion of Eq. (7), together with Table I, can conveniently be used for the evaluation of the spin-orbit energies of two-nucleon configurations. If, however, we are to consider the interaction between an almost-closed shell and an external nucleon, we must express the operator in terms of tensor products of single-nucleon tensor operators. To do this, we may rewrite Eq. (1) and expand the distance function in a form in which the radial and orbital components are not separate:

$$J(r_{12})=\sum_{k,i,j}(i\mathcal{C}_{(1)}{}^k\cdot j\mathcal{C}_{(2)}{}^k), \qquad (11)$$

in which $i\mathcal{C}_{(1)}{}^k=f_i(r_1)C_{(1)}{}^k$, where we have made a formal expansion of the function $J_k(r_1,r_2)$ of the type

$$J_k(r_1,r_2)=\sum_{i,j}f_i(r_1)f_j(r_2).$$

We note that the operators $\mathcal{C}^k$ are radial-orbit tensor operators, in that

$$(nl\|\mathcal{C}^k\|n'l')=(l\|C^k\|l')(nl|f(r)|n'l'),$$

where $(nl|f(r)|n'l')$ is a single-nucleon radial integral.

We can now write the spin-orbit radial-orbit operator in the form

$$J(r_{12})L^1=-i\sqrt{2}\sum_{s,t=1}^{2}(-1)^{s+t}\times\sum_{k,i,j}(i\mathcal{C}_{(1)}{}^k\cdot J\mathcal{C}_{(2)}{}^k)(R_{(s)}{}^1\odot{}^1P_{(t)}{}^1),$$

and we are interested in the evaluation of the amplitude elements of this operator for s not equal to t.

For convenience of notation, we shall consider an operator of the form

$$A_{12}{}^k(PR)=-i\sqrt{2}\sum_{i,j}(i\mathcal{C}_{(1)}{}^k\cdot j\mathcal{C}_{(2)}{}^k)(P_{(1)}{}^1\odot{}^1R_{(2)}{}^1), \qquad (12)$$

and note in passing that, for s not equal to t, the complete operator may be written

$$J(r_{12})L^1{}_{(s\neq t)}=A_{12}{}^k(PR)-A_{12}{}^k(RP). \qquad (13)$$

[12] For convenience we have here omitted the primes, writing n', l' as n, l.

TABLE I. The function $\{\Psi(PR)-\Psi(RP)\}$.

λ_1	λ_2	ν_1	ν_2	$\{\Psi(PR)-\Psi(RP)\}$
l_1+1	l_2+1	n_1+1	n_2	$-\{(l_1+1)(l_2+1)(n_1+\frac{1}{2})(n_2-l_2-1)\}^{\frac{1}{2}}$
		n_1	n_2+1	$+\{(l_1+1)(l_2+1)(n_1-l_1-1)(n_2+\frac{1}{2})\}^{\frac{1}{2}}$
	l_2-1	n_1+1	n_2-1	$-\{(l_1+1)l_2(n_1+\frac{1}{2})(n_2-\frac{1}{2})\}^{\frac{1}{2}}$
		n_1	n_2	$2^{\frac{1}{2}}\times+\{(l_1+1)l_2(n_1-l_1-1)(n_2-l_2)\}^{\frac{1}{2}}$
l_1-1	l_2+1	n_1	n_2	$-\{l_1(l_2+1)(n_1-l_1)(n_2-l_2-1)\}^{\frac{1}{2}}$
		n_1-1	n_2+1	$+\{l_1(l_2+1)(n_1-\frac{1}{2})(n_2+\frac{1}{2})\}^{\frac{1}{2}}$
	l_2-1	n_1	n_2-1	$-\{l_1l_2(n_1-l_1)(n_2-\frac{1}{2})\}^{\frac{1}{2}}$
		n_1-1	n_2	$+\{l_1l_2(n_1-\frac{1}{2})(n_2-l_2)\}^{\frac{1}{2}}$

