Isobar Model for Meson Production in Proton-Proton Collisions*

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(Received December 26, 1956)

A model is considered for single- and double-pion production in which the production takes place via an intermediate state wherein either one or both of the initial nucleons is excited to the isobaric state of $J=I=\frac{3}{2}$. The treatment is phenomenological and comparison is made with recent experiments in the 0.5- to 1.5-Bev range. Two striking features of the experiments, the strong preference for the emission of mesons with kinetic energies of 50 to 150 Mev, and the rapid increase in the two-meson processes at bombarding energies above 1 Bev, are exhibited by the calculation.

INTRODUCTION

RECENT experiments on meson production in nucleon-nucleon collisions have thrown some light on the usefulness of the $J=\frac{3}{2}$, $I=\frac{3}{2}$ isobar as an intermediate state in high energy processes involving one or more nucleons. At the Cosmotron, Yuan and Lindenbaum[1] have observed the energy spectra of positive and negative pions produced in $p-\mathrm{Be}$ and $p-p$ collisions at 1 Bev and at 2.3 Bev. At the higher energy, double production appears to predominate. At both 1 and 2.3 Bev, plots of the relative meson-production cross sections per Mev per unit solid angle *versus* meson kinetic energy in the nucleon-nucleon center-of-mass system exhibit strong peaks between 100 and 150 Mev. In the case of π^+ production, these peaks are surprisingly similar in shape to the peak that appears in the π^+-p total-interaction cross section when plotted *versus* meson kinetic energy in the π^+-p center-of-mass system.[2] At 1 Bev the curve is shifted somewhat toward the lower energies. At 2.3 Bev the curve is considerably broadened as compared to the π^+-p curve. The peaks in the negative pion spectra are markedly similar to that which appears in a plot of the π^--p interaction cross section *versus* meson kinetic energy. The π^- spectra also show the effects described above, at 1 and 2.3 Bev, respectively.

These facts may be explained[1] in a qualitative manner as follows. In the collision of the two nucleons, a mechanism whose detailed nature is probably quite complicated operates to form an intermediate state in which either one or both of the nucleons has been raised to the $(J=\frac{3}{2}, I=\frac{3}{2})$ isobaric state. If the initial nucleons have momenta $\pm\mathbf{p}$ in the center-of-mass system and if the excitation process involves a transfer of momentum $\mathbf{g}$ between the particles, the particles in the intermediate state have momenta $\pm(\mathbf{p}-\mathbf{g})$. It is then supposed that these particles separate somewhat, and one or both decay by emission of a pion. The decay of one isobar is to be thought of as being independent of the presence of another isobar or nucleon. It may be that the excitation occurs near the edge of a region of strong interaction, with not too great a momentum transfer. This would make the separation and independent decay more plausible in the light of the very short lifetime of the isobar ($\sim 10^{-23}$ second).[1]

In this calculation we shall describe the excitation of the intermediate state by amplitudes which we shall define as functions of the bombarding energy. The decay of the isobar will be described by an amplitude which will be defined as a function of the total energy in the total center-of-mass system of the pion-nucleon system resulting from the decay. These amplitudes are in general also functions of the relative momentum and energy of the intermediate-state particles. As a first approximation we shall neglect this dependence in the amplitudes which we define. That is not to say that we shall consider the intermediate-state particles to be literally brought to rest by the excitation process. Being massive, their relative momentum will not be negligible, although their relative kinetic energy may be small. We shall rather consider that the final nucleons account for most of the conservation of momentum and hence tend to follow the directions of the intermediate-state particles, moving off with approximately equal and opposite momenta in the total center-of-mass system. We thus neglect the part of the meson momenta (or equivalently the motion of the center of mass of the two-nucleon system) in total momentum conservation. Each of the final-state nucleons will have about one-half of their energy of relative motion and the latter will be given by the meson energies and energy conservation. With this kinematic picture of the collision we may construct from our approximate amplitudes the transition probabilities for single and double meson production.

In Sec. A under the Calculations, we give the charge ratios in single and double production on the basis of the model. These were derived originally by Peaslee[3] and are presented here for clarity in the ensuing discussion. In Sec. B, we define the fundamental amplitudes to be used in calculating the transition probabilities for meson production. These amplitudes were

* The work was performed under the auspices of the U. S. Atomic Energy Commission.
[1] L. C. L. Yuan and S. J. Lindenbaum, Phys. Rev. 103, 404 (1956).
[2] L. C. L. Yuan and S. J. Lindenbaum, Phys. Rev. 100, 306 (1955).
[3] D. C. Peaslee, Phys. Rev. 94, 1085 (1954); 95, 1580 (1954).

defined by Austern[4] and were utilized by him in his study of the photodisintegration of the deuteron on the basis of the isobar model. The reader is referred to his paper for a lucid discussion of the model. In Sec. C, we derive the cross section for the process $p+p\to n+p+\pi^+$. The energy spectrum of the emitted pion is of particular interest for comparison with experiment. In Sec. D, we derive the cross sections for double-pion production and present the energy spectrum of the pions. An estimate of the double-to-single ratio as a function of bombarding energy is given. In Sec. E, we discuss the modification of the angular distributions of the two pions that would occur if a hypothetical meson-meson interaction were present. The predictions of the model are compared in the Discussion with the recent experiments on meson production in $p-p$ collisions.

CALCULATION

A. Charge Ratios in Single and Double Production

An expansion of the initial two-proton state of total isotopic spin $I=1$ and total z-component $I_z=1$ in terms of an $I=\frac{3}{2}$ isobar state and an $I=\frac{1}{2}$ nucleon state leads to the following result for the single production cross section:

$$\sigma=\frac{9}{12}\sigma(\pi^+p;n)+\frac{1}{12}\sigma(\pi^+n;p)+\frac{2}{12}\sigma(\pi^0p;p). \quad (1)$$

A semicolon stands between the pion-nucleon pair resulting from the isobar decay and the second nucleon. The π^+/π^0 ratio is 5. It is important to note that the total cross sections $\sigma(\pi^+p;n)$ and $\sigma(\pi^+n;p)$ represent physically distinguishable processes, in that the π^+-p pair and the π^+-n pair will show the characteristic Q-value of the isobar decay.

By expanding the initial state in terms of two $I=\frac{3}{2}$ isobar states, one obtains for the double-production cross section:

$$\sigma=\frac{1}{45}\{18\sigma(\pi^+p;\pi^0n)+8\sigma(\pi^0p;\pi^+n)+9\sigma(\pi^+p;\pi^-p)+8\sigma(\pi^0p;\pi^0p)+2\sigma(\pi^+n;\pi^+n)\}. \quad (2)$$

This gives for total π^+, π^0, π^- production the relative weights 13, 14, 3, respectively. A prediction of this model is that in the dominant double-production process $p+p\to\pi^0+\pi^++n+p$ the π^+ should be correlated by Q-value to the proton 9/4 as often as to the neutron.

B. Fundamental Amplitudes for Calculating Transition Probabilities

Following Austern's method, we now define an amplitude Π in the following manner. Consider the scattering of positive P-wave pions by protons, at a total energy E. We assume that the scattering proceeds through an intermediate state that involves formation of the $(\frac{3}{2},\frac{3}{2})$ isobar with energy $E_0=m+\mu+0.16=1.24$ Bev, where m and μ are the nucleon and meson rest energies, respectively. The P-wave part of the incident meson plane wave of momentum $\mathbf{k}$ may be written as

$$\sum_{m=-1}^{m=1} Y_{1,m}^*(\mathbf{k}\cdot\mathbf{z})Y_{1,m}(\mathbf{r}\cdot\mathbf{z}), \quad (3)$$

where $\mathbf{z}$ is the axis of directional quantization and the $Y_{1,m}$ are normalized spherical harmonics. Boldface symbols which appear in the arguments of spherical harmonics denote unit vectors. The $J=\frac{3}{2}$ part of the product of the above expression with the initial proton spinor with spin z-component s, N^s, is given by

$$\sum_{m=-1}^{m=1} Y_{1,m}^*(\mathbf{k}\cdot\mathbf{z})\langle 1,\tfrac{1}{2},m,s|\tfrac{3}{2},m+s\rangle\,|\tfrac{3}{2},m+s\rangle. \quad (4)$$

The definition[4] of Π is then achieved by stating that a positive-meson plane wave of unit amplitude incident upon a proton forms the $J=\frac{3}{2}$ isobar with spin z-component $\sigma=m+s$, with the amplitude

$$\Pi(E)Y_{1,m}^*(\mathbf{k}\cdot\mathbf{z})\langle 1,\tfrac{1}{2},m,s|\tfrac{3}{2},m+s\rangle. \quad (5)$$

If the initial pion-nucleon state is not pure $I=\frac{3}{2}$, say π^+-n, then an additional Clebsch-Gordan coefficient appears in the above amplitude, i.e., $\langle 1,\frac{1}{2},1,-\frac{1}{2}|\frac{3}{2},\frac{1}{2}\rangle$, representing that part of the state that couples into the $I=\frac{3}{2}$ isobar.

The scattering cross section may then be evaluated in terms of $\Pi(E)$ and the result is[4]

$$d\sigma_E(\pi^++p\to\pi^++p)=(8\pi)^{-1}\Pi^4\frac{\rho_E(\pi N)}{V_E(\pi)}|D|^{-2}(3\cos^2\theta+1)\frac{d\Omega}{4\pi}, \quad (6)$$

where D is an energy denominator given by

$$D=E-E_0-i\Gamma/2. \quad (7)$$

The quantity Γ is introduced as an imaginary part to the isobar energy owing to its decay and is given by[5]

$$\Gamma=\frac{1.36(k/\mu)^3\times 58\text{ Mev}}{1+0.77(k/\mu)^2}, \quad (8)$$

where k is the meson momentum, $[(E-m)^2-\mu^2]^{\frac{1}{2}}$. The quantity $V_E(\pi)$ is the incident pion velocity $=k/\omega$, where $\omega=E-m$; $\rho_E(\pi N)$ is the density of final states $=(2/\pi)k\omega$. The amplitude $\Pi(E)$ as a function of the

[4] N. Austern, Phys. Rev. **100**, 1522 (1955).

[5] M. Gell-Mann and K. M. Watson, in *Annual Review of Nuclear Science* (Annual Reviews, Inc., Stanford, 1954), Vol. 4, p. 219.

total cross section is then

$$\Pi^4(E)=4\pi\frac{V_E(\pi)}{\rho_E(\pi N)}|D|^2\sigma_E(\pi^++p\rightarrow\pi^++p). \quad (9)$$

In Austern's work, use was made of this amplitude with the experimental cross section on the right-hand side. In order to perform certain phase space integrations when investigating meson-production phenomena, we shall use a theoretical expression for $\sigma_E(\pi^++p\rightarrow\pi^++p)$ given by the Chew-Low theory[6]:

$$\sigma(E)=8\lambda_3{}^2k^4\left\{\frac{1}{\omega^2(1-\omega/\omega_3)^2+\lambda_3{}^2k^6}\right\}, \quad (10)$$

where ω is the meson energy $=[k^2+\mu^2]^{\frac{1}{2}}=E-m$, $\lambda_3{}^2=(16/9)(f^4/\mu^4)=29.5$ for the pseudovector coupling constant $f^2=0.08$, and $\omega_3=0.3$ Bev.

