PHYSICAL REVIEW VOLUME 106, NUMBER 3 MAY 1, 1957

Fine Structure of the Ground States of N^{15} and O^{17}†*

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(Received November 2, 1956)

An attempt has been made to fit the experimental values of the ground-state splittings for N^{15} and O^{17} by using the same nucleon-nucleon spin-orbit force for both cases. It is found that the spin-orbit force constants for the N^{15} case must be made about twice as large as those for the O^{17} case. The magnitudes of the spin-orbit force parameters required to account satisfactorily for the N^{15} and O^{17} ground-state splittings have been found to be from 5 to 10 times larger than those necessary to obtain agreement between the calculated and the experimental values of the ground state splitting in Li^7.

The exchange-integral contributions to the splittings have been found to be from $\frac{1}{4}$ to $\frac{1}{2}$ of the direct integral contributions; therefore, the concept of a one-body spin-orbit force is not accurately applicable in the cases of the N^{15} and O^{17} ground state splittings.

Gaussian error potentials and wave functions have been used. The wave-function range parameters have been adjusted by means of experimental Coulomb energies and examined by means of binding energies. Effects of excited configurations have not been included and no claim of an exhaustive consideration of all possible adjustments of wave function and potential parameters is made.

I. INTRODUCTION

MANY calculations have been made of the level structure of light nuclei, using mixtures of central, tensor, and spin-orbit types of interactions.[1] One especially interesting problem of level structure is that of nuclear fine structure. It was observed by Breit[2] in 1936 that the experimentally observed $p_{\frac{3}{2}}$ and $p_{\frac{1}{2}}$ states of Li^7 appeared to be members of an inverted doublet, analogous to that of atomic spectra, except that the level order would be normal if one used the usual spin-orbit interaction. Inglis[3] pointed out that by using just the Thomas-type spin-orbit term the required inverted order could be obtained. Breit and Stehn[4] in 1938 were able to account for the 479-kev energy difference of the Li^7 ^2P states, using the approximately relativistic spin-orbit interactions of Breit,[5] which were obtained by requiring relativistic invariance (to order v^2/c^2, in the nucleon velocities) of the n-nucleon wave equation.

Recently Feingold[6] has been able to explain the ground-state splitting of Li^7 by means of a tensor force alone. He takes the viewpoint that nuclear fine structure may be due to just a tensor force; however, he does not exclude the possibility that the effect may be due to a combination of tensor and spin-orbit forces. The words "spin-orbit forces" are here used in the sense of forces arising from potentials containing particle spins only in scalar products with vectors expressible in terms of displacements (distances) and gradient operators.

The present investigation will be limited to a consideration of spin-orbit forces. Partial justification for this restriction comes from the fact that spin-orbit splitting is a first order effect of spin-orbit forces and does not necessarily require consideration of excited configurations, while tensor-force splitting is essentially a second order effect and requires consideration of excited configurations,[6] at least to obtain agreement with the sign of the ^2P splitting of Li^7.

Evidence for the existence of a nucleon-nucleon spin-orbit force comes from several sources. Meson field theories, as is well known, point to it, even though the exact form is left in doubt. A phenomenological spin-orbit interaction has been introduced by Case and Pais[7] in their interpretation of high-energy nucleon-nucleon scattering data. A more recent investigation[8] seems to indicate that it is not possible to account for both the p-p differential cross section and polarization, at high energies, if one assumes only tensor and central forces to be present. From the above it may be inferred that a nucleon-nucleon spin-orbit force may not be ruled out as a possible form of interaction.

A one-body spin-orbit interaction has also played an important role in recent years in nucleon-nucleus scattering problems as well as in nuclear shell theory. Sack, Biedenharn, and Breit[9] have been able to account for

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† Part of a dissertation submitted in partial fulfillment of requirements for the degree of Doctor of Philosophy at Yale University.

* This research was supported by the U. S. Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command under Contract No. AF18(600)771.

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[4] G. Breit and J. R. Stehn, Phys. Rev. **53**, 459 (1938).

[5] G. Breit, Phys. Rev **51**, 248 (1937); **53**, 153 (1938); see also L. Rosenfeld, *Nuclear Forces* (North-Holland Publishing Company, Amsterdam, 1948), Sec. 15.22.

[6] A. M. Feingold and E. P. Wigner, Phys. Rev. **79**, 221(A) (1950); A. M. Feingold, Phys. Rev. **101**, 258 (1956); Bull. Am. Phys. Soc. Ser. II, **1**, 181 (1956).

[7] K. M. Case and A. Pais, Phys. Rev. **80**, 203, 138(A) (1950).

[8] Private communication by Professor G. Breit on the basis of work done in collaboration with M. S. Wertheim.

[9] Sack, Biedenharn, and Breit, Phys. Rev. **93**, 321 (1953).

the $p_{\frac{3}{2}}$ and $p_{\frac{1}{2}}$ phase shift *vs* energy relationships, required by low-energy $p-\alpha$ scattering data, by means of a phenomenological one-body spin-orbit force arriving at a factor of $\sim$25 that must be applied to the simple Thomas term in order to secure agreement with experiment. Recently, van der Spuy,[10] in a similar calculation, has obtained reasonable agreement with $n-\alpha$ phase shift *vs* energy data. Massey *et al.*[11] have considered the 5-nucleon system and nucleon-nucleon central and spin-orbit forces in explaining the low-energy $n-\alpha$ and $p-\alpha$ phase shift *vs* energy relationships and have determined the relevant nucleon-nucleon spin-orbit force constant.

Fermi[12] found that a one-body spin-orbit force 15–20 times larger than that of a Thomas-type term was adequate to account for the experimentally measured polarization of 340-Mev protons scattered by carbon. Similar considerations have been used, by several other authors,[13] in predicting nucleon polarization in scattering produced by several other nuclei.

The possibility of representing a one-body spin-orbit force as the collective interactions of a nucleon with the nucleons of a core has been investigated by Hughes and Le Couteur.[14] If the Case and Pais nucleon-nucleon spin-orbit force is assumed, they have shown by averaging over all core nucleons, that about the correct order of magnitude of splitting, as is required by shell theory, results. In taking an average over the core nucleons, exchange integral effects are not taken into account by them. Because these terms cannot be represented in terms of an equivalent central potential it will be of interest in the present investigation to compare the magnitude of exchange integral contributions with direct integral contributions, since the latter can be represented by an equivalent one-body spin-orbit interaction.