We can now expand Eq. (12) in single-nucleon form, using II, Eq. (8), by writing

$$A_{12}{}^k(PR)=-i\sqrt{2}\sum_{i,j,x,y}(-1)^k\{[x][y]\}^{\frac{1}{2}}\times W(xk11;1y)(i\mathcal{P}_{(1)}{}^x\odot{}^1j\mathcal{R}_{(2)}{}^y),$$

in which $i\mathcal{P}^x=(i\mathcal{C}^k\odot{}^xP^1)$, $j\mathcal{R}^y=(j\mathcal{C}^k\odot{}^yR^1)$. From this, we evaluate the general element of the amplitude matrix for this operator, which is

$$A^k(PR)=(n_1l_1,n_2l_2,L\|A_{12}{}^k(PR)\|n_1'l_1',n_2'l_2',L'),$$

by using first the relation I, Eq. (3) to write the element in terms of single-nucleon amplitude elements together with a χ function, and then breaking down the single-nucleon amplitude elements by means of II, Eq. (13). We arrive at the result

$$A^k(PR)=\sum_{\nu,\lambda}\Psi(PR)\Omega^k\Phi^k, \qquad (14)$$

in which $\Psi(PR)$ is as in Eq. (6),

$$\Omega^k=\sum_{i,j}(n_1l_1\|i\mathcal{C}^k\|\nu_1\lambda_1)(n_2l_2\|j\mathcal{C}^k\|\nu_2\lambda_2),$$

and

$$\Phi^k=\sum_{x,y}(-1)^{x+y+k}\{[1][x][y]\}^{\frac{1}{2}}\times\chi(l_1l_2l_1'l_2';LL';xy;1)W(xk11;1y)\times W(k1l_1l_1';x\lambda_1)W(k1l_2l_2';y\lambda_2).$$

The summation over ν, λ is to serve as a reminder that we have made use of a dummy suffix notation, as we did before.

The factor Ω^k can be written

$$\Omega^k=(l_1\|C^k\|\lambda_1)(l_2\|C^k\|\lambda_2)(n_1l_1,n_2l_2|J_k(r_1,r_2)|\nu_1\lambda_1,\nu_2\lambda_2),$$

by using Eq. (11), and we can now replace Eq. (7) by the equation

$$(n_1l_1,n_2l_2,L\|J(r_{12})L^1{}_{(s\neq t)}\|n_1'l_1',n_2'l_2',L')=\sum_{\nu,\lambda}(l_1\|C^k\|\lambda_1)(l_2\|C^k\|\lambda_2)\Phi^k\{\Psi(PR)-\Psi(RP)\}\times(n_1l_1,n_2l_2|J_k(r_1,r_2)|\nu_1\lambda_1,\nu_2\lambda_2). \qquad (15)$$

In the next section we shall consider the application of this result.

6. TWO-NUCLEON AND ALMOST-CLOSED SHELL CONFIGURATIONS

The factor Φ^k of Eq. (15), which is for a two-nucleon configuration, can be reduced as follows.

It can be shown that[13]

$$\sum_r(-1)^r[r]W(eckd;rf)W(ecbg;ra)W(bgkd;rh) = (-1)^{e+g-a}W(abfk;eh)W(hadc;fg), \quad (16)$$

and the χ function can be expanded in terms of W functions[14]

$$\chi(abcd;ef;gh;k)=\{[e][f][g][h]\}^{\frac{1}{2}}\sum_r[r] \times W(eckd;rf)W(ecbg;ra)W(gbkd;rh). \quad (17)$$

If we expand the χ function in Φ^k using Eq. (17), then apply Eq. (16) to the summation over each of x and y, and finally apply Eq. (17) again to the summation over the remaining parameter, we obtain the result

$$\Phi^k=(-1)^{\lambda_2+l_1-L}3^{-\frac{1}{2}}W(l_1l_2\lambda_1\lambda_2;Lk) \times\chi(\lambda_1\lambda_2l_1'l_2';LL';11;1), \quad (18)$$

which, in conjunction with Eq. (15), brings us back to Eq. (7).