We now need to evaluate one more basic amplitude. We recapitulate Austern's argument in brief. Consider the mesonic disintegration of the deuteron, $\pi^++d\rightarrow2p$, at a total energy E. We consider the process to go through an intermediate state involving one isobar and one nucleon. We neglect the energy of relative motion of the particles in the intermediate state, and we consider them to be in an S state. It is then readily seen that, in order to conserve angular momentum and parity, the final-state protons must be in a ${}^1D_2$ state. Now the D part of the two-proton plane wave of momentum $\mathbf{p}$ is

$$\sum_{m=-2}^{m=2}Y_{2,m}{}^*(\mathbf{p}\cdot\mathbf{z})Y_{2,m}(\mathbf{r}\cdot\mathbf{z}). \quad (11)$$

We define the amplitude[4] $T_2(E)$ by stating that the two-nucleon plane wave of unit amplitude forms the $J=2$, $J_z=m$ part of the isobar-nucleon state with amplitude

$$T_2(E)Y_{2,m}{}^*(\mathbf{p}\cdot\mathbf{z}). \quad (12)$$

The cross section for the process may then be evaluated in terms of $\Pi(E)$ and $T_2(E)$, and the result is[4]

$$\sigma_E(\pi^++d\rightarrow2p)=\frac{5}{9\pi}\rho_E(2N)/V_E(\pi)|D|^{-2}\Pi^2(E-m)T_2{}^2(E), \quad (13)$$

where

$$V_E(\pi)=k/\omega \quad \text{with} \quad \omega=[k^2+\mu^2]^{\frac{1}{2}}=E-2m,$$
$$\rho_E(2N)=(2/\pi)pE/2 \quad \text{with} \quad p=[\tfrac{1}{4}E^2-m^2]^{\frac{1}{2}}.$$

We solve this for $T_2(E)$, using Eq. (9) for $\Pi(E)$.

$$|T_2(E)|^2=\frac{9\pi}{10\sqrt{2}}\frac{k}{p^2}\{(E-m-E_0)^2+\tfrac{1}{4}\Gamma^2\}^{\frac{1}{2}}\times\frac{\sigma_E(\pi^++d\rightarrow2p)}{\sigma_{(E-m)}{}^{\frac{1}{2}}(\pi^++p\rightarrow\pi^++p)}$$
$$=\frac{9\pi}{10\sqrt{2}}\frac{4}{3k}\{(E-m-E_0)^2+\tfrac{1}{4}\Gamma^2\}^{\frac{1}{2}}\times\frac{\sigma_E(p+p\rightarrow\pi^++d)}{\sigma_{(E-m)}{}^{\frac{1}{2}}(\pi^++p\rightarrow\pi^++p)}, \quad (14)$$

where we have used the relation from detailed balancing:

$$\sigma_E(\pi^++d\rightarrow2p)=\tfrac{4}{3}(p^2/k^2)\sigma_E(p+p\rightarrow\pi^++d). \quad (15)$$

Our quantity Γ here is defined by Eq. (8) with $k=[(E-2m)^2-\mu^2]^{\frac{1}{2}}$.

C. Cross-Section Derivation for $p+p\rightarrow\pi^++p+n$

We are now in a position to apply the model to calculate some single and double production processes involving unbound nucleons in the final state. We consider first $p+p\rightarrow\pi^++n+p$. The transition is to the ${}^3S_1$ state of the final nucleons, because they cannot emerge in a ${}^1S_0$ state with a P-wave meson and yet conserve angular momentum and parity. The calcu-

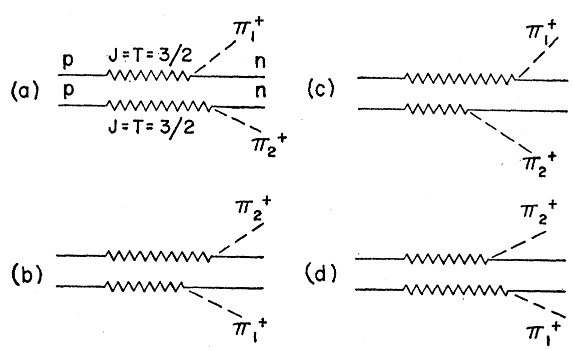

FIG. 1. Feynman graphs for double meson production through an intermediate two-isobar state.

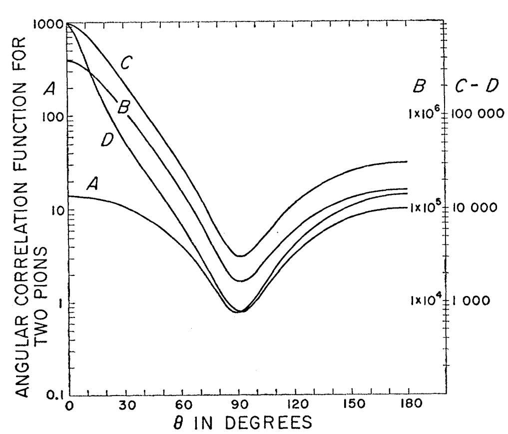

FIG. 2. (a) Plot of Eq. (35) *versus* θ with $\theta_1=45°$. (b) Plot of Eq. (35) times $\sin^2\delta/q^2$ *versus* θ with $\theta_1=45°$, $k_1=20$ Mev/c, $k_2=30$ Mev/c, $\epsilon_r=190$ Mev, $\Gamma=10$ Mev. (c) Same as (b) with $k_1=70$ Mev/c, $k_2=50$ Mev/c. (d) Same as (b) with $k_1=150$ Mev/c, $k_2=130$ Mev/c.

[6] G. F. Chew and F. Low, Phys. Rev. **101**, 1570 (1956).

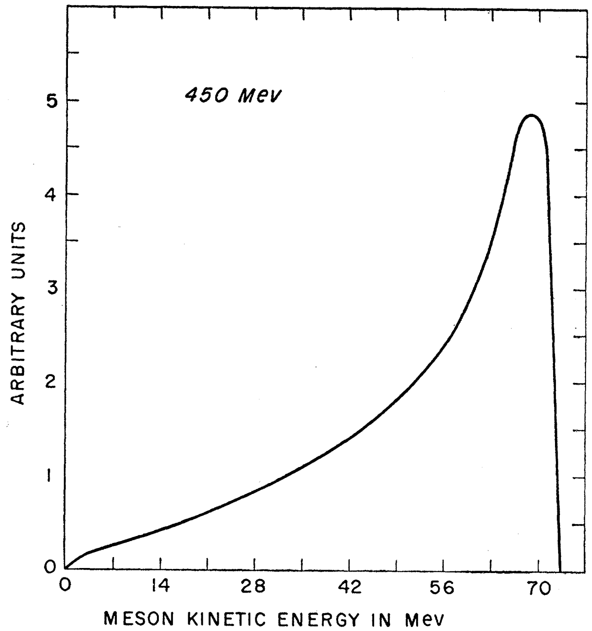

FIG. 3. Energy spectra for the meson produced in the reaction $p+p\to\pi^++n+p$ at 450 Mev. The final-state n-p force is included in terms of the low-energy triplet scattering length.

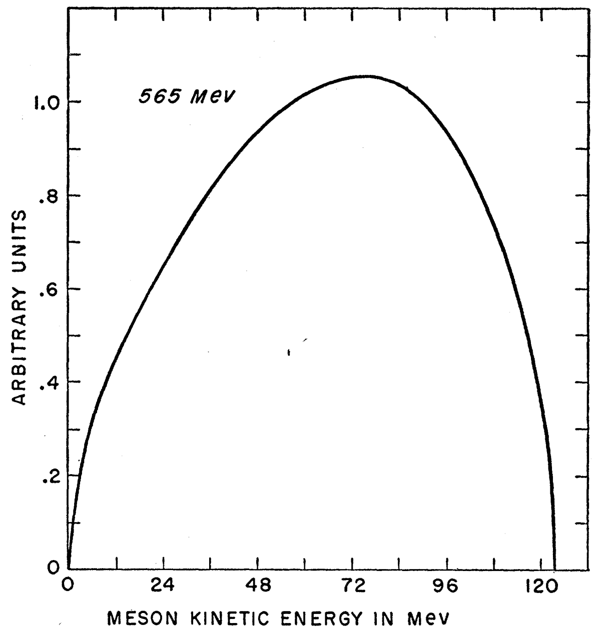

FIG. 4. Energy spectra for the meson produced in the reaction $p+p\to\pi^++n+p$ at 565 Mev.

lation proceeds in a manner similar to that of reference 4. We construct the final n-p states of $S=1$, $S_z=\pm1, 0$; $I=0$, and $I_z=0$, and from these we obtain the intermediate state of isobar and nucleon by coupling the pion to either the proton or the neutron. We then extract the $J=2$, $J_z=0$; $I=1$, and $I_z=1$ part of this state, and project it upon the initial two-proton state with an amplitude T_2', as only the ${}^1D_2$ part of the initial state conserves angular momentum and parity with the intermediate state. The quantity $|T_2'|^2=|T_2|^2|\psi_d(r=0)|^{-2}$, where $\psi_d(r=0)$ is the deuteron space function at $r=0$. (See Appendix B.) In computing the cross section for this process, we add the cross sections for the separate processes in which the pion is coupled to the proton and to the neutron to form the isobar, remembering that these two final states are distinguishable by Q-values measurements. The results for the two cross sections are

$$d\sigma_E(p+p\to\pi^++p+n)=a|T_2'(E)|^2|\Pi(E-m)|^2 \times(3\cos^2\theta+1)|D|^{-2}\frac{\rho_E(2N\pi)}{V_E(N)}\frac{d\Omega}{4\pi}, \quad (16)$$

where $a=\frac{1}{8}$ when the π^+-p result from the isobar and $a=1/72$ when the π^+-n result from the isobar. The quantity $V_E(N)={}^2P/E$. The meson angular distribution given here, as well as those presented in Sec. B for the double production processes, is obtained with neglect of the effect of the transformation from the isobar rest system to the total center-of-mass system on the meson momentum.