The plan of this paper is as follows. In Sec. II the ground state splittings of two especially simple configurations will be considered; that of a single or valence nucleon outside of closed shells and that of a "hole" in a closed shell with a closed shell. The splittings will be evaluated by considering the approximately relativistic spin-orbit interactions of Breit[4] to be a perturbation of the otherwise degenerate ground state configurations. The splitting evaluations are given in terms of arbitrary orbital quantum numbers of the shell and the valence nucleon, arbitrary radial wave functions and central nucleon-nucleon potentials. Similar calculations to these have been performed using the various forms of the phenomenological spin-orbit interactions given by Wigner and Eisenbud.[15]

In Sec. III the splitting expressions relevant to the configurations $(s^2s^2p^6p^5)$ or $(1p^{-1})$ and $(s^2s^2p^6p^6d)$ or $(1d)$, assumed to represent the N^{15} and O^{17} ground states respectively, are evaluated using Gaussian wave functions and a Gaussian potential.

Section IV contains the expressions of the binding energy difference (taken to be due to Coulomb forces only) of $N^{15}-O^{15}$ and $O^{17}-F^{17}$ and the binding energy of the "last" neutron of O^{17} and of the "hole" in N^{15}. These quantities are then compared with experimental values to obtain a set of wave function parameters to be used in evaluating splittings. The strength and range of the Gaussian potential to be used is also discussed here.

In Sec. V the spin-orbit interaction constants obtained by comparison with the experimental splitting of N^{15} and O^{17} are given for the different sets of wave function parameters of Sec. IV. A comparison is made of the relative contributions of direct and exchange integrals to the splittings.

Section VI is a discussion of the present results, and a comparison is made of the interaction constants obtained with those required in the Li^7 calculation of Breit and Stehn.[4] Also a comparison is made, insofar as is possible, with the fine structure calculations of other authors[16] relevant to N^{15}, O^{17}, and Li^7. The work reported below differs from many other papers quoted above in that the whole interaction is formally covariant to order v^2/c^2 in the nucleon velocities.[5]

Notation

$\Delta E_Y{}^X$ signifies the difference in energy between two doublet levels. X may be M, W, or H depending on whether the nucleon-nucleon potential is of Majorana, Wigner, or Heisenberg type. Y may be dir or ex, indicating whether the splitting arises from direct or exchange integral contributions.

$$u_{m,a}{}^L(1)\equiv R_L(r)Y_L{}^m(\theta,\varphi)\begin{Bmatrix}\alpha_1\\ \beta_1\end{Bmatrix}\equiv\text{wave function of individual nucleon,}$$

when the latter is in a closed shell; here L, m, a denote the orbital, azimuthal and spin quantum numbers of the nucleon state

$$R_L(r)\equiv N_Lr^L\exp(-\tfrac{1}{2}cr^2);$$

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$c\equiv\bar{\mu}$, $\bar{\nu}$, or $\bar{\gamma}$ for s, p, or d shell nucleons.

$$N_L^2=\frac{2^{L+2}c^{L+3/2}}{1\cdot 3\cdot 5\cdots(2L+1)\sqrt{\pi}}.$$

$$Y_L^m(\theta,\varphi)\equiv\frac{(-1)^{L+m}}{2^L(L!)}\left[\frac{(2L+1)(L-m)!}{4\pi(L+m)!}\right]^{\frac{1}{2}}(\sin\theta)^m \times\frac{d^{L+m}}{d(\cos\theta)}(\sin\theta)^{2L}e^{im\varphi}.$$

α_i, $\beta_i\equiv$spin functions for nucleon i corresponding to spin being up or down, respectively. $v_{\mu\alpha}{}^{\Lambda}\equiv$eigenfunction of a valence nucleon. $\bar{\alpha}$, $A\equiv$range and strength parameters, respectively, of a Gaussian potential. $a\equiv\bar{\mu}/\bar{\alpha}$; $b\equiv\bar{\nu}/\bar{\alpha}$; $c\equiv\bar{\gamma}/\bar{\alpha}$. $P_L'(u)\equiv dP_L(u)/du$, where $P_L(u)$ =Legendre polynomial. $\lambda\!\!\!^{-}\equiv\hbar/Mc$; $M\equiv$mass of a nucleon; $c\equiv$velocity of light. $\mathcal{A}\equiv\bar{\alpha}\lambda\!\!\!^{-2}A$. $D_{xy}\equiv[(x+y)/2]^2+x+y$. $F_{xy}\equiv xy+x+y$.

II. EVALUATIONS OF SPLITTINGS FOR A SINGLE NUCLEON AND A SINGLE HOLE

The approximately relativistic spin-orbit interactions,[4] to be used, are as follows for Wigner, Heisenberg, and Majorana nucleon-nucleon potentials:

$$H'^W=\sum_{jk}\mathbf{B}_{jk}\cdot[a^W\boldsymbol{\sigma}_j+(a^W-1)\boldsymbol{\sigma}_k],\qquad(1)$$

$$H'^H=\sum_{jk}\mathbf{B}_{jk}\cdot[a^H\boldsymbol{\sigma}_j+(a^H-1)\boldsymbol{\sigma}_k]P_{jk}{}^H,\qquad(2)$$

where

$$\mathbf{B}_{jk}=-\frac{\hbar}{4M^2C^2}[\mathbf{p}_j\times\boldsymbol{\nabla}_jV(r_{jk})];$$

$$H'^M=\sum_{jk}\mathbf{A}_{jk}\cdot[a^M\boldsymbol{\sigma}_j+(a^M-1)\boldsymbol{\sigma}_k],\qquad(3)$$

where

$$\mathbf{A}_{jk}=-\frac{i}{4M^2C^2}[\mathbf{p}_j\times V(r_{jk})P_{jk}{}^M\mathbf{p}_j],$$

$V(r_{jk})$=ordinary scalar potential energy of interaction of nucleon j with nucleon k, $P_{jk}{}^M\equiv$Majorana exchange operator, and $P_{jk}{}^H\equiv$Heisenberg exchange operator.