If, on the other hand, we are concerned with the direct term for the interaction between an almost-closed shell and an external nucleon, we must, following the methods of Sec. 4 of I, insert a factor $(-1)^{x+1}$ under the summation in the expansion of Φ^k. After having expanded the χ function, we are faced with a sum over six W functions which can be reduced as follows. We note that l_1 is now equal to l_1', and make use of Eq. (16) in the summation over y to reduce the expression to a sum over five W functions. We then use Eq. (17) on the summation over x to write this as a sum over two W functions and a χ function. After some tedious manipulation, the expression finally reduces to a simple result containing a Wigner $12-j$ coefficient.[15] We find, using a notation similar to that employed in I and II,

$$(D)\Phi^k{}_{(m_1-1)}=(-1)^{l_1+l_2+\lambda_1+\lambda_2}\{[1][L][L']\}^{\frac{1}{2}} \times\begin{Bmatrix} l_1 & l_2 & L & L' \\ \lambda_1 & k & l_1 & 1 \\ 1 & \lambda_2 & 1 & l_2' \end{Bmatrix} \quad (19)$$

for the value of Φ^k appropriate to the direct "normal" terms.

The exchange "normal" term can be evaluated by the methods of Sec. 4 of I, and reduces to an explicit expression. We here quote the result for the general exchange "normal" amplitude element for the operator $J(r_{12})L^1{}_{(s\neq t)}$. It is as in Eq. (15) except that the function $\{\Psi(PR)-\Psi(RP)\}$ is to be replaced by the function $(E)\{\Psi(PR)-\Psi(RP)\}$, which can be obtained from $\{\Psi(PR)-\Psi(RP)\}$ by interchanging $n_1'l_1'$ with $n_2'l_2'$ and then putting $n_1'l_1'$ equal to n_1l_1, and the function Φ^k is to be replaced by $(E)\Phi^k{}_{(m_1-1)}$, where[16]

$$(E)\Phi^k{}_{(m_1-1)}=(-1)^{l_1+l_2+L+L'}\{[1][L][L']\}^{\frac{1}{2}} \times W(11LL';1k)W(k1l_1l_2';L'\lambda_1)W(k1l_2l_1;L\lambda_2). \quad (20)$$

Corresponding expressions for the almost-closed shell "null" terms, and for the closed shell configurations considered in I and II, are easily derived, and are not given here.

7. COMMENTS ON THE RESULTS

The typical element of the two-nucleon amplitude matrix is expressed in Eq. (7) in a more simple form than in the corresponding result of II, Eq. (16). The functions $P(a,b,c)$ which involve a sum over two W functions do not appear, and the coefficient of the radial integral is, on the whole, less complex than the function Θ, defined in I. Also, in place of the radial integrals J^k, C^k, and D^k we have only radial integrals of the type I^k which are of the central-force kind.

On the other hand, in place of a summation over x and y we have a summation extending over all of ν and λ, with a different radial integral for each set of values of these parameters. However, as can be seen from Table I, this can involve at the most eight terms, and we note that if either $n_1'=l_1'+1$ or $n_2'=l_2'+1$, the number of terms reduces to six and if both conditions are complied with, this reduces to four. Also we would remark that in any series of calculations extended over a number of different configurations the same radial integrals will occur many times, whereas with the method of II different integrals are required for each configuration.

The direct "normal" terms for the interaction of an almost-closed shell with an external nucleon are not hard to evaluate for lighter nuclei, and the exchange "normal" terms are, as usual, relatively simple.

The elements of the operator C^k can be calculated by using I, Eq. (8), and the W function has been tabulated in convenient form.[17]

The writer has carried out a series of calculations involving the nuclear shells $(1s)$, $(2p)$, $(2s)$, and $(3d)$ by a method similar to the above in order to check values obtained using the results of II, and has found this method to be the more tractable of the two.

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[13] This result was demonstrated to the writer by J. P. Elliott. It is also given by L. C. Biedenharn, Oak Ridge National Laboratory Report No. 1098, 1952 (unpublished).

[14] See, for example, H. A. Jahn and J. Hope, Phys. Rev. **93**, 318 (1954).

[15] For the relevant relation see the addendum to reference 14.

[16] This result includes the exchange phase factor $(-1)^{l_1'+l_2'-L}$.

[17] The tables of Shin-ya Obi *et al.*, Ann. Tokyo Astron. Observ. Ser. II, **3**, No. 3 (1953), for integral variables, are very useful.