by Lee, Yang, and Oehme,[9] Gatto,[10] and Gell-Mann.[11] The results are that if the product CP (charge conjugation times parity) is conserved, one obtains again the Gell-Mann and Pais $\theta_1-\theta_2$ scheme; but if CP is not conserved, the eigenstates of the system (i.e., the particles with definite mass and definite lifetime) are, in vacuum as follows:

$$K_+^0=\frac{1}{C}(pK^0+qK^0),\qquad K_-^0=\frac{1}{C}(pK^0-qK^0), \tag{13}$$

[9] Lee, Yang, and Oehme, Phys. Rev. (to be published).

[10] R. Gatto, "The K^0 decay modes and the question of time reversal of weak interactions," UCRL-3658, January, 1957; also Phys. Rev. (to be published).

[11] M. Gell-Mann, Nuovo cimento (to be published).

where $C=[|p|^2+|q|^2]^{\frac{1}{2}}$, and the term K^0 implies a single particle, with θ^0 and τ^0 decay modes.

We must now ask the following question: what is the behavior, in this most general case, of the neutral K-meson complex in an absorber?

If we use Eq. (13) for the eigenstates in rederiving the relations stated in this paper, the equations of motion of a_+ and a_- (the amplitudes of K_+^0 and K_-^0) turn out to be precisely Eq. (5) again, except that a_1 is replaced by a_+, a_2 by (ia_-), and ω_1, ω_2, λ_1, λ_2 by ω_+, ω_-, λ_+, λ_-.

It follows that the regeneration of the short-lived component (say K_+^0) from the long-lived one (K_-^0) would then be identical in detail to that predicted in the original Gell-Mann and Pais scheme; in particular, Eq. (12) would hold here wherever it did there.

PHYSICAL REVIEW VOLUME 106, NUMBER 3 MAY 1, 1957

Relation of Isotopic Spin Space to Space-Time*

R. L. Ingraham
Institute for Fluid Dynamics, University of Maryland, College Park, Maryland

(Received November 21, 1956)

The customary form of physical space as the direct product of space-time and isotopic spin space can be replaced by a certain fusion of these two spaces into one space if one widens the group from the Lorentz group to the conformal space-time group which endows particles with an intrinsic finite size. It is shown that the familiar space-time transformations continue to induce their familiar Dirac spinor transformations, while the new space-time transformations in general induce isotopic spin transformations as well. The detailed correspondence is worked out for a theory of $T=1$ bosons coupled to $T=\frac{1}{2}$ fermions.

In elementary particle theory, physical space has long been considered as simply the direct product of space-time and of a 3-dimensional vector space called isotopic spin space. This is physically not very satisfying; for example, the operational prescriptions which take one from one frame to an equivalent one in space-time are at the foundations of special relativity, but the corresponding prescriptions for a change of frame in isotopic spin space are lacking. This note suggests a possible fusion of the two spaces on the basis of the conformal sphere geometry in physics. This group, containing the conventional inhomogeneous Lorentz group, permits a covariant description of finite-size sources in both classical and quantum theories,[1,2] as one of its interesting features.

1. DETAILS

Six homogeneous coordinates X^μ ($\mu=0, 1, \cdots 5$) are linked to space-time x^m ($m=1, \cdots 4$) and to the coordinate of finite size λ by[3]

$$\tau X^0=1,\quad \tau X^m=x^m,\quad \tau X^5=\tfrac{1}{2}(\boldsymbol{x}^2+\lambda^2),\qquad (\boldsymbol{x}^2\equiv g_{mn}x^mx^n), \tag{1.1}$$

where g_{mn} (signature $+++-$) is the Lorentz metric. Taking τ to have the dimensions of length like the x^m (we write $\tau\sim L$), we note that the X^μ have different dimensions: $X^0\sim L^{-1}$, $X^m\sim 1$, $X^5\sim L$. The points ($\lambda=0$) of space-time are then given by the locus

$$G_{\mu\nu}X^\mu X^\nu\equiv \boldsymbol{X}^2-2X^0X^5=0, \tag{1.2}$$

where italic boldface shall always refer to the four space-time components. The transformations of the group are now simply rotations $\bar{X}^\mu=L^\mu{}_\nu X^\nu$ ($L^\mu{}_\nu$= constants) with respect to the metric $G_{\mu\nu}$ of signature $(+++-, +-)$. The corresponding spinor algebra is generated by six 8×8 γ matrices Γ^μ satisfying

$$\Gamma^\mu\Gamma^\nu+\Gamma^\nu\Gamma^\mu=2G^{\mu\nu}1, \tag{1.3}$$

where 1 means the 4×4 spinor identity.