Each of these interactions can be seen to have two types of terms. One type (obtained by setting a^M, a^W or $a^H=1$) gives an interaction between the spin $\boldsymbol{\sigma}_j$ and momentum $\mathbf{p}_j$ of the jth nucleon, while the other (setting a^M, a^W, or $a^H=0$) yields an interaction between the jth nucleon's spin $\boldsymbol{\sigma}_j$ and the kth nucleon's momentum $\mathbf{p}_k$.

A. Single Nucleon Outside of a Closed Shell

The zero-order wave function of a single nucleon outside of a closed shell will be taken to be a Slater determinant:

$$\psi_0=\sum_P(-1)^pPu_a(1)u_b(2)\cdots u_n(n),\qquad(4)$$

where $\sum$ is a summation over all permutations P of the n nucleons with $(-1)^p=-1$ for P odd and $(-1)^p=+1$ for P even. Let the perturbation term of an n nucleon system be given by

$$H'=\tfrac{1}{2}\sum_{i,j}{}'H_{ij};\qquad(5)$$

where $H_{ij}=H_{ji}$ and the prime on $\sum$ means $i\neq j$. Then, for a "closed shell plus one nucleon" system, one obtains for the diagonal matrix element of H':

$$(\psi_0,H'\psi_0)=\sum_a(u_a(1)u_c(2),\times H_{12}[u_a(1)u_c(2)-u_c(1)u_a(2)]),\qquad(6)$$

where $\sum_a$ is a summation over the nucleon states making up the closed shell and the letter c designates the valence nucleon state. In what follows, the first term of Eq. (6) will be referred to as the direct integral and the second as the exchange integral. By an application of the well-known diagonal sum rule, it may be shown[17] that the doublet splitting can be given by

$$\Delta E_{L\Lambda}=(\psi_0{}^h,H'\psi_0{}^h)(2\Lambda+1)/\mu,\qquad(7)$$

where L and Λ designate the orbital quantum numbers of the closed shell and valence nucleons respectively and $\psi_0{}^h$ denotes the zero order wave function which contains the valence nucleon's wave function of higher j value (i.e., μ and the spin have the same sign).

(a) Majorana Potential

For the splitting due to the direct term of H'^M of Eq. (3) with $a_M=1$, one has

$$\Delta_{L\Lambda}E_{\rm dir}{}^{M1}=(2\Lambda+1)(\psi_0{}^h,H'^M(a^M=1)\psi_0{}^h)_{\rm dir}/m$$
$$=(2\Lambda+1)\sum_{ma}\frac{1}{\mu}\int u_{ma}{}^{*L}(1)v_{\mu\alpha}{}^{*\Lambda}(2)\times(\mathbf{A}_{12}\cdot\boldsymbol{\sigma}_1+\mathbf{A}_{21}\cdot\boldsymbol{\sigma}_2)u_{ma}{}^L(1)v_{\mu\alpha}{}^{\Lambda}(2)d\mathbf{r}_1d\mathbf{r}_2,\qquad(8)$$

where $\sum_{ma}$ indicates a sum on m from $-L$ to $+L$ and on a over both spin states, and $M1$ indicates that $a^M=1$. Applying the spin operators and summing over both spin states gives

$$\Delta_{L\Lambda}E_{\rm dir}{}^{M1}=2(2\Lambda+1)\sum_m\frac{1}{\mu}\int u_m{}^*(2)v_\mu{}^*(1)\times A_{12}{}^zu_m(2)v_\mu(1)d\mathbf{r}_1d\mathbf{r}_2.\qquad(9)$$

Since Eq. (9) can be shown to be independent of the value of μ, one can multiply it by μ^2 and then summing on μ from $-\Lambda$ to Λ yields

$$[\Lambda(\Lambda+1)(2\Lambda+1)/3]\Delta_{L\Lambda}E_{\rm dir}{}^{M1}$$
$$=2(2\Lambda+1)(\lambda\!\!\!^{-2}/4)i\sum_m\sum_\mu\mu\int u_m{}^*(2)v_\mu{}^*(1)\times[\boldsymbol{\nabla}_1\times V(r_{12})P_{12}{}^M\boldsymbol{\nabla}_1]_zu_m(2)v_m(1)d\mathbf{r}_1d\mathbf{r}_2\qquad(10)$$

[17] G. Breit, Phys. Rev. 35, 1447 (1930).

with

$$ƛ^2=\hbar^2/M^2c^2.$$

Making use of the addition theorem for spherical harmonics, one finds in a straightforward manner that

$$\Delta_{L\Lambda}E_{\mathrm{dir}}{}^{M1}=\frac{(2L+1)(2\Lambda+1)ƛ^2}{32\pi^2\Lambda(\Lambda+1)}\int V_{12}R_L(1)R_L(2)\times P_L(\cos\theta_{12})\cdot\{\ \}d\mathbf{r}_1d\mathbf{r}_2, \quad (11)$$

where

$$\{\ \}=\Big\{\Lambda(\Lambda+1)\cos\theta_{12}P_\Lambda(\cos\theta_{12})\Big[r_2\frac{R_\Lambda(1)}{r_1}\frac{d}{dr_2}\Big(\frac{R_\Lambda(2)}{r_2}\Big)+r_1\frac{R_\Lambda(2)}{r_2}\frac{d}{dr_1}\Big(\frac{R_\Lambda(1)}{r_1}\Big)+3\frac{R_\Lambda(1)}{r_1}\frac{R_\Lambda(2)}{r_2}\Big]+\sin^2\theta_{12}P_\Lambda'(\cos\theta_{12})\Big[2r_1\frac{R_\Lambda(2)}{r_2}\frac{d}{dr_1}\Big(\frac{R_\Lambda(1)}{r_1}\Big)+2r_2\frac{R_\Lambda(1)}{r_1}\frac{d}{dr_2}\Big(\frac{R_\Lambda(2)}{r_2}\Big)+r_1r_2\frac{d}{dr_1}\Big(\frac{R_\Lambda(1)}{r_1}\Big)\times\frac{d}{dr_2}\Big(\frac{R_\Lambda(2)}{r_2}\Big)+[4+\Lambda(\Lambda+1)]\frac{R_\Lambda(1)}{r_1}\frac{R_\Lambda(2)}{r_2}\Big]\Big\}.$$

Equation (11) may be found in the literature[4] and is given again here for completeness.

The remaining splittings for the Majorana potential may be evaluated in a similar manner and are as follows:

$$\Delta_{L\Lambda}E_{\mathrm{dir}}{}^{M0}=\frac{(2L+1)(2\Lambda+1)ƛ^2}{32\pi^2\Lambda(\Lambda+1)}\int V_{12}P_\Lambda'(1,2)\times R_\Lambda(2)\sin^2\theta_{12}[\ ]d\mathbf{r}_1d\mathbf{r}_2, \quad (12)$$

where

$$[\ ]=\Big[R_L'(1)R_L'(2)P_L(1,2)-P_L'(1,2)\times\cos\theta_{12}\Big(\frac{R_L(1)}{r_1}R_L'(2)+R_\Lambda(1)\frac{R_L(2)}{r_2}-\frac{R_L(1)}{r_1}\frac{R_L(2)}{r_2}\Big)+\frac{R_L(1)}{r_1}\frac{R_L(2)}{r_2}\times P_L''(1,2)\cos^2\theta_{12}\Big]; \quad (13)$$

$$\Delta_{L\Lambda}E_{\mathrm{ex}}{}^{M1}=-(2L+1)(2\Lambda+1)\frac{ƛ^2}{(32\pi^2)}\int(V_{12}/r_2)\times R_L(1)R_L'(1)R_\Lambda{}^2(2)\cos\theta_{12}d\mathbf{r}_1d\mathbf{r}_2;$$

$$\Delta_{L\Lambda}E_{\mathrm{ex}}{}^{M0}=-\Delta_{L\Lambda}E_{\mathrm{ex}}{}^{M1}. \quad (14)$$

(b) Wigner Potential

The single-particle splittings for the Wigner potential are as follows:

$$\Delta_{L\Lambda}E_{\mathrm{dir}}{}^{W1}=-\frac{(2L+1)(2\Lambda+1)ƛ^2}{32\pi^2}\int\Big[V_{21}'R_L{}^2(1)\times R_\Lambda{}^2(2)\frac{r_2{}^2-r_1r_2\cos\theta_{12}}{r_{21}r_2{}^2}\Big]d\mathbf{r}_1d\mathbf{r}_2; \quad (15)$$

$$\Delta E_{\mathrm{dir}}{}^{W0}=0; \quad (16)$$

$$\Delta_{L\Lambda}E_{\mathrm{ex}}{}^{W1}=\frac{(2L+1)(2\Lambda+1)ƛ^2}{64\pi^2\Lambda(\Lambda+1)}\Big\{\int\frac{V_{12}'}{r_{12}}\frac{R_L(1)}{r_2}\frac{R_L(2)}{r_2}\times P_L(1,2)P_\Lambda'(1,2)R_\Lambda(2)\{\ \}d\mathbf{r}_1d\mathbf{r}_2+\int\frac{V_{12}'}{r_{12}}R_L(1)R_L'(2)r_1\sin^2\theta_{12}P_L(1,2)\times P_\Lambda'(1,2)R_\Lambda(1)R_\Lambda(2)d\mathbf{r}_1d\mathbf{r}_2+\int\frac{V_{12}'}{r_{12}}\frac{R_L(1)}{r_1}\frac{R_L(2)}{r_2}r_1{}^2r_2{}^2\sin^2\theta_{12}\times R_\Lambda(1)R_\Lambda(2)[\ ]d\mathbf{r}_1d\mathbf{r}_2\Big\}, \quad (17)$$

where

$$\{\ \}=\{[(2r_1r_2\cos\theta_{12}-2r_2{}^2)+r_2{}^2\sin^2\theta_{12}]R_\Lambda(1)-r_1r_2{}^2\sin^2\theta_{12}R_\Lambda'(1)\}$$

and

$$[\ ]=\Big[\Big(\frac{1}{r_1r_2}-\frac{\cos\theta_{12}}{r_2{}^2}\Big)P_L'(1,2)P_\Lambda'(1,2)-\Big(\frac{1}{r_1r_2}-\frac{\cos\theta_{12}}{r_2{}^2}\Big)P_L(1,2)P_\Lambda''(1,2)\Big];$$

$$\Delta_{L\Lambda}E_{\mathrm{ex}}{}^{W0}=-\Delta E_{\mathrm{ex}}{}^{W1}. \quad (18)$$

(c) Heisenberg Potential

The single-particle splittings for the Heisenberg potential can be expressed in terms of those for the Wigner potential as follows:

$$\Delta E_{\mathrm{dir}}{}^{H1}=-\Delta E_{\mathrm{dir}}{}^{W1}, \quad (19)$$

$$\Delta E_{\mathrm{dir}}{}^{H0}=-\Delta E_{\mathrm{dir}}{}^{W0}, \quad (20)$$

$$\Delta E_{\mathrm{ex}}{}^{H1}=-\Delta E_{\mathrm{dir}}{}^{W1}, \quad (21)$$

$$\Delta E_{\mathrm{ex}}{}^{H1}=-\Delta E_{\mathrm{dir}}{}^{W0}=0. \quad (22)$$

B. System Consisting of a "Hole" in a Closed Shell and Another Closed Shell

As is well known, the splitting (with opposite sign) of this system can be obtained in terms of single-particle splittings by assuming both shells to be filled and identifying the "hole" state with that of a valence nucleon. So, for example, to determine the $(1p^{-1})$ state splitting of N^{15} it suffices to assume a $p_{\frac{1}{2}}$ nucleon state outside of closed s and p shells, and to apply the corresponding formulas of Sec. II-A, with the signs reversed.

III. SPLITTING EXPRESSIONS ASSUMING GAUSSIAN WAVE FUNCTIONS AND GAUSSIAN POTENTIALS

The Gaussian wave functions to be used in evaluating the formulas of Sec. II, A are as follows:

$$R_0(r)=N_0\exp(-\tfrac{1}{2}\bar{\mu}r^2);\quad R_1(r)=N_1r\exp(-\tfrac{1}{2}\bar{\nu}r^2);\quad R_2(r)=N_2r^2\exp(-\tfrac{1}{2}\bar{\gamma}r^2);$$

where

$$N_0^2=4\bar{\mu}^{\frac{3}{2}}/\sqrt{\pi};\quad N_1^2=(8/3)(\bar{\nu}^{\frac{5}{2}}/\sqrt{\pi});\quad N_2^2=(16/15)(\bar{\gamma}^{7/2}/\sqrt{\pi}).$$

The Gaussian potential to be used is

$$V(r_{12})=-A\exp(-\bar{\alpha}r_{12}^2).$$

To calculate, say, the direct Majorana ($a^M=1$) contribution to the ($d_{\frac{5}{2}}-d_{\frac{3}{2}}$) splitting of O^{17}, one has

$$\Delta E_{\mathrm{dir}}^{M1}(O^{17})=2[\Delta_{01}E_{\mathrm{dir}}^{M1}+\Delta_{12}E_{\mathrm{dir}}^{M1}],$$

and for the ($d_{\frac{3}{2}}-d_{\frac{1}{2}}$) splitting of N^{15} one has

$$\Delta E_{\mathrm{dir}}^{M1}(N^{15})=-2[\Delta_{01}E_{\mathrm{dir}}^{M1}+\Delta_{11}E_{\mathrm{dir}}^{M1}],$$

the 2 arising from the fact that both neutrons and protons contribute to direct integral contributions, if one assumes charge independence of nuclear forces. By making use of integrals in Appendix A and Eqs. (11)–(18), the O^{17} and N^{15} splitting expressions may be evaluated and the results are given in Secs. A–D below.

A. ($d_{5/2}-d_{3/2}$)O^{17} Splittings for the Majorana Potential

$$\Delta E_{\mathrm{dir}}^{M1}(O^{17})=-10\mathfrak{A}\left\{\frac{c^2(ac)^{\frac{3}{2}}}{D_{ac}^{\frac{5}{2}}}\left[1-\frac{c(\frac{3}{4}c+\frac{1}{2}a+1)}{D_{ac}}\right]+\frac{c(bc)^{\frac{5}{2}}}{D_{bc}^{\frac{5}{2}}}\left[1-\frac{7c(\frac{1}{4}c+\frac{1}{2}b+1)}{D_{bc}^2}+\frac{5-c(\frac{1}{4}c+\frac{1}{2}b+1)}{D_{bc}}\right]\right\}, \tag{23}$$

where $a\equiv\bar{\mu}/\bar{\alpha}$, $b\equiv\bar{\nu}/\bar{\alpha}$, $c\equiv\bar{\gamma}/\bar{\alpha}$, $\mathfrak{A}\equiv\bar{\alpha}\lambda^2A$, and $D_{xy}\equiv[(x+y)/2]^2+x+y$;

$$\Delta E_{\mathrm{dir}}^{M0}(O^{17})=-5\mathfrak{A}\left\{\frac{1}{2}\left[\frac{ac}{D_{ac}}\right]^{7/2}+b\left[\frac{bc}{D_{bc}}\right]^{7/2}(D_{bc}+7)\right\}, \tag{24}$$

$$E_{\mathrm{ex}}^{M1}(O^{17})=-\frac{5}{2}\mathfrak{A}\left\{\frac{c(ac)^{5/2}}{F_{ac}^{7/2}}(a+1)-\frac{c(bc)^{5/2}}{F_{bc}^{7/2}}\left[(2-3b)-\frac{7b}{F_{bc}}\right]\right\}, \tag{25}$$

where $F_{xy}\equiv xy+x+y$ and

$$\Delta E_{\mathrm{ex}}^{M0}(O^{17})=-\Delta E_{\mathrm{ex}}^{M1}. \tag{26}$$

B. ($d_{5/2}-d_{3/2}$)O^{17} Splittings for the Wigner Potential

$$\Delta E_{\mathrm{dir}}^{W1}(O^{17})=-10\mathfrak{A}\left\{\frac{c(ac)^{5/2}}{F_{ac}}(a+1)+\frac{c(bc)^{5/2}}{F_{bc}^{7/2}}\left[3(b-1)+\frac{7b}{F_{bc}}\right]\right\}, \tag{27}$$

$$\Delta E_{\mathrm{dir}}^{W0}(O^{17})=0, \tag{28}$$

$$\Delta E_{\mathrm{ex}}^{W1}(O^{17})=5\mathfrak{A}\left\{\frac{c^2(ac)^{\frac{3}{2}}}{D_{ac}^{5/2}}\left[\frac{(c-3a)}{4D_{ac}}-\frac{(a+c+6)}{10(a+c+2)}\right]+\frac{c(bc)^{5/2}}{D_{bc}^{7/2}}\left[(2-b)-\frac{7b}{D_{bc}}\right]\right\}, \tag{29}$$

$$\Delta E_{\mathrm{ex}}^{W0}(O^{17})=-\Delta E_{\mathrm{ex}}^{W1}. \tag{30}$$

C. ($p_{3/2}-p_{1/2}$)N^{15} Splittings for the Majorana Potential

$$\Delta E_{\mathrm{dir}}^{M1}(N^{15})=6\mathfrak{A}\left\{\frac{(a+4)}{4}\left[\frac{ab}{D_{ab}}\right]^{5/2}+\frac{b^6[1-(3b/4)]}{D_{bb}^{7/2}}\right\}, \tag{31}$$

$$\Delta E_{\mathrm{dir}}^{M0}(N^{15})=\frac{3}{2}\mathfrak{A}\left\{a\left[\frac{ab}{D_{ab}}\right]^{\frac{5}{2}}+\frac{5b^7}{D_{bb}^{7/2}}\right\}, \tag{32}$$

$$\Delta E_{\mathrm{ex}}^{M1}(N^{15})=\frac{3}{2}\mathfrak{A}\left\{\left[\frac{ab}{F_{ab}}\right]^{\frac{5}{2}}+\frac{b^6(3b+1)}{F_{bb}^{7/2}}\right\}, \tag{33}$$

$$\Delta E_{\mathrm{ex}}^{M0}(N^{15})=-\Delta E_{\mathrm{ex}}^{M1}. \tag{34}$$

D. ($p_{3/2}-p_{1/2}$)N^{15} Splittings for the Wigner Potential

$$\Delta E_{\mathrm{dir}}^{W1}(N^{15})=6\mathfrak{A}\left\{\left[\frac{ab}{F_{ab}}\right]^{\frac{5}{2}}+\frac{b^6(3b+1)}{F_{bb}^{7/2}}\right\}, \tag{35}$$

$$\Delta E_{\mathrm{dir}}^{W0}(N^{15})=0, \tag{36}$$

$$\Delta E_{\mathrm{ex}}^{W1}(N^{15})=3\mathfrak{A}\left\{\frac{1}{2}\left[\frac{ab}{D_{ab}}\right]^{\frac{5}{2}}-\frac{b^6(b-\frac{1}{2})}{D_{bb}^{7/2}}\right\}, \tag{37}$$

$$\Delta E_{\mathrm{ex}}^{W0}(N^{15})=-\Delta E_{\mathrm{ex}}^{W1}. \tag{38}$$

IV. DETERMINATION OF WAVE FUNCTIONS AND POTENTIAL PARAMETERS

To obtain an estimate of the wave function parameters a, b, and c the binding energy differences (assumed to be due to Coulomb forces only) of $F^{17}-O^{17}$ and $O^{15}-N^{15}$ have been evaluated using Gaussian wave functions. If the ground state wave functions of O^{17} and F^{17} are assumed to be identical, then the Coulomb energy difference of $F^{17}-O^{17}$ can easily be shown to be given by

$$\Delta E_c(F^{17}-O^{17})=\sum_{L=0,1}\frac{(2L+1)}{(4\pi)^2}e^2\left[2\int\frac{R_L^2(1)R_2^2(2)}{r_{12}}d\mathbf{r}_1d\mathbf{r}_2-\int\frac{R_L(1)R_L(2)R_2(2)R_2(1)}{r_{12}}P_L(\cos\theta_{12})P_2(\cos\theta_{12})d\mathbf{r}_1d\mathbf{r}_2\right]$$

$$=e^2\bar{\alpha}^{\frac{1}{2}}\frac{8}{15\sqrt{\pi}}\left\{c^2(ac)^{\frac{3}{2}}\left[\frac{[(15/2)+10(a/c)+4(a^2/c^2)]}{ac(a+c)^{\frac{5}{2}}}-\frac{3}{(a+c)^{9/2}}\right]\right.$$
$$\left.+c(bc)^{\frac{5}{2}}\left[\frac{1}{bc(b+c)^{\frac{5}{2}}}\left\{\frac{5[(15/2)+10(b/c)+4(b^2/c^2)]}{b+c}+\frac{15}{b}-\frac{8b}{c^2}\right\}-\frac{31}{(b+c)^{11/2}}\right]\right\}, \tag{39}$$

where $e\equiv$charge on an electron and use has been made of the integrals in Appendix B. The Coulomb energy difference of $O^{15}-N^{15}$ is given by

$$\Delta E_c(O^{15}-N^{15})=\frac{e^2}{(4\pi)^2}\left\{\left[2\frac{R_0^2(1)R_1^2(2)}{r_{12}}d\mathbf{r}_1d\mathbf{r}_2-\int R_0(1)R_0(2)P_0(1,2)P_1(1,2)R_1(1)R_1(2)d\mathbf{r}_1d\mathbf{r}_2\right]\right.$$
$$\left.+\left[6\int\frac{R_1^2(1)R_1^2(2)}{r_{12}}\left(1-\frac{P_1^2(\cos\theta_{12})}{2}\right)d\mathbf{r}_1d\mathbf{r}_2\right]\right\} \tag{40}$$
$$=\frac{e^2\bar{\alpha}^{\frac{1}{2}}}{\sqrt{\pi}}\frac{4}{3}\left[b(ab)^{\frac{3}{2}}\left(\frac{1}{ab(a+b)^{\frac{3}{2}}}+\frac{2}{ab^2(a+b)^{\frac{1}{2}}}-\frac{2}{(a+b)^{7/2}}\right)+\frac{43}{16}(2b)^{\frac{1}{2}}\right],$$

again using Appendix B.

To afford an independent estimate of the wave function parameters, the binding energy of the "last" neutron of O^{17} has been calculated using a potential energy of the form

$$V(r_{12})=-A\exp(-\bar{\alpha}r_{12}^2)[g+(1-g)P_{12}^M] \tag{41}$$

and making the assumptions that the s and p shell wave function parameters (a and b) of O^{16} and O^{17} are the same. A straightforward calculation gives

$$(\text{B.E.})_{LN}(O^{17})=\frac{28\hbar^2\bar{\alpha}}{17M}c-A\left\{c^2(ac)^{\frac{3}{2}}\left[\frac{(5g-1)(a+1)^2}{F_{ac}^{7/2}}+\frac{(4-5g)}{D_{ac}^{7/2}}\right]\right.$$
$$\left.+c(bc)^{\frac{5}{2}}\left[\frac{(5g-1)(b+1)(3F_{bc}+7)}{F_{bc}^{9/2}}+\frac{(4-5g)(2D_{bc}+7)}{D_{bc}^{9/2}}\right]\right\}. \tag{42}$$

Similarly, assuming that a and b are the same for N^{15} and O^{16}, one finds that the binding energy of the hole in N^{15} is given by

$$(\text{B.E.})_{\text{hole}}(N^{15})=\frac{7\hbar^2\bar{\alpha}}{6M}b-A\left\{(5g-1)\left[\frac{b(a+1)(ab)^{\frac{3}{2}}}{F_{ab}^{\frac{5}{2}}}+\frac{3b^7+6b^6+5b^5}{F_{bb}^{7/2}}\right]+(4-5g)\left[\frac{b(ab)^{\frac{3}{2}}}{D_{ab}^{\frac{5}{2}}}+\frac{b^7+2b^6+5b^5}{D_{bb}^{7/2}}\right]\right\}. \tag{43}$$

The potential constants shall be taken to be

$$\bar{\alpha}=22(Mmc^2/\hbar^2)\,(1/\bar{\alpha}^{\frac{1}{2}}=1.92\times10^{-13}\text{ cm})$$

and $A=91\ mc^2$. These constants are based upon the parameters of Hatcher, Arfken, and Breit,[18] which were obtained by securing agreement for the low-energy p-p scattering cross section. The depth of 91 mc^2 is the weighted mean of the triplet and singlet depths with the latter being taken to be 51 mc^2. By way of comparison, the potential constants used by Massey *et al.*[11] are $A=88.2\ mc^2$ and $1/\bar{\alpha}^{\frac{1}{2}}=1.91\times10^{-13}$ cm, which give the correct binding energy of He^4.

[18] Hatcher, Arfken, and Breit, Phys. Rev. **75**, 1389 (1949).

Equating the experimental binding energy difference of $O^{15}-N^{15}$ of 3.547 Mev[19] to $\Delta E_c(O^{15}-N^{15})$ of Eq. (40) and setting $a=b$, one obtains approximately 1.2 for $a=b$. $\Delta E_c(O^{15}-N^{15})$ is found to be relatively insensitive to variations of either a or b about 1.2, so for simplicity $a=b$ will be assumed[20] in what follows.

Equating the 3.549-Mev binding energy difference of

[19] All experimental quantities have been taken from F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).

[20] The variational binding energy calculations of E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937), show that the condition $a=b$ gives reasonable results for light-nuclei binding energies. The same may be seen from related earlier calculations of E. Feenberg and E. Wigner, Phys. Rev. **51**, 95 (1937).

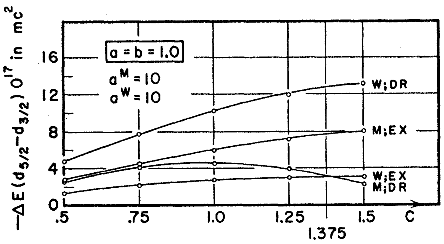

FIG. 1. Graphs of contributions to ground-level splitting of O^{17} plotted against range parameter c for fixed values of range parameters $a=b=1.0$.

$F^{17}-O^{17}$ to $\Delta E_c(F^{17}-O^{17})$ of Eq. (39), and using the previously found value of $a=b=1.2$ for the N^{15} s- and p-shell constants, one obtains $c=1.37$ for the O^{17} d-shell constant.

Alternatively, estimates of a, b, and c may be obtained by consideration of Eqs. (42) and (43). If one assumes $g=0.2$, Eq. (43) with $a=b=1.2$ gives $-42.7\ mc^2$ for the binding energy of the hole in N^{15}, which is seen to be about 40% greater than the experimental value of $-30.6\ mc^2$. If again one uses $g=0.2$, Eq. (42) with $a=b=1.2$, $c=1.37$ gives $+12.0\ mc^2$ (no binding) for the binding energy of the "last" neutron of O^{17} and the latter value is to be compared with the experimental value of $-8.11\ mc^2$.

Upon examining the dependence of Eq. (42) ($g=0.2$) upon a, b, and c (assuming $a=b$), from 0.1 to 3.0, it was found that maximum "binding" of the "last" neutron of O^{17} occurs when $a=b\approx 0.7$ and $c\approx 0.9$ and is $+2.4\ mc^2$. It is found also that Eq. (43) ($a=b$, $g=0.2$) gives the experimental value ($-30.6\ mc^2$) for the "hole" binding energy of N^{15} when $a=b\approx 0.75$. It thus appears that the potential given by Eq. (41) is not adequate to account simultaneously for the binding energy of the "last" neutron of O^{17} and of the hole of N^{15}, provided the s- and p-shell paramaters of N^{15}, O^{16}, and O^{17} are assumed to be approximately equal.

Using $a=b=0.75$ and $c=0.9$, which give the experimental "hole" binding and maximum "last" neutron binding, it is found that $\Delta E_c(O^{15}-N^{15})=2.97$ Mev and $\Delta E_c(F^{17}-O^{17})=2.94$ Mev, which are seen to be about 0.6 Mev lower than the respective experimental binding energy differences.

If the rms radii of N^{15} and O^{17} are calculated using Gaussian functions and are set equal to the radii as determined from $r=r_0A^{\frac{1}{3}}$ ($r_0=1.2\times 10^{-13}$ cm), the resulting p- and s-shell parameters are found to be $b=1.04$ and $c=1.35$. These values are seen to be in

TABLE I. Values of spin-orbit interaction parameters for two sets of wave function range parameters a, b, and c.

$a=b$	c	$a^M(N^{15})$	$a^W(N^{15})$	$a^M(O^{17})$	$a^W(O^{17})$
1.2	1.37	12.4	8.15	6.7	4.3
0.9	0.9	19.0	13.5	10.8	8.7

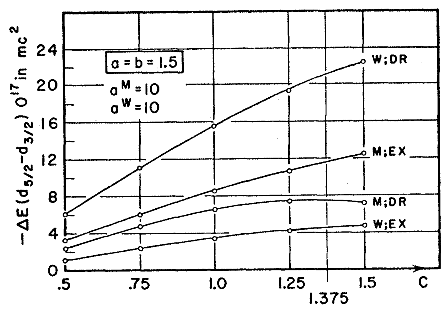

FIG. 2. Graphs of contributions to ground-level splitting of O^{17} plotted against range parameter c for fixed values of range parameters $a=b=1.5$.

rough agreement with those determined by the Coulomb energy difference method.

As a compromise of the values determined by the foregoing methods, the values $a=b=0.9$ and $c=0.9$ shall be used in obtaining splitting estimates in Sec. V.

V. DETERMINATION AND COMPARISON OF SPIN-ORBIT INTERACTION CONSTANTS

The total splitting ΔE^X ($X=M$, W or H), from the two parts of Eqs. (1), (2) and (3) is given in terms of Eqs. (23) through (38) and may be written:

$$\Delta E^X = a^X(\Delta E_{\rm dir}{}^{X1}-\Delta E_{\rm dir}{}^{X0})+a^X(\Delta E_{\rm ex}{}^{X1}-\Delta E_{\rm ex}{}^{X0}) +\Delta E_{\rm dir}{}^{X0}+\Delta E_{\rm ex}{}^{X0}. \quad (44)$$

The dependence of ΔE^X upon a, b, and c, for the $(d_{\frac{5}{2}}-d_{\frac{3}{2}})O^{17}$ and $(p_{\frac{3}{2}}-p_{\frac{1}{2}})N^{15}$ splittings (for a^X arbitrarily taken to be 10) is shown in Figs. 1, 2, 3. It may be seen that these quantities are quite sensitive to variations in a, b, and c and also that the exchange integral contributions are relatively large.

If the experimental values of the $(p_{\frac{3}{2}}-p_{\frac{1}{2}})N^{15}$ splitting ($+12.4\ mc^2$) and the $(d_{\frac{5}{2}}-d_{\frac{3}{2}})O^{17}$ splitting ($-10\ mc^2$) are compared with the calculated splittings, the constants a^X, which are needed to give agreement, are as in Table I.

It may be noted that the Heisenberg interaction of Eq. (2) has not been considered since it gives an inverted ordering of levels from that which is required in the N^{15} and O^{17} splittings.

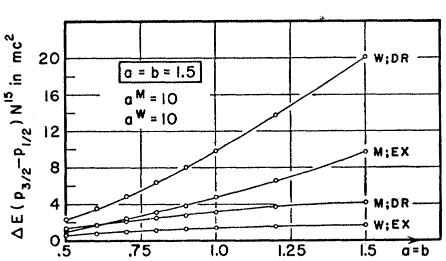

FIG. 3. Graphs of contributions to ground-level splitting of $^1N^5$ plotted against range parameter c for fixed values of range parameters $a=b=1.5$.

It is seen that $a^M(N^{15})$ or $a^W(N^{15})$ must be about twice as large as $a^M(O^{17})$ or $a^W(O^{17})$. Elliot and Lane,[16] using a phenomenological spin-orbit potential of the type

$$V_{s.o.}'=J_{12}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot[(\mathbf{p}_1-\mathbf{p}_2)\times(\mathbf{r}_1-\mathbf{r}_2)], \quad (45)$$

where

$$J_{12}=V_0'\frac{e^{-r_{12}/r_0}}{(r_{12}/r_0)},$$

and using Gaussian wave functions, have found that $V_0'(N^{15})$ must be about 1.8 times larger than $V_0'(O^{17})$ to give agreement with the experimental data. There does not seem to be a direct connection between their result and the present one because their spin-orbit force is purely phenomenological while the one used in the present calculation is not.

Breit and Stehn[4] have given an expression for the $(p_{\frac{3}{2}}-p_{\frac{1}{2}})$ splitting [their Eqs. (8)] of Li^7. The constants a^M, a^W, and a^H needed to give the 479-kev splitting were found by them to be between 2 and 4, depending upon the potential mixture assumed. Their constants were based upon $V(r_{12})=-A'\exp(-\bar{\alpha}'r_{12}^2)$; $A'=72\ mc^2$ and $\bar{\alpha}'=16M\ mc^2/\hbar^2$; and $\bar{\mu}=\bar{\nu}=1.6\ \bar{\alpha}'$. Abraham[16] has obtained the Gaussian wave function parameters for Li^7 from a consideration of Be^7-Li^7 binding energy difference and also from calculating the binding energy of Li^7. The equivalent constants which he finds by these methods are, respectively, $\bar{\mu}=\bar{\nu}=1.396\bar{\alpha}$ and $\bar{\mu}=\bar{\nu}=1.16\bar{\alpha}$ [$\bar{\alpha}=22\ (M\ mc^2/\hbar^2)$]. Considering the splitting to be due to either a Majorana or Wigner potential, one finds from Eq. (8) of the paper of Breit and Stehn that $a^M(Li^7)=3.36$ or 4.47 and $a^W(Li^7)=2.3$ or 3.06 using the above criteria of the binding energy difference or binding energy, respectively.

Assuming a Gaussian spin-orbit force of the type given by Eq. (44) and Gaussian wave functions, one may easily obtain $\Delta E(Li^7)$ [Abraham,[16] first term of Eqs. (16a) or (16b)] and $\Delta E(N^{15})$ and $\Delta E(O^{17})$ (Talmi[16]; Secs. 10B and 10D; making allowance for factor of $\frac{1}{2}$ in the interaction used). The constants found, which are required to give the experimental splittings, are as follows: $V_0(Li^7)=-0.75$ Mev (an average of Abraham's two values); $V_0(N^{15})=-3.4$ Mev and $V_0(O^{17})=-1.34$ Mev (found by setting Talmi's $\bar{\nu}=0.9\times22(M\ mc^2/\hbar^2)$.

These interaction constants are seen to be in the approximate ratio

$$V_0(Li^7):V_0(N^{15}):V_0(O^{17})=0.56:2.54:1.$$

The approximate ratios of the presently determined interaction constants are

$$a^W(Li^7):a^W(N^{15}):a^W(O^{17})=0.264:1.56:1$$

and

$$a^M(Li^7):a^M(N^{15}):a^M(O^{17})=0.31:1.76:1,$$

using in the Li^7 calculation the Be^7-Li^7 binding energy radius. There is thus an approximate correspondence in the ratios of effective interaction constants for these nuclei even though the approaches used are quite different. The considerations of Abraham and Talmi are in a sense more phenomenological than those employed in the present work since the latter makes use of relativistic invariance through Breit's invariant forms of order v^2/c^2.

CONCLUSIONS

The present investigation seems to indicate that it is not possible to account for both the N^{15} and the O^{17} splittings, using the same value of spin-orbit interaction constant in Eqs. (1) and (2). By comparing the constants, a^X, needed to account for the N^{15} and O^{17} splitting with those needed to account for the Li^7 splitting,[4] it is seen that they are in the approximate ratio $a^X(Li^7):a^X(N^{15}):a^X(O^{17})=\frac{1}{4}:\frac{3}{2}:1$. This lack of agreement does not seem to be attributable to the form of interaction given by Eqs. (1), (2), and (3), for a similar result was obtained by Elliot and Lane[16] using a Yukawa spin-orbit force given by Eq. (45). Also the calculations of Abraham[16] and Talmi[16] enable one to conclude that a Gaussian spin-orbit force of the type given by Eq. (45) is inadequate to account for the Li^7, N^{15}, and O^{17} splittings with the same strength of spin-orbit interaction.

It may be mentioned that the existence of excited configurations has not been considered here, which may effect the above conclusions.

The approximation of representing two-body spin-orbit forces in terms of an equivalent one-body force does not appear to be valid in the cases of N^{15} and O^{17} as is evidenced by the relatively large exchange-integral contributions to the total splittings.

ACKNOWLEDGMENTS

It is with pleasure that the author thanks Professor G. Breit for suggesting