Before using this result, we would like to modify it to take into account somewhat the final state n-p interaction. In Appendix A, it is shown that consideration of the relative motion of the two nucleons in the final state approximately modifies the above matrix element by the multiplicative factor

$$e^{i\delta}\frac{\sin\delta}{q}\int d^3\mathbf{r}f(r)=\text{constant}\times e^{i\delta}\frac{\sin\delta}{q}, \quad (17)$$

where δ is the n-p scattering phase shift in the triplet state, q is the magnitude of the relative momentum, and $f(r)$ is a function of the magnitude of the relative coordinate between the nucleons. We approximate δ by $q\cot\delta=\alpha$ where $\alpha^{-1}=5.39\times10^{-13}$ cm is the triplet-scattering length. The density of final states $\rho_E(2N\pi)$ is given by

$$(2/\pi)k\omega(2\pi)^{-3}2\pi m^{3/2}\epsilon^{1/2}d\epsilon, \quad (18)$$

where $\epsilon=q^2/m$ is the relative energy of the nucleons. Inserting the expressions for T_2, Π, and ρ_E into Eq. (16) appropriately modified for the final-state interaction, we obtain

$$\frac{d\sigma_E(p+p\to n+p+\pi^+)}{\sigma_E(p+p\to\pi^++d)}\propto a|\psi_d(r=0)|^{-2}E/2[\tfrac{1}{4}E^2-m^2]^{1/2} \times\frac{\{(E-2m-\epsilon)^2-\mu^2\}^{1/2}\{(E-m-\frac{1}{2}\epsilon-E_0)^2+\frac{1}{4}\Gamma^2\}^{1/2}}{\{(E-2m)^2-\mu^2\}^{1/2}\{(E-m-E_0)^2+\frac{1}{4}\Gamma^2\}^{1/2}} \times\frac{m^{3/2}\epsilon^{1/2}\sigma_{E1}^{1/2}(\pi^++p\to\pi^++p)d\epsilon}{\sigma_{(E-m)}^{1/2}(\pi^++p\to\pi^++p)\{\alpha^2+m\epsilon\}}. \quad (19)$$

The proportionality sign indicates that this is the ratio to within a constant factor involving $\int d^3\mathbf{r} f(r)$. The quantity $\sigma_{E1}^{\frac{1}{2}}$ arises from the vertex at which the isobar decays. The energy E_1 is taken as equal to the sum of the kinetic energies and rest energies of the pion and nucleon resulting from the decay. If we denote the pion kinetic energy by t, then $E_1=\mu+m+t+\frac{1}{2}\epsilon$. In this expression we have set the nucleon kinetic energy equal to one-half the energy of relative motion of the two nucleons, $\frac{1}{2}\epsilon$. In this approximation we neglect the momentum of the meson and assume that the massive nucleons account for most of the conservation of momentum, and hence, tend to move off in opposite directions with about equal momenta in the total center-of-mass system. The meson energy is given by energy conservation: $t+\mu=E-2m-\epsilon$. These substitutions into the matrix element and the three-body phase space take into some account the motion of the intermediate state particles which we have neglected in the amplitudes describing the production and decay of the isobar.

We now use Eq. (10) giving $\sigma_{E1}^{\frac{1}{2}}$ as a function of meson energy $\omega=E_1-m=E-2m-\frac{1}{2}\epsilon$:

$$\sigma_{E1}^{\frac{1}{2}}(\pi^++p\rightarrow\pi^++p)=[(E-m-\tfrac{1}{2}\epsilon)^2-\mu^2]\left\{\frac{8\lambda_3^2}{\{(E-2m-\frac{1}{2}\epsilon)^2[1-(E-2m-\frac{1}{2}\epsilon)/\omega_3]^2+\lambda_3^2[(E-2m-\frac{1}{2}\epsilon)^2-\mu^2]^3\}}\right\}^{\frac{1}{2}}. \tag{20}$$

Inserting this value into Eq. (19) and putting $\epsilon=E-2m-\mu-t$, we obtain the energy spectrum of the produced pion for a given value of E. The ratio of the cross sections as a function of E is obtained by integrating over ϵ for $0\leq\epsilon\leq E-2m-\mu$. However, it must be pointed out that this ratio will be badly distorted at the higher bombarding energies by the approximation that was used in writing the amplitude $T_2'(E)$ as an energy-independent multiple of $T_2(E)$. (See Appendix B.) This approximation neglects the high-momentum components in the deuteron. In the single meson production processes at high bombarding energies (above, say, 800 Mev), the low-momentum components of the final two-nucleon system will be less important and deteron formation will not occur except in rare instances as a consequence of the high-momentum tail. At these energies the quantity $|\psi_d(r=0)|^{-2}$ should be replaced by some energy-dependent factor such that this factor times $\sigma_E(p+p\rightarrow\pi^++d)$ is approximately energy-independent. The approximation should not be too bad for single meson production below 800 Mev and for double meson production in the range 1 to 1.5 Bev, where the low-momentum components of the final two-nucleon system are still important.

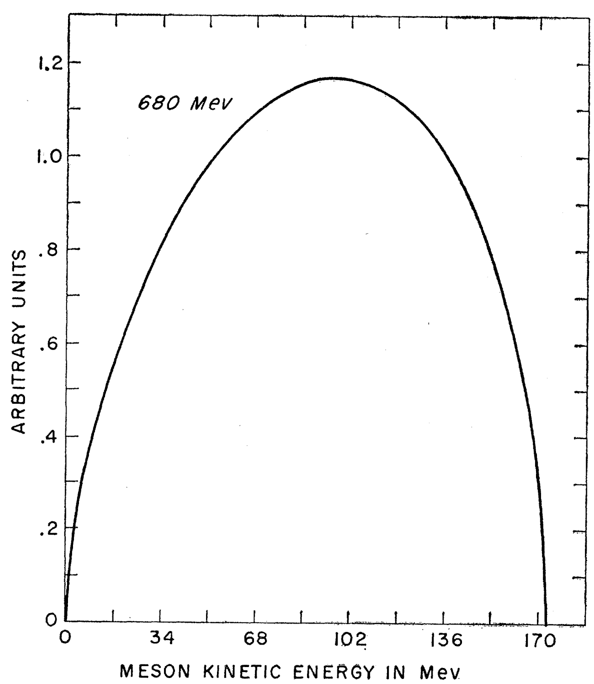

FIG. 5. Energy spectra for the meson produced in the reaction $p+p\rightarrow\pi^++n+p$ at 680 Mev.

D. Cross-Section Derivation for $p+p\rightarrow\pi^++\pi^++n+n$

Turning our attention to the double-pion production processes, we shall illustrate the method of calculation for the process $p+p\rightarrow\pi^++\pi^++n+n$. We consider only

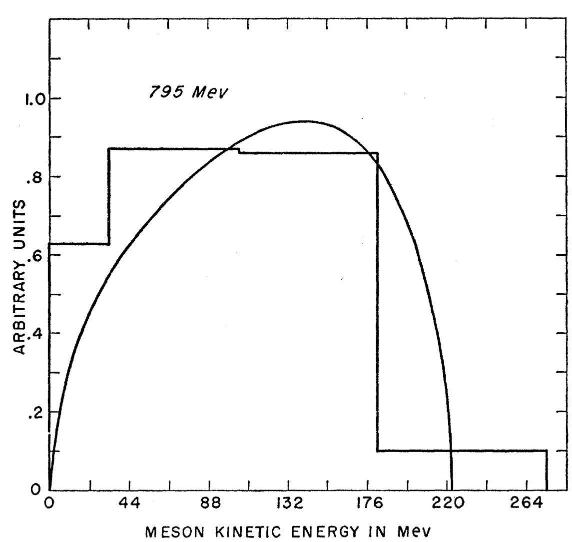

FIG. 6. energy spectra for the meson produced in the reaction $p+p\rightarrow\pi^++n+p$ at 795 Mev. Histogram is from the Brookhaven experiment at 810±100 Mev.

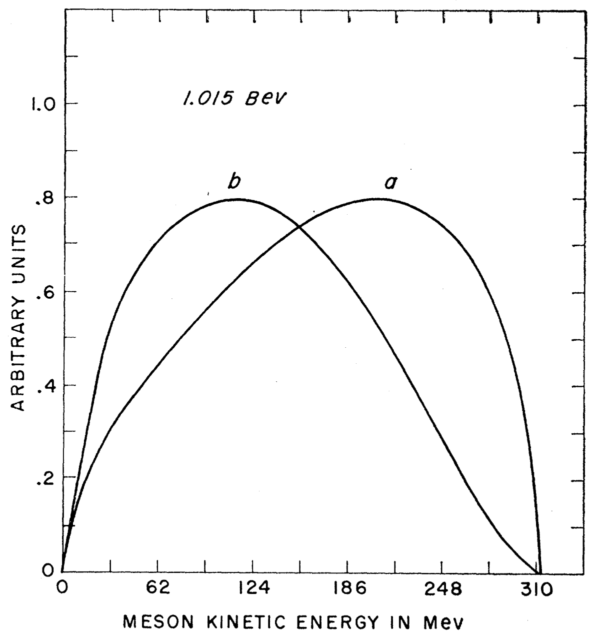

FIG. 7. Energy spectra for the meson produced in the reaction $p+p\rightarrow\pi^{+}+n+p$ at 1.015 Bev. Curve (a) nucleons are in an S state. Curve (b) nucleons are in a P state.

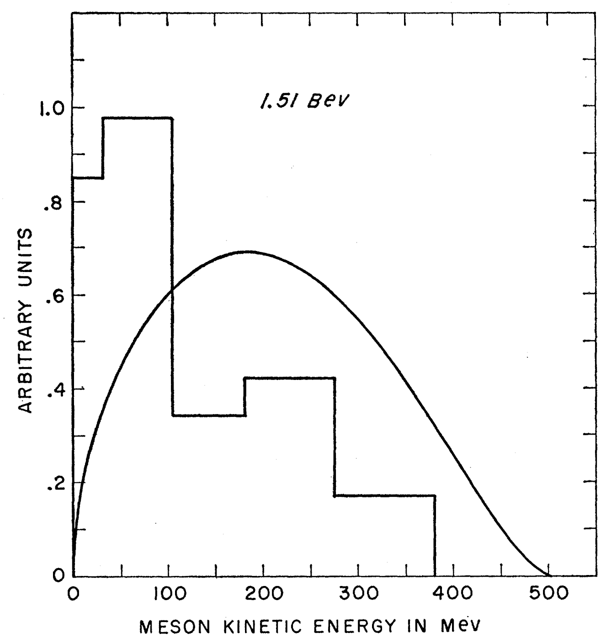

FIG. 8. Energy spectra for the meson produced in the reaction $p+p\rightarrow\pi^{+}+n+p$ at 1.51 Bev. The nucleons are in a P state. Histogram is from the Brookhaven experiment at 1.5 ± 0.1 Bev.

S states of relative motion for the final two nucleons. The final state of the $2n$ is then the ${}^1S_0$. The production process is now considered to go through an intermediate state involving two $J=\frac{3}{2}$, $I=\frac{3}{2}$ isobars. We neglect the energy of motion of the isobars and we consider them to be in an S state. We have two identical fermions in the intermediate state with total $I=1$, total $I_z=1$. These are in a symmetric space state and because the total state must be antisymmetric, we see that the total angular momentum of the intermediate state must be $J=2$ or $J=0$. Hence the ${}^1S_0$ and ${}^1D_2$ parts of the incident two-proton state couple into the two-isobar state, with amplitudes that we shall call A_0 and A_2, respectively. In Appendix B it is shown that in the approximation in which we neglect the dependence of these amplitudes on the motion of the intermediate state isobars, A_2 is related to the amplitude T_2' derived earlier, by

$$|A_2(E)|^2\sim|\psi_{2X}(R)/\psi_{X,N}(R)|^2|T_2'|^2=\beta|T_2'(E)|^2, \quad (21)$$

where $\psi_{2X}(R)$ is the two-isobar wave function evaluated at some relative coordinate R characteristic of the interaction region for meson production, $\psi_{X,N}(R)$ is the isobar-nucleon wave function, and the factor β is energy-independent. An approximation such as this is of very limited validity. It will be used only to get an idea of the behavior of the two-meson excitation function from 0.8 to about 1.3 Bev. We are not able to relate A_0 to a simpler reaction, so we leave it as an undetermined parameter. However, the amplitudes A_0 and A_2 bear the following relationships to the S-matrix elements for the corresponding transitions, $\mathcal{S}_0$ and $\mathcal{S}_2$

$$A_0=\mathcal{S}_0, \quad A_2=(i)^2 5^{\frac{1}{2}}\mathcal{S}_2. \quad (22)$$

$|A_2|^2$ carries a statistical factor of 5 relative to $|A_0|^2$.

The process to be calculated may be represented by the Feynman diagrams in Fig. 1. The energy denominators that will enter into the matrix elements may be read off from the diagrams. The first intermediate state contributes $D^{-1}=(E-2E_0-i\Gamma_0)^{-1}$. The second intermediate state contributes $D_{1,2}^{-1}=(E-E_0-\mu-m-\frac{1}{2}\epsilon-t_{1,2}-\frac{1}{2}i\Gamma_0)^{-1}$. Here the subscripts 1 and 2 refer to Diagrams a and b and Diagrams c and d respectively. In the former case, the meson $\mathbf{k}_1$ is emitted first; in the latter case, the meson $\mathbf{k}_2$ is emitted first. The $t_{1,2}$ refer to the meson kinetic energies. The kinetic energy of the nucleon in the second intermediate state is taken as $\frac{1}{2}\epsilon$, where ϵ is the energy of relative motion of the final two nucleons. The width Γ_0 is given by Eq. (8) with $k=\{[(E-2m)^2/4]-\mu^2\}^{\frac{1}{2}}$. We must now evaluate the remainder of the matrix element. The wave function for the ${}^1S_0$ state of the nucleons and the two P-wave mesons is

$$\psi=2^{-\frac{1}{2}}(N_1^{\frac{1}{2},-\frac{1}{2}}N_2^{-\frac{1}{2},-\frac{1}{2}}-N_1^{-\frac{1}{2},-\frac{1}{2}}N_2^{\frac{1}{2},-\frac{1}{2}})\sum_{m,m'}Y_{1,m}^*(\mathbf{k}_1\cdot\mathbf{z})Y_{1,m'}^*(\mathbf{k}_2\cdot\mathbf{z})Y_{1,m}(\mathbf{r}_1\cdot\mathbf{z})Y_{1,m'}(\mathbf{r}_2\cdot\mathbf{z}). \quad (23)$$

We combine meson $\mathbf{k}_1$ with nucleon 1 in the $J=I=\frac{3}{2}$ state, and meson $\mathbf{k}_2$ with nucleon 2 in this state, with amplitudes $\Pi(E_1)$ and $\Pi(E_2)$, respectively. To take into account Diagrams *a* and *b* of Fig. 1 (for a particular time ordering), we also combine meson $\mathbf{k}_1$ with nucleon 2 and meson $\mathbf{k}_2$ with nucleon 1. We then take the $J=2$ and 0, $I=1$, $I_z=1$ parts of this two-isobar state and project them on the initial two-proton state with amplitudes A_2 and A_0, respectively. The matrix element is then

$$\begin{aligned} M(m,m') = {} & [\Pi(E_1)\Pi(E_2)A_2 Y_{1,m}^*(\mathbf{k}_1\cdot\mathbf{z})Y_{1,m}^*(\mathbf{k}_2\cdot\mathbf{z})D^{-1}D_1^{-1}2^{-1}\{\langle 1,\tfrac{1}{2},m,\tfrac{1}{2}|\tfrac{3}{2},m+\tfrac{1}{2}\rangle\langle 1,\tfrac{1}{2},m',-\tfrac{1}{2}|\tfrac{3}{2},m'-\tfrac{1}{2}\rangle \\ & \times\langle\tfrac{3}{2},\tfrac{3}{2},m+\tfrac{1}{2},m'-\tfrac{1}{2}|2,0\rangle+\langle 1,\tfrac{1}{2},m',\tfrac{1}{2}|\tfrac{3}{2},m'+\tfrac{1}{2}\rangle\langle 1,\tfrac{1}{2},m,-\tfrac{1}{2}|\tfrac{3}{2},m-\tfrac{1}{2}\rangle\langle\tfrac{3}{2},\tfrac{3}{2},m'+\tfrac{1}{2},m-\tfrac{1}{2}|2,0\rangle \\ & -\langle 1,\tfrac{1}{2},m,-\tfrac{1}{2}|\tfrac{3}{2},m-\tfrac{1}{2}\rangle\langle 1,\tfrac{1}{2},m',\tfrac{1}{2}|\tfrac{3}{2},m'+\tfrac{1}{2}\rangle\langle\tfrac{3}{2},\tfrac{3}{2},m-\tfrac{1}{2},m'+\tfrac{1}{2}|2,0\rangle-\langle 1,\tfrac{1}{2},m',-\tfrac{1}{2}|\tfrac{3}{2},m'-\tfrac{1}{2}\rangle \\ & \times\langle 1,\tfrac{1}{2},m,\tfrac{1}{2}|\tfrac{3}{2},m+\tfrac{1}{2}\rangle\langle\tfrac{3}{2},\tfrac{3}{2},m'-\tfrac{1}{2},m+\tfrac{1}{2}|2,0\rangle\}\{\langle 1,\tfrac{1}{2},1,-\tfrac{1}{2}|\tfrac{3}{2},\tfrac{1}{2}\rangle^2\langle\tfrac{3}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}|1,1\rangle\} \\ & +\text{same expression with } D_1^{-1}\to D_2^{-1}]+\text{same expression with } A_2\to A_0 \text{ and } J=2\to J=0 \\ & \text{in the Clebsch-Gordan coefficients.} \end{aligned} \tag{24}$$

In the line written out explicitly, the first bracket contains the angular-momentum vector addition and the second bracket contains the isotopic-spin vector addition. It should be noted that the angular-momentum bracket is antisymmetric with respect to the interchanges $m\leftrightarrow m'$, $s=\frac{1}{2}\leftrightarrow s'=-\frac{1}{2}$, which are equivalent to interchanging the spin z components of the two isobars. The total transition amplitude is obtained by summing the expression (24) over the spin z components, σ and σ', of each intermediate state isobar, subject to $\sigma+\sigma'=J_z=0$. This is equivalent to summing over m and m' in Eq. (36), subject to $m+m'=0$. The result for the transition amplitude is

$$\begin{aligned} M = {} & \Pi(E_1)\Pi(E_2)A_2(E)D^{-1}\{D_1^{-1}+D_2^{-1}\}\{\tfrac{1}{3}(2/5)^{\frac{1}{2}}\}(3/4\pi)\{\tfrac{2}{3}\cos\theta_1\cos\theta_2+\tfrac{1}{3}\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\} \\ & +\Pi(E_1)\Pi(E_2)A_0(E)D^{-1}\{D_1^{-1}+D_2^{-1}\}\{\tfrac{1}{3}(2/5)^{\frac{1}{2}}\}(3/4\pi)\{-\tfrac{2}{3}\cos\theta_1\cos\theta_2+\tfrac{2}{3}\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\}. \end{aligned} \tag{25}$$

The amplitude is properly symmetric in the two final-state mesons. The quantities $\Pi(E_1)$ and $\Pi(E_2)$ are the isobar decay amplitudes evaluated at the total energies of the resulting pion-nucleon systems, $E_1=\mu+m+t_1+\frac{1}{2}\epsilon$, and $E_2=\mu+m+t_2+\frac{1}{2}\epsilon$, respectively. The square of the transition amplitude is given by

$$\begin{aligned} |M|^2 = {} & (2/45)(3/4\pi)^2(1/9)|\Pi(E_1)|^2|\Pi(E_2)|^2|D|^{-2}\{|D_1|^{-2}+|D_2|^{-2}+2\,\mathrm{Re}(D_1D_2)^{-1}\} \\ & \times[|A_2|^2\{4\cos^2\theta_1\cos^2\theta_2+\sin^2\theta_1\sin^2\theta_2\cos^2(\phi_1-\phi_2)+\sin(2\theta_1)\sin(2\theta_2)\cos(\phi_1-\phi_2)\} \\ & \times|A_0|^2\{4\cos^2\theta_1\cos^2\theta_2+4\sin^2\theta_1\sin^2\theta_2\cos^2(\phi_1-\phi_2)-2\sin(2\theta_1)\sin(2\theta_2)\cos(\phi_1-\phi_2)\} \\ & +2|A_2||A_0|\cos\Delta\{-4\cos^2\theta_1\cos^2\theta_2+2\sin^2\theta_1\sin^2\theta_2\cos^2(\phi_1-\phi_2)+\tfrac{1}{2}\sin(2\theta_1)\sin(2\theta_2)\cos(\phi_1-\phi_2)\}]. \end{aligned} \tag{26}$$

Here $\Delta=\delta_0-\delta_2$, the difference in the phases of the amplitudes A_0 and A_2. The differential cross section is obtained by multiplying Eq. (26) by $2\pi\rho_E(2\pi,2N)/V_E(N)$ where

$$\rho_E(2\pi,2N)=(2/\pi)^2k_1\omega_1d\omega_1k_2\omega_2d\omega_2(2\pi)^{-2}m^{\frac{3}{2}}\epsilon^{\frac{1}{2}}d\epsilon d\Omega_1d\Omega_2. \tag{27}$$

The salient features of the angular distribution of either meson are best seen by integrating over one of the solid-angle elements. The angular dependence is then given by

$$\sigma(\theta)\propto|A_2|^2\{3\cos^2\theta+1\}+8|A_0|^2+4|A_2||A_0|\cos\Delta\{1-3\cos^2\theta\}. \tag{28}$$

Little can be said about the interference term because the phase factor, $\cos\Delta$, is not given by this analysis. We see that if $A_0=0$, then the angular distribution of each meson is given by the familiar $1+3\cos^2\theta$ characteristic of emission from a $J=\frac{3}{2}$ state. On the other hand, if $A_2=0$, the angular distribution of each meson is isotropic, as it must be for any initial state with $J=0$. Neglecting the interference term and using Eq. (22) to write $\sigma(\theta)$ in terms of the S-matrix elements, we have

$$\sigma(\theta)\propto 5|S_2|^2+8|S_0|^2+15|S_2|^2\cos^2\theta. \tag{29}$$

If $|S_0|\sim|S_2|$, the angular distribution can be somewhat more isotropic than that which is characteristic of the $J=\frac{3}{2}$ state.

We now discuss the energy spectrum of the mesons and the total cross section as a function of energy on the basis of Eq. (26). For simplicity, the angular integrations are performed with A_0 set equal to zero. Substituting from Eqs. (9), (14), and (21) into Eq. (26), we obtain

$$\begin{aligned} V_E(N)d^2\sigma(p+p\to\pi^++\pi^++n+n)/d\epsilon dt_1 \propto {} & \beta|\psi_d(r=0)|^{-2}\{t_1(t_1+2\mu)(E-2m-2\mu-t_1-\epsilon)(E-2m-t_1-\epsilon)\epsilon\}^{\frac{1}{2}} \\ & \times\{(E-2m)^2-\mu^2\}^{-\frac{1}{2}}\{(E-m-E_0)^2+\tfrac{1}{4}\Gamma^2\}^{\frac{1}{2}}\{(E-2E_0)^2+\Gamma_0^2\}^{-1}[\{(E-E_0-t_1-\tfrac{1}{2}\epsilon-\mu-m)^2+\tfrac{1}{4}\Gamma_0^2\}^{-1} \\ & +\{-E_0+m+\mu+t_1+\tfrac{1}{2}\epsilon)^2+\tfrac{1}{4}\Gamma_0^2\}^{-1}+2\{(E-E_0-t_1-\tfrac{1}{2}\epsilon-\mu-m)(-E_0+m+\mu+t_1+\tfrac{1}{2}\epsilon)+\tfrac{1}{4}\Gamma_0^2\} \\ & \times\{(E-E_0-t_1-\tfrac{1}{2}\epsilon-\mu-m)^2+\tfrac{1}{4}\Gamma_0^2\}^{-1}\{(-E_0+m+\mu+t_1+\tfrac{1}{2}\epsilon)^2+\tfrac{1}{4}\Gamma_0^2\}^{-1}]\{(t_1+\mu+m+\tfrac{1}{2}\epsilon-E_0)^2+\tfrac{1}{4}\Gamma_1^2\}^{\frac{1}{2}} \\ & \times\{(E-t_1-\tfrac{1}{2}\epsilon-\mu-m-E_0)^2+\tfrac{1}{4}\Gamma_2^2\}^{\frac{1}{2}}\sigma_{E1}^{\frac{1}{2}}(\pi^++p\to\pi^++p)\sigma_{E2}^{\frac{1}{2}}(\pi^++p\to\pi^++p) \\ & \times\sigma_E(p+p\to\pi^++d)/\sigma_{(E-m)}^{\frac{1}{2}}(\pi^++p\to\pi^++p). \end{aligned} \tag{30}$$

In this expression $\sigma_{E1,2}(\pi^+ + p \to \pi^+ + p)$ is given by Eq. (10) with $E_1 = t_1 + \frac{1}{2}\epsilon + \mu + m$, $E_2 = E - m - \mu - t_1 - \frac{1}{2}\epsilon$, and

$$k_1^2 = (t_1 + \tfrac{1}{2}\epsilon)(t_1 + \tfrac{1}{2}\epsilon + 2\mu), \quad k_2^2 = (E - 2m - 2\mu - t_1 - \tfrac{1}{2}\epsilon)(E - 2m - t_1 - \tfrac{1}{2}\epsilon).$$

The $\Gamma_{1,2}$ are defined by Eq. (8) with $k = k_{1,2}$, respectively. The quantities Γ and Γ_0 are defined by Eq. (8) with $k = \{(E-2m)^2 - \mu^2\}^{\frac{1}{2}}$ and $k = \{[(E-2m)^2/4] - \mu^2\}^{\frac{1}{2}}$, respectively.

The energy spectra are obtained by integrating over ϵ for $0 \leq \epsilon \leq E - 2m - 2\mu - t_1$, and plotting the resulting function *vs* t_1 for given values of the total energy E. Finally one integrates over t_1 for $0 \leq t_1 \leq E - 2m - 2\mu$ to obtain $\sigma(E)$. It should be noted that in the energy region above 1 Bev, where double meson production begins to occur to some extent, $\sigma(p + p \to \pi^+ + d)$ is essentially energy-independent. The right-hand side of Eq. (44) should then give the shape of the double-production cross section as a function of the total energy E. For this purpose $\sigma_{(E-m)^{\frac{1}{2}}}(\pi^+ + p \to \pi^+ + p)$ will be evaluated from the experimental work of Yuan and Lindenbaum.[2] As a final test of the analysis, the ratio of double to single production as a function of energy may be obtained by using Eqs. (19) and (30). Normalizing this ratio to its experimental value at 1.5 Bev, we may calculate it at lower and at higher energies and compare the results with those of recent experiments at around 800 Mev and 2 Bev.

In a similar manner, we may calculate the processes

$$\begin{aligned} &(a)\ p + p \to p + p + \pi^0 + \pi^0, \\ &(b)\ p + p \to p + p + \pi^+ + \pi^+, \\ &(c)\ p + p \to n + p + \pi^+ + \pi^0, \end{aligned} \tag{31}$$

whose relative weights were given in Sec. A. In connection with Process (c), which is predicted by the model to be the dominant one, and which, experimentally, seems to be so, we note that the transition amplitude into the $^3S_1$ state of the final nucleons tends to be suppressed. This condition results from several factors. The total mesonic isotopic spin must be one. In the approximation in which we may consider the center-of-mass of the two-meson system to be in an S-state, the relative orbital angular momentum of the two mesons gives their total angular momentum in the total center-of-mass system. However, the odd-orbital states are forbidden by parity conservation, and the even-orbital states are forbidden by the requirement of symmetry for the two-meson wave function. The effect implies that deuteron formation in the final state of reaction (c) should be suppressed. Such an absence of deuteron formation is not contradicted by the present preliminary experimental results.

E. Modification of the Angular Distribution for Two Pions Produced by a Meson-Meson Interaction

In making this phenomenological analysis of the meson-production problem, we have so far taken into account the strong pion-nucleon interaction in the $J = I = \frac{3}{2}$ state and the nucleon-nucleon final-state interaction in the S state. There is also the possibility that a meson-meson interaction may affect the momentum and angular distributions of the final-state pions. In this section, we introduce a hypothetical effect of a particular form, and observe its consequences for the angular correlation of the two mesons. In the same manner in which we separate out the relative

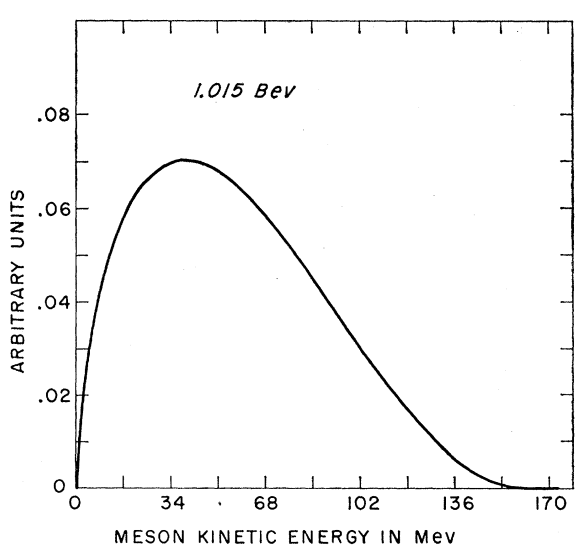

FIG. 9. Energy spectra of the mesons produced in $p + p \to 2N + 2\pi$ at 1.015 Bev with final nucleons in an S state.

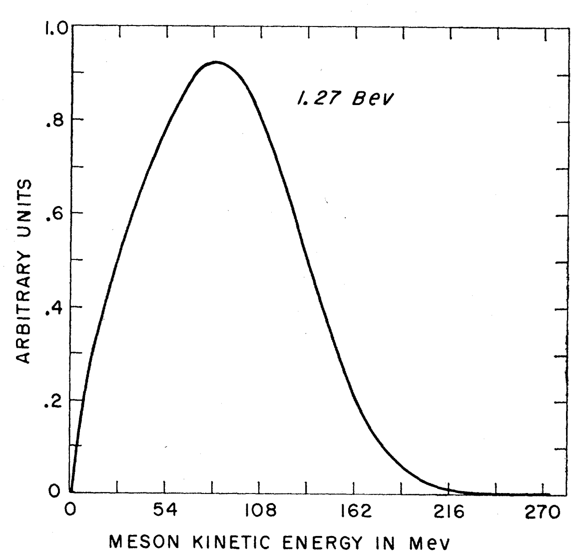

FIG. 10. Energy spectra of the mesons produced in $p + p \to 2N + 2\pi$ at 1.27 Bev with final nucleons in an S state.

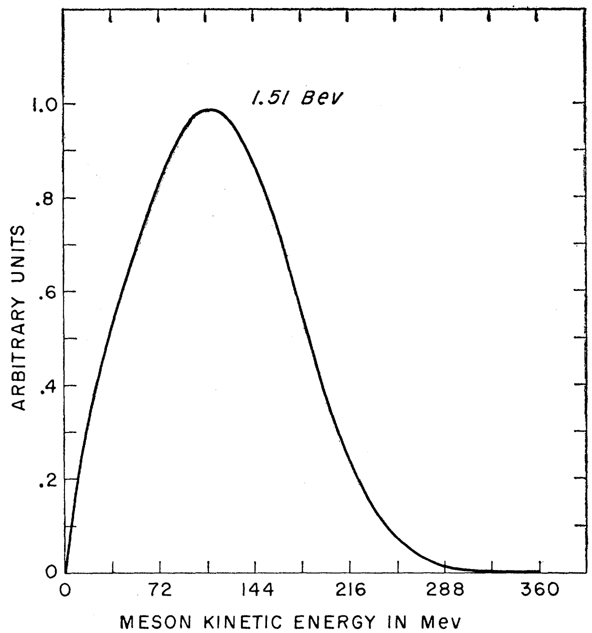

FIG. 11. Energy spectra of the mesons produced in $p+p\rightarrow 2N+2\pi$ at 1.51 Bev with final nucleons in an S state.

motion of the two final-state nucleons from the matrix element, and replace the plane wave by a wave function of the form $e^{i\delta}\sin\delta f(r)/q$ for an S state, we may separate out the relative motion of the two final-state mesons and describe the interaction in terms of a similar modification of the matrix element, where δ is now the meson-meson S-wave phase shift and q is the relative momentum of the two mesons. This separation of the nucleon-nucleon and meson-meson final-state effects is the simplest manner of observing the modifications brought about by each. In reality, the two effects may interfere, and one may be obscured by the other.

For the meson-meson phase shift, we choose a Breit-Wigner form

$$\sin\delta=\frac{\Gamma/2}{\{\epsilon-\mu-\epsilon_r\}-i\Gamma/2}, \qquad (32)$$

where ϵ_r is the resonant kinetic energy in the meson-meson center-of-mass system, Γ is the resonance width, and ϵ is the total energy of either of the two mesons in their center-of-mass system. The ϵ is given in terms of the meson momenta in the total center-of-mass system, k_1 and k_2, and the angle between them, $\theta_{1,2}$, by

$$4\epsilon^2=\{\omega_1+\omega_2\}^2-k_1^2-k_2^2-2k_1k_2\cos\theta_{1,2}, \qquad (33)$$

where

$$\omega_{1,2}=\{k_{1,2}^2+\mu^2\}^{\frac{1}{2}}.$$

Also

$$q=\{k_1^2+k_2^2-2k_1k_2\cos\theta_{1,2}\}^{\frac{1}{2}}.$$

The modification of the angular distribution is then in the factor $\sin^2\delta/q^2$. The angular function of the double-production cross section is given by Eq. (26) in terms of the polar angles of each meson with respect to the direction of the incident nucleon.

$$\sigma(\theta_1\phi_1,\theta_2\phi_2)\propto\{4\cos^2\theta_1\cos^2\theta_2+\sin^2\theta_1\sin^2\theta_2\cos^2(\phi_1-\phi_2) +\sin(2\theta_1)\sin(2\theta_2)\cos(\phi_1-\phi_2)\}d\Omega_1d\Omega_2. \qquad (34)$$

An exercise in spherical trigonometry allows one to transform this distribution into a function of the polar angle, θ_1, of one meson with respect to the incident direction, and the polar angle, θ, of a second meson with respect to the first.

The result is

$$\sigma(\theta_1,\theta)\propto\{4\cos^4\theta_1\cos^2\theta+\sin^4\theta_1\cos^2\theta +2.5\sin^2\theta_1\cos^2\theta_1\sin^2\theta +\sin^2\theta_1(2\cos^2\theta-\sin^2\theta)\}d\Omega_1d\Omega. \qquad (35)$$

In Fig. 2 we plot this function, as well as the function modified by $\sin^2\delta/q^2$, *vs* θ for $\theta_1=45°$, with parameters, and $\epsilon_r=190$ Mev and $\Gamma=10$ Mev, for several pairs of meson-momenta values. Because the mesons account for a portion of the momentum conservation, there is a kinematic tendency for angles $\theta>90°$. This effect has not been taken into account in the figure.

DISCUSSION

Figures 3 to 8 contain the energy spectra for the pion produced in the process $p+p\rightarrow n+p+\pi^+$. The final-state nucleon-nucleon interaction is included in the 450-Mev spectrum, but is neglected in the spectra at the higher energies. This final state interaction should be most important when $\hbar/q\gg a$,[7] where q is the relative

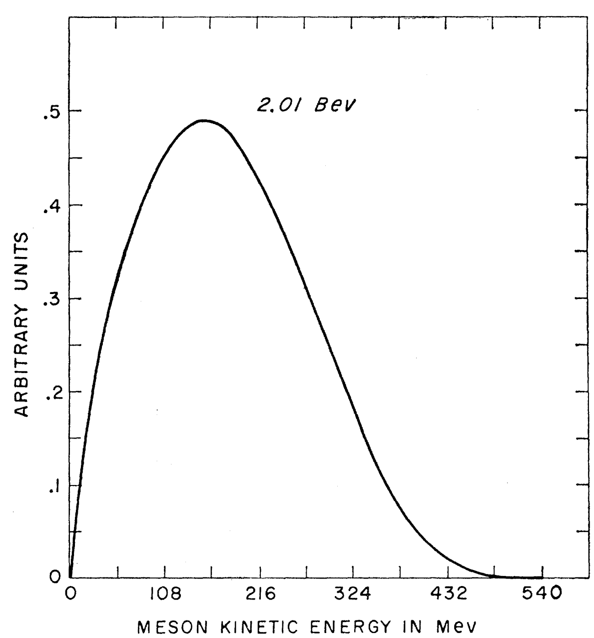

FIG. 12. Energy spectra of the mesons produced in $p+p\rightarrow 2N+2\pi$ at 2.01 Bev with final nucleons in an S state.

[7] K. M. Watson, Phys. Rev. 88, 1163 (1952).

momentum of the final nucleons and a is the radius of the region of the primary interaction (excitation of the isobar). Experiment seems to indicate for the proton a region of very strong interaction of radius $\sim 0.5\times 10^{-13}$ cm surrounded by a region of weaker interaction of $\sim 1\times 10^{-13}$ cm.[8] Blokhintsev[9] has termed the former region the kernel of the nucleon, and the latter region, the meson shell. In nucleon-nucleon collisions, one may speak of interactions between the meson shells, between a meson shell and a kernel, and between the kernels. Barring a strong meson-meson effect the first-mentioned interaction, which is of the longest range, is probably not responsible for the excitation of the isobaric states. However, the interaction between the meson shell of one nucleon and the kernel of the second may account for this excitation. The region of the primary interaction may therefore be $\sim 1\times 10^{-13}$ cm. The final-state nucleon-nucleon interaction should play a decreasing role in the reaction as the bombarding energy is raised and q may be $\sim\mu$. In any event, in this region the interaction will not be describable in terms of the low-energy scattering parameters. The curve labeled (b) in Fig. 7 and the curve in Fig. 8 represent the meson spectra when the final nucleons are in a P state. These are obtained by replacing $\epsilon^{\frac{1}{2}}$ in the phase space of the final nucleons by $\epsilon^{\frac{3}{2}}$. In Figs. 6 and 8 the experimental histograms at 810 Mev[10] and 1.5 Bev,[11] respectively, are superposed on the theoretical spectra. Agreement, especially at the lower energy, appears to be fair. At 1.5 Bev a certain amount of double production may be included in the histogram.[11] Further experimental evidence on the shape of the energy spectra in the region 0.5 to 1 Bev is supplied by the Russian experiments on the process $p+p\rightarrow n+p+\pi^+$ at 560 and 660 Mev.[12] The mean energies of the experimental spectra were 82 and 110 Mev, respectively. These are to be compared with the peak energies in Figs. 4 and 5 of about 75 and 100 Mev, respectively. The neglected nucleon-nucleon interaction in the final state would tend to raise these peak energies somewhat. Figures 9 to 12 contain the energy spectra for the pions produced in the processes $p+p\rightarrow 2N+2\pi$ with the final nucleons in an S state. These spectra all exhibit pronounced peaks at relatively low meson kinetic energies. Such a marked preference for the emission of low-energy mesons is indeed one of the striking features of the experimental situation in the Bev range.[1,13] Another striking feature

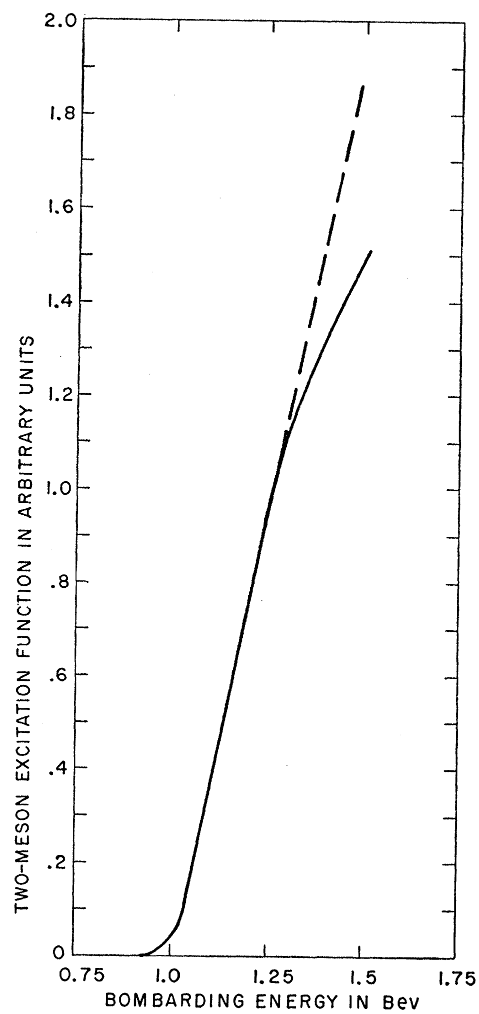

FIG. 13. Two-pion excitation function *versus* bombarding energy. The curvature of the upper portion of the solid curve is probably due to the neglect of the energy dependence of the factor $|\psi_{X,N}(R)|^{-2}$ which is not justified at the higher bombarding energies. The dashed line is an extrapolation of the essentially linear portion of the solid curve.

of the experiments, the rapid increase above 1 Bev of the double meson production processes,[13,14] is evidenced in Fig. 13, where we have plotted a rough estimate of the two-meson excitation function *versus* bombarding energy. A rough estimate of the ratio of two-pion to one-pion production cross sections *versus* bombarding energy is plotted in Fig. 14.

The situation as to the charge ratios has been covered in some detail in the recent series of papers by the workers at the Cosmotron.[13] We mention only that the charge ratios in double production are not yet well established. In single production, the isobar model predicts a π^+/π^0 ratio of 5. The experimental ratio at present is between 5 and 17. In connection with this ratio, it should be remembered that the process $p+p\rightarrow$

[8] Eisberg, Fowler, Lea, Shephard, Shutt, Thorndike, and Whittemore, Phys. Rev. **97**, 797 (1955).
[9] D. Blokhintsev, *Proceedings of the CERN Conference, Geneva, 1956* (European Organization of Nuclear Research, Geneva, 1956).
[10] Morris, Fowler, and Garrison, Phys. Rev. **103**, 1472 (1956).
[11] Fowler, Shutt, Thorndike, and Whittemore, Phys. Rev. **103**, 1479 (1956).
[12] Mescheryakov, Zrelov, Neganov, Vzorov, and Shabudin, *Proceedings of the CERN Conference, Geneva, 1956* (European Organization of Nuclear Research, Geneva, 1956).
[13] Fowler, Shutt, Thorndike, Whittemore, Cocconi, Hart, Block, Harth, Fowler, Garrison, and Morris, Phys. Rev. **103**, 1489 (1956).
[14] Chen, Leavitt, and Shapiro, Phys. Rev. **103**, 211 (1956).

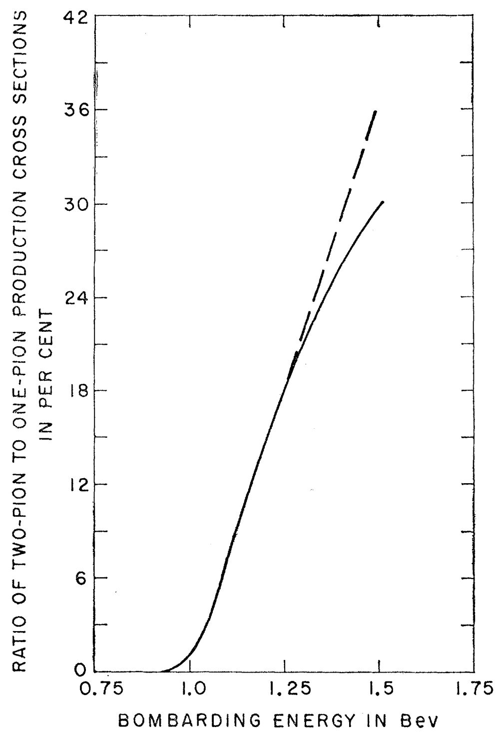

FIG. 14. Ratio of two-pion to one-pion cross sections as a function of bombarding energy. The ratio is normalized by 30% at 1.5 Bev. The dashed line is an extrapolation of the essentially linear portion of the solid curve.

$p+p+\pi^0$ is forbidden by the requirements of angular momentum conservation and the Pauli exclusion principle whenever the meson is in a P state and the final nucleons are in an S state.[5] The process will be suppressed during that portion of the time in which the final nucleons are in an S state.

Little can be said as to meson-meson effects between two final-state low-energy pions. A marked forward-backward asymmetry in the angle between the mesons might be detectable if enough events could be observed. No such correlation was observed in the few double-production events analyzed in the early Brookhaven work.[15]

In conclusion, we may say that the current experiments on meson production in the Bev range seem to strongly indicate an important role is being played by the $J=I=\frac{3}{2}$ isobar. This is particularly true when examining meson-energy spectra and Q-values between final meson-nucleon pairs. The mechanism of excitation is unknown, but it seems to involve a sort of peripheral collision in which the final nucleons retain much of their incident momenta. The calculations presented here are meant to give a rough idea of how far this model can go in correlating the data on meson production.

ACKNOWLEDGMENTS

I would like to thank Dr. Joseph V. Lepore for his guidance during the course of these studies. It is a pleasure to thank Dr. William B. Fowler for several stimulating discussions. I am indebted to Mr. James Baker and Mr. Robert Freeman for their aid in getting the computations done.

APPENDIX A

We here separate the nucleon-nucleon final-state interaction from the meson-production matrix element. If we consider single-meson production, the part of the total transition amplitude in which we are interested is the matrix element that describes the transition from the isobar-nucleon intermediate state $|\psi_{X,N}\rangle$ to the meson-two-nucleon final state, $|\psi_{N1,N2,\pi}\rangle$. We shall call the operator that brings about this transition U. It operates on the isobar with momentum $\mathbf{s}$ and energy E_s to form a meson of momentum $\mathbf{k}$ and energy ω and a nucleon of momentum $\mathbf{p}_1$ and energy E_1. The second nucleon has momentum $\mathbf{p}_2$ and energy E_2 in the final state, and $\mathbf{p}'$ and E' in the intermediate state. With all particles in plane-wave states, the matrix element is the following:

$$\begin{aligned}\langle\psi_{N1,N2,\pi}|U|\psi_{X,N}\rangle &= \int d^3\mathbf{r}d^3\mathbf{r}'\langle\psi_{N1,N2,\pi}|\mathbf{r}\rangle\langle\mathbf{r}|U|\mathbf{r}'\rangle\langle\mathbf{r}|\psi_{X,N}\rangle \\ &= \int d^3\mathbf{r}_\pi d^3\mathbf{r}_{N1}d^3\mathbf{r}_{N2}d^3\mathbf{r}'d^3\mathbf{r}_X \exp[-i(\mathbf{k}\cdot\mathbf{r}_\pi+\mathbf{p}_1\cdot\mathbf{r}_1+\mathbf{p}_2\cdot\mathbf{r}_2)] \\ &\qquad\times\exp[i(\mathbf{s}\cdot\mathbf{r}_X+\mathbf{p}'\cdot\mathbf{r}')]\delta(\mathbf{r}_1-\mathbf{r}_X)\delta(\mathbf{r}_\pi-\mathbf{r}_X)\delta(\mathbf{r}_2-\mathbf{r}')\langle U\rangle \\ &= (2\pi)^6\delta(\mathbf{p}_2-\mathbf{p}')\delta(\mathbf{s}-\mathbf{p}_1-\mathbf{k})\langle U\rangle. \end{aligned} \quad \text{(A-1)}$$

In the second line, the notation for the position vectors of the particles is self-evident. The δ functions arise from the definition of an operator U in the $\mathbf{r}$ representation and the assumed local nature of the interaction. The quantity $\langle U\rangle$ is the isobar-decay operator between initial and final states after the space dependence has been extracted. This quantity is, in general, a function of $\mathbf{k}$, ω, $\mathbf{p}_1$, E_1, $\mathbf{s}$, E_s and, possibly, even the spin orientation of the isobar and the resulting nucleon. In this work the isobar motion has been neglected, and we have considered this function to be given by our quantity $\Pi(E)$, with $E=\omega+E_1$, times a P-wave spherical harmonic for the meson. The relative

[15] Fowler, Shutt, Thorndike, and Whittemore, Phys. Rev. 95, 1026 (1954).

motion of the final-state nucleons may be separated out from Eq. (1) by writing the matrix element in terms of $\mathbf{R}=\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$, $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$; $\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2$, and $\mathbf{q}=\frac{1}{2}(\mathbf{p}_1-\mathbf{p}_2)$. The factor $e^{-i\mathbf{q}\cdot\mathbf{r}}$, which represents the plane-wave relative motion of the particles, is then replaced by an S state wave function of the form

$$\psi(r)\propto e^{i\delta}\frac{\sin\delta}{q}f(r). \tag{A-2}$$

The matrix element now becomes the following:

$$\begin{aligned}&(2\pi)^3\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1-\mathbf{p}')\int d^3\mathbf{r}_\pi d^3\mathbf{r}_1 d^3\mathbf{r}_2 d^3\mathbf{r}' d^3\mathbf{r}_X \exp[-i\mathbf{P}\cdot(\mathbf{r}_1+\mathbf{r}_2)/2]\exp(-i\mathbf{k}\cdot\mathbf{r}_\pi)e^{i\delta}\frac{\sin\delta}{q}\\ &\times f(|\mathbf{r}_1-\mathbf{r}_2|)\exp[i(\mathbf{s}\cdot\mathbf{r}_X+\mathbf{p}'\cdot\mathbf{r}')]\delta(\mathbf{r}_1-\mathbf{r}_X)\delta(\mathbf{r}_2-\mathbf{r}')\delta(\mathbf{r}_\pi-\mathbf{r}_X)\langle U\rangle\\ &\qquad=(2\pi)^3\delta(\mathbf{p}_2-\mathbf{p}')e^{i\delta}\frac{\sin\delta}{q}\int d^3\mathbf{R}d^3r\exp[i(\mathbf{s}-\mathbf{k}-\mathbf{p}_1)\cdot\mathbf{R}]\exp[i(\mathbf{s}-\mathbf{k}-\mathbf{p}_2)\cdot(\mathbf{r}/2)]f(r)\langle U\rangle\\ &\qquad=(2\pi)^6\delta(\mathbf{p}_2-\mathbf{p}')\delta(\mathbf{s}-\mathbf{k}-\mathbf{p}_1)e^{i\delta}\frac{\sin\delta}{q}\langle U\rangle\int f(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3\mathbf{r}.\end{aligned} \tag{A-3}$$

Comparing this result with the last line of Eq. (A-1), we see that the matrix element is modified by the factor $e^{i\delta}[(\sin\delta)/q]f(q)$, where

$$f(q)=\int f(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3\mathbf{r}\sim\int f(r)d^3\mathbf{r}=\text{constant}. \tag{A-4}$$

APPENDIX B

In Sec. B, we defined an amplitude $T_2(E)$ for the transition from a $^1D_2$ two-nucleon state of energy, E, to an isobar-nucleon state. This amplitude was defined in terms of the cross section, $\sigma_E(\pi^++d\to 2p)$, and the previously defined amplitude, $\Pi(E)$, for creation of the isobar. In order to modify $T_2(E)$ so that we may use it in calculating $\sigma_E(p+p\to\pi^++n+p)$, we need to observe that, by defining it in terms of the deuteron mesodisintegration, we include it in a factor of the deuteron wave function evaluated at the origin, $\psi_d(r=0)$. This is because the production of an isobar-nucleon intermediate state involves a matrix element of the form

$$\langle\pi\psi_d|U|\psi_{X,N}\rangle=\int d^3\mathbf{p}\langle\pi\psi_d|\mathbf{p}\rangle\langle\mathbf{p}|U|X,N\rangle, \tag{B-1}$$

where the integration is over the momentum distribution of the deuteron. The matrix element $\langle\mathbf{p}|U|\psi_{X,N}\rangle$ is $\Pi(E')$ at the total energy E' of the incident pion and the nucleon within the deuteron that absorbs it to form the isobar. If we neglect the deuteron momentum distribution, Eq. (1) becomes

$$\Pi(E)\int d^3\mathbf{p}\langle\psi_d|\mathbf{p}\rangle=\Pi(E)\psi_d(r=0), \tag{B-2}$$

where E is now the energy of the pion-nucleon system when one considers the nucleon to be at rest. For use in calculating the unbound reaction, we define $|T_2'(E)|^2=|T_2(E)|^2|\psi_d(r=0)|^{-2}$.

We now show that the amplitude $T_2'(E)$ can be of use in describing the transition from the two-nucleon state of energy, E, to the two-isobar state, in the approximation that we neglect the effect of the motion of the isobars upon the amplitude. The operator, T', may be defined formally in terms of a potential operator, V, by the relation

$$\langle u_{X,N}{}^{E'}|T'|u_{2N}{}^E\rangle=\langle\psi_{X,N}{}^{E'}|V|u_{2N}{}^E\rangle. \tag{B-3}$$

The $|u\rangle$ are plane-wave eigenstates of the free-field Hamiltonian, H_0,

$$H_0|u^E\rangle=E|u^E\rangle. \tag{B-4}$$

The $|\psi\rangle$ are eigenstates of the total Hamiltonian, H:

$$H|\psi^{E'}\rangle=(H_0+V)|\psi^{E'}\rangle=E'|\psi^{E'}\rangle. \tag{B-5}$$

They satisfy the Lippmann-Schwinger equation,

$$\psi^E=u^E+(E-H_0-i\xi)^{-1}V\psi^E. \tag{B-6}$$

Introducing an $\mathbf{r}$ representation, we obtain

$$\begin{aligned}\langle u_{X,N}{}^{E'}|T'|u_{2N}{}^E\rangle&=T'(E,E')\\ &=\int d^3\mathbf{r}d^3\mathbf{r}'\langle\psi_{X,N}{}^{E'}|\mathbf{r}\rangle\langle\mathbf{r}|V|\mathbf{r}'\rangle\langle r'|u_{2N}{}^E\rangle.\end{aligned} \tag{B-7}$$

In the center-of-mass system, $\mathbf{r}$ and $\mathbf{r}'$ are the relative position coordinates of the two particles. The interaction is assumed to be local:

$$\langle\mathbf{r}|V|\mathbf{r}'\rangle=V(r)\delta(\mathbf{r}-\mathbf{r}'),$$

$$\begin{aligned}T'(E,E')&=\int d^3r\langle\psi_{X,N}{}^{E'}(r)|V(r)|u_{2N}{}^E(r)\rangle\\ &\sim\psi_{X,N}{}^{E'}(R)\int\langle V(r)\rangle e^{i\mathbf{p}\cdot\mathbf{r}}d^3\mathbf{r}.\end{aligned} \tag{B-8}$$

We have taken out of the integral the isobar-nucleon wave function and have evaluated it at a relative coordinate, R, characteristic of the region in which $|\psi_{X,N}{}^{E'}(R)|^2$ is large. In addition, we have written the two-nucleon plane wave explicitly, where $\mathbf{p}$ is the relative momentum. The quantity $\langle V(r)\rangle$ is the potential operator between the two states after the position dependence of these states has been extracted. This is in general a function of E, E', $\mathbf{p}$, and $\mathbf{p}'$ (the relative momentum of the isobar and nucleon). As a first approximation, we neglect the dependence on the motion of the isobar-nucleon system.

In this approximation, the right-hand side of Eq. (B-8) defines a function of E,

$$T'(E)=\psi_{X,N}{}^{E_0+m}(R)f(E). \tag{B-9}$$

A similar argument leads to the amplitude for the transition from the two-nucleon state to the two-isobar state in this approximation.

$$A(E)=\psi_{2X}{}^{2E_0}(R)f'(E). \tag{B-10}$$

We have

$$|f(E)-f'(E)|\sim|(E_0-m)/E_0|=0.3/1.24=0.24; \tag{B-11}$$

therefore

$$A(E)\sim\psi_{2X}{}^{2E_0}(R)f(E), \tag{B-12}$$

and

$$|A(E)|^2\sim\left|\frac{\psi_{2X}{}^{2E_0}(R)}{\psi_{X,N}{}^{E_0+m}(R)}\right|^2|T'(E)|^2. \tag{B-13}$$

The separation into amplitudes for various total J states is obtained by writing

$$T'=\Sigma_J T_J'P_J,\quad V=\Sigma_J V_J P_J, \tag{B-14}$$

where the P_J are projection operators and Σ_J means the sum over J. An attempt to improve this crude approximation may be made by constructing an energy-dependent wave function for the isobar-nucleon state. For example, if the isobar-nucleon system is considered to be in an S state characterized by a scattering length of $\alpha^{-1}\sim0.5$ to 1.0×10^{-13} cm, we may approximate

$$|\psi_{X,N}{}^{E}(R)|^2 \quad\text{by}\quad f(R)/(\alpha^2+q^2),$$

where q is the relative momentum. The relative momentum may be determined by considering the isobar-nucleon system to be approximately on the energy shell or by assigning a certain average momentum transfer to the excitation process.

In the next appendix we show that one can somewhat correct the angular distributions of this simple model for the motion of the intermediate-state particles.

APPENDIX C

Consider the double production process,

$$p+p\rightarrow2\pi^++2n, \tag{C-1}$$

at an energy E. We have up to now considered the process as going through an intermediate state involving two isobars in an S state. These isobars move relative to each other with a momentum $2\mathbf{s}$ in their center-of-mass system. The momentum $\mathbf{s}$ is given in terms of the momenta of the final-state pions and nucleons by

$$\mathbf{s}=\mathbf{k}_1+\mathbf{p}_1=-(\mathbf{k}_2+\mathbf{p}_2),$$

where $\mathbf{k}_{1,2}$ and $\mathbf{p}_{1,2}$ are the pion and nucleon momenta of the correlated pairs. If the isobars are in a P state, the above reaction will take place from the ${}^3P_{0,1,2}$ states of the initial protons into the P states of the final two neutrons. The creation of the two-isobar state is described by the amplitudes $A({}^3P_0)$, $A({}^3P_1)$, and $A({}^3P_2)$, whose possible dependence on the motion of the isobars we continue to neglect. The decay of the two-isobar system was described by the amplitudes $\Pi(E_1)$ and $\Pi(E_2)$ times P-wave spherical harmonics for each meson, where E_1 and E_2 were total center-of-mass energies of the pion-nucleon system resulting from the decays. We shall neglect the effect of the isobar motion on the amplitudes Π; however, we may modify the matrix element by adding a P-wave harmonic that describes the relative motion of the two pion-nucleon systems, $Y_{1,n}(\mathbf{s}\cdot\mathbf{z})$. Because the massive nucleons usually carry considerable more momentum than the pions, we may approximate $\mathbf{s}$, the intermediate-isobar momentum, by $\mathbf{p}$, the final-state nucleon momentum. We may compute the modified angular angular distributions as before. For example, for the Process (C-1), the final state, with nucleon spin z component, $S_z=1$, is

$$(N_1{}^{\frac{1}{2}\,-\frac{1}{2}}N_2{}^{\frac{1}{2}\,-\frac{1}{2}})\sum_{m,m',n}Y_{1,m}{}^*(\mathbf{k}_1\cdot\mathbf{z})Y_{1,m}(\mathbf{r}_1\cdot\mathbf{z})$$
$$\times Y_{1,m'}{}^*(\mathbf{k}_2\cdot\mathbf{z})Y_{1,m'}(\mathbf{r}_2\cdot\mathbf{z})Y_{1,n}{}^*(\mathbf{p}\cdot\mathbf{z})Y_{1,n}(\mathbf{r}\cdot\mathbf{z}). \tag{C-2}$$

We perform the vector addition, combining the meson-nucleon systems into isobars with amplitudes Π, then adding the two isobar angular momenta, and finally combining this with the orbital angular momentum of the isobars to a total $J=0$, 1, and 2 with the amplitudes $A({}^3P_{0,1,2})$, respectively. Computing the matrix for $S_z=0$, and weighting the absolute square of the matrix elements for $S_z=1$ and $S_z=0$ with 2 and 1, respectively, we obtain, for the angular part of the cross section from the ${}^3P_0$ state of the initial protons, after integration over azimuthal angles,

$$\begin{aligned}\sigma(\theta_1,\theta_2,\theta)\propto\{&2\cos^2\theta_1\sin^2\theta_2+2\cos^2\theta_2\sin^2\theta_1\\&+8\cos^2\theta_1\cos^2\theta_2+16\sin^2\theta_1\sin^2\theta_2\}\cos^2\theta\\&+\{32\cos^2\theta_1\cos^2\theta_2+11\sin^2\theta_1\sin^2\theta_2\\&\quad+\cos^2\theta_1\sin^2\theta_2+\cos^2\theta_2\sin^2\theta_1\}\sin^2\theta.\end{aligned} \tag{C-3}$$

From the ${}^3P_2$ state, we have

$$\begin{aligned}\sigma(\theta_1,\theta_2,\theta)\propto\{&32\cos^2\theta_1\cos^2\theta_2+64\sin^2\theta_1\sin^2\theta_2\\&+8\cos^2\theta_1\sin^2\theta_2+8\cos^2\theta_2\sin^2\theta_1\}\cos^2\theta\\&+\{\cos^2\theta_1\sin^2\theta_2+\cos^2\theta_2\sin^2\theta_1\\&+32\cos^2\theta_1\cos^2\theta_2+11\sin^2\theta_1\sin^2\theta_2\}\sin^2\theta.\end{aligned} \tag{C-4}$$

The angle θ is the polar angle of the nucleon relative momentum with respect to the incident-proton direction in the total center-of-mass system. The angles θ_1 and θ_2 are the polar angles of the two mesons with respect to this direction, also in the total center-of-mass system, if one neglects the transformation from the rest system of each isobar. By integration over the meson angles in Eqs. (3) and (4), we can get an idea of the angular distribution of the final-state nucleons that could arise from inclusion of the P-wave motion of the intermediate-state isobars. For the transition from the $^3P_0$ state, we have

$$2\pi d\sigma(\theta)/d\Omega \propto (\cos^2\theta+\sin^2\theta); \qquad \text{(C-5)}$$

from the $^3P_2$ state, we have the familiar

$$2\pi d\sigma(\theta)/d\Omega \propto (3\cos^2\theta+1). \qquad \text{(C-6)}$$

Because the transition amplitudes from $^3P_{0,1,2}$ states give rise to interference terms that depend on their relative phases, a more detailed account of the excitation of the isobaric states will be necessary in order to obtain quantitative angular distributions. However, the observed forward-backward preference for the nucleons is not beyond the reach of the model.

PHYSICAL REVIEW VOLUME 106, NUMBER 3 MAY 1, 1957

Photofission Cross Sections of U^{235}, U^{238}, Th^{232}, Bi^{209}, and Au^{197} at Energies of 150 to 500 Mev*

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(Received January 17, 1957)

Photofission cross sections for U^{238}, U^{235}, Th^{232}, Bi^{209}, and Au^{197} have been measured by use of bremsstrahlung spectra whose maximum energies ranged from 150 to 500 Mev. The fissions were detected in 2π geometry with a double ionization chamber. A suggested correlation of the resulting cross sections with those for proton fission and for photomeson production is made.

I. INTRODUCTION

SEVERAL experiments have been performed to investigate photofission cross sections in the energy region 100 to 300 Mev.[1–5] In the experiment here presented we have investigated the photofission cross sections of U^{238}, U^{235}, Th^{232}, Bi^{209}, and Au^{197} for photons produced in bremsstrahlung spectra whose maximum energies ranged from 150 to 500 Mev. The energy region 150 to 335 Mev was investigated for the most part at the University of California synchrotron, whereas the higher energy data were obtained at the synchrotron of the California Institute of Technology. In a previous paper[6] (hereafter referred to as A) we have reported the high-energy proton-induced fission cross sections of the above elements. The same apparatus and essentially the same methods were used in the measurement of the photofission cross sections.

* This work was done under the auspices of the U. S. Atomic Energy Commission.

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II. APPARATUS AND METHOD

A. Fission Chamber

The ionization chamber used in this experiment is described in A. The beam was passed through the chamber in the direction *CBA* in order to minimize the effect of the electron-positron pairs produced in the sample backing. The distance from the thin entrance window to the sensitive region of the ionization chamber was approximately 4 inches, so that any pairs produced in the entrance window had only a small chance of producing uncanceled pulses in the sensitive region of the ionization chamber. In order to minimize pair production in the gas, the chamber was filled with 1 atmosphere of hydrogen. Finally, pair production in the electrodes was kept small by making them of 140-μg/cm^2 aluminum foil.

Chronologically, most of the photofission experiments were performed prior to the proton experiments described in A. Throughout most of the photofission runs only one scaler was used to record the number of pulses from the ionization chamber. However, in the last photofission run at the Berkeley synchrotron we switched to a system of using six scalers simultaneously in order to obtain an integral bias curve for each