We now assert that the fusion of space-time and

* This research was partially supported by the USAF, Contract No. AF18(600)1315.

[1] R. Ingraham, Phys. Rev. **101**, 1411 (1956).

[2] R. Ingraham and J. Ford (to be published).

[3] See, for example, R. Ingraham, Nuovo cimento **12**, 825 (1954).

isotopic spin space can be recognized by choosing the following representation for the Γ^μ:

$$\Gamma^m=\gamma^m\times\tau^3,\quad \Gamma^0=l^{-1}1\times\tau^+,\quad \Gamma^5=-l1\times\tau^-. \tag{1.4}$$

Here γ^m are the 4×4 Dirac matrices,

$$\gamma^m\gamma^n+\gamma^n\gamma^m=2g^{mn}1, \tag{1.5}$$

the τ's are the 2×2 Pauli matrices

$$\tau^\pm=\frac{1}{\sqrt{2}}(\tau^1\pm i\tau^2),\ \tau^3, \tag{1.6}$$

and l is an arbitrary but fixed length. That is, in any wave equation involving the Γ's, the four components m refer to the space-time degrees of freedom, while the $(+,-,3)$ label the charge states of bosons having $T=1$. As for the spin indices, we assume that the rows of the γ^m and 4×4 1 label spin and particle-antiparticle, while the rows of the τ's label the isotopic spin of fermions with $T=\frac{1}{2}$. (Thus at this stage it looks as if only boson-fermion interaction of the pion-nucleon type could be described by this formalism.) To support this interpretation, we must now see whether the familiar transformations continue to induce their familiar spin transformations, and whether the new coordinate transformations induce isotopic spin transformations (and if so, just what is the connection between them).

Lorentz transformations are given by $\bar{X}^m=L^m{}_nX^n$, $\bar{X}^0=X^0$, $\bar{X}^5=X^5$, whence the corresponding spinor transformation S_{Lor} obviously does not involve Γ^5 or Γ^0. But $\Gamma^m\propto\tau^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and hence does not mix N and P (the neutron-like and proton-like fermion isotopic spin brothers). Moreover, the spin transformation induced in each of the 4-dimensional N and P subspaces is obviously of the same form $[(\tau^3)^2=1]$ as that induced by this Lorentz transformation in the Dirac theory. Hence this situation is satisfactory.

Let us now look at the single *improper* transformation

$$\bar{X}^m=X^m,\quad \bar{X}^0=l^{-2}X^5,\quad \bar{X}^5=l^2X^0. \tag{1.7}$$

This means the inversion

$$\bar{k}^m=2x^m/(x^2+\lambda^2),\quad \bar{\kappa}=2\lambda/(x^2+\lambda^2), \tag{1.8}$$

in the space-time sphere $\xi^2=2$ of radius $\sqrt{2}$ around the space-time origin, followed by the (dimensional) dilatation $\bar{x}^m=l^2k^m$, $\bar{\lambda}=l^2\bar{\kappa}$, as can be easily worked out from (1.1). We have $L^0{}_5=l^2$, $L^5{}_0=l^2$, other $L^\mu{}_\nu=\delta_\nu{}^\mu$, and hence from the fundamental equation

$$L^\mu{}_\nu\Gamma^\nu=S\Gamma^\mu S^{-1}, \tag{1.9}$$

we obtain $S_{\mathrm{inv}}\Gamma^0=l^{-2}\Gamma^5S_{\mathrm{inv}}$, $\Gamma^0S_{\mathrm{inv}}=l^{-2}S_{\mathrm{inv}}\Gamma^5$, and $[\Gamma^m,S_{\mathrm{inv}}]=0$. This gives

$$S_{\mathrm{inv}}=\gamma^5\times\tau^2, \tag{1.10}$$

which, since $\tau^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, exactly switches the N and P subspaces. Since the indices 0 and 5 refer to the $+$ and $-$ boson isotopic degrees of freedom, we see that the physical meaning of the improper conformal sphere transformation (1.7) is the interchange of isotopic spin brothers in both spaces. Equation (1.8) also induces an interchange of particle and antiparticle within the N and P subspaces via the γ^5 factor.

The one-parameter subgroup

$$\bar{x}=\frac{x+\frac{1}{2}\alpha(x^2+\lambda^2)}{D},\quad \bar{y}=\frac{y}{D},\quad \bar{z}=\frac{z}{D},\quad \bar{t}=\frac{t}{D},\quad \bar{\lambda}=\frac{\lambda}{D},$$
$$D\equiv1+\alpha x+\tfrac{1}{4}\alpha^2(x^2+\lambda^2), \tag{1.11}$$

where $\alpha\sim L^{-1}$, is nonlinear and so must involve inversions. Hence we expect that it will transform the isotopic spin in some way. The precise form of S will be worked out in a moment. These transformations have been studied for some time now[4]; they apparently mean physically a (relativistic) uniform acceleration $a=\alpha c^2$ of observers. For example, (1.11) goes into ordinary uniform acceleration $\bar{x}=x-\frac{1}{2}at^2$, y, etc., unchanged, in the nonrelativistic limit $c\to\infty$. We can now give them a new interpretation as transformations taking N's and P's continuously into each other in the sense originally envisaged by the introduction of the τ's to describe proper rotations in isotopic spin space. As a transformation of the X^μ, the infinitesimal elements of (1.11) correspond to $L^\mu{}_\nu=\delta_\nu{}^\mu+\epsilon^\mu{}_\nu$, $\epsilon_{\mu\nu}=-\epsilon_{\nu\mu}$, with only $\epsilon_{15}(=\epsilon^0{}_1=\epsilon^1{}_5)=\alpha\neq0$. [This is the convenience of the non-principal-axis form (1.2).] The solution of (1.9) for $L^\mu{}_\nu$ infinitesimal is

$$S=1-\tfrac{1}{4}\epsilon_{\mu\nu}\Gamma^\mu\Gamma^\nu. \tag{1.12}$$

Hence,

$$S_{\mathrm{acc}}=1-\tfrac{1}{2}\epsilon_{15}\Gamma^1\Gamma^5=1-\tfrac{1}{2}\alpha l\gamma^1\times\tau^-. \tag{1.13}$$

The accelerative transformation (1.11) is not geometrically simple. Nor is it physically very simple, for by (1.4) and the fact that $\epsilon^1{}_5=\epsilon^0{}_1$ are nonzero, the new π^0 and π^+ get small admixtures of the old π^- and old π^0, respectively. In the fermion spin space we see that

$$\psi=c_1P+c_2N\to c_1P+[c_2-(\alpha l/\sqrt{2})c_1\gamma^1]N, \tag{1.14}$$

so that the neutronic component gets an increment $-2^{-\frac{1}{2}}\alpha lc_1\gamma^1N$, a neutron with certain spin and particle-antiparticle interchanges given by γ^1.

The uniform dilatations

$$\bar{x}^m=e^\psi x^m,\quad \bar{\lambda}=e^\psi\lambda,\quad (\psi\text{ real and dimensionless}) \tag{1.15}$$

are given simply by

$$\bar{X}^m=X^m,\quad \bar{X}^0=e^{-\psi}X^0,\quad \bar{X}^5=e^\psi X^5, \tag{1.16}$$

i.e., rotations through the imaginary angle $\theta=-i\psi$ in the $2^{-\frac{1}{2}}(X^5+X^0)$, $-i2^{-\frac{1}{2}}(X^5-X^0)$ plane. Since these transformations mean simply a change of the unit of

[4] J. Haantjes, Proc. Ned. Akad. Wetenschap **43**, 1288 (1940); E. L. Hill, Phys. Rev. **72**, 143 (1947).

length, we should expect no isotopic spin transformation therefrom. And indeed, $\epsilon_{50}=\psi$, other $\epsilon_{\mu\nu}=0$, characterizes the infinitesimal elements of (1.16), whence, via (1.12) and (1.4),

$$S_{\text{dil}}=1-\tfrac{1}{2}\epsilon_{50}(\Gamma^5\Gamma^0+1)=1+\tfrac{1}{2}\psi(\tau^-\tau^+-1)\times 1 \\ =1-\tfrac{1}{2}\psi 1\times\tau^3, \quad (1.17)$$

which induces no N, P transformation. Of course, since only $\epsilon^0{}_0=-\epsilon^5{}_5\neq 0$, there is no transformation among the $\pi^\pm$, π^0 either.

Finally, we must consider the space-time translations $\bar{x}^m=x^m+a^m$, $\bar{\lambda}=\lambda$. These, for an infinitesimal x-translation a, say, are

$$\bar{X}^1=X^1+aX^0,\quad \bar{X}^0=X^0,\quad \bar{X}^5=X^5+aX^1, \\ \bar{X}^i=X^i\ (i=2,3,4). \quad (1.18)$$

These are intimately related to the x accelerations (1.11), in fact, they are obtained from them by the substitutions $X^0\rightleftarrows X^5$ and $a\rightleftarrows\alpha$. Hence we can write down directly from (1.13) and (1.4)

$$S_{\text{trans}}=1-(a/2l)\gamma^1\times\tau^+. \quad (1.19)$$

Hence a translation *does* induce an isotopic spin transformation: (1.19) augments the protonic component of ψ by a certain amount, etc. [see the discussion following Eq. (1.13)], as well as mixing the boson isotopic spin degrees of freedom.

2. REMARKS

This last fact may seem anomalous. Translations, however, must be more complicated than Lorentz transformations in the sense that they bring in the coordinate X^0 together with the space-time X^m by the very nature of the projective coordinates X^μ. Another point perhaps worth making is the following: one has the possibility here of describing all the physical variables of a particle by operators "of the same kind"[5]; in particular, the space-time position operator becomes

$$\xi_m\equiv(1/i)\mathfrak{L}_{0m}\equiv(1/i)(X_0\partial/\partial X^m-X_m\partial/\partial X^0), \quad (2.1)$$

rather than the x_m themselves. Note that $\xi_m\sim L$, as it should be. Under a Lorentz transformation, ξ^m transforms as a Lorentz vector, but now under the infinitesimal translation (1.18), we get

$$\bar{\xi}_m=\xi_m-a(M^1{}_m/\hbar)+\delta_m{}^1 a\Delta, \quad (2.2)$$

where M_{mn} is the 4-angular momentum operator, and

$$\Delta\equiv(1/i)\mathfrak{L}_{50}\equiv(1/i)(X_5\partial/\partial X^0-X_0\partial/\partial X^5), \quad (2.3)$$

relating in some way to the finite extent of the particle. Thus (apart from the angular momentum term) particle trajectories are x-translated by amounts $a\Delta'$ in states in which Δ is sharp $=\Delta'$. The point is, that $\bar{\xi}_m$, being linear homogeneous in the particle operators, must involve as the coefficient of $\delta_m{}^1 a$ a space-time-independent operator, that is, one referring to another, "internal" degree of freedom. Hence the behavior of the particle on translating the space axes depends not only on its position but on an internal state as well, which would make (1.19) less surprising.

One final remark. In field theories built on the Lorentz group, the status of the improper transformation is not completely settled in everybody's mind: for example, the uneasy truce with time reversal.[6] If this geometry were used, the improper transformation could be taken to be the inversion (1.7) whose physical meaning, as the interchange of isotopic spin brothers, is clear.

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[5] I. Segal, Duke Math. J. **18**, 221 (1951); see especially p. 260 ff. See also C. Yang, Phys. Rev. **72**, 874 (1947).

[6] See J. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Press, Cambridge, 1955), Chap. 5, Sec. 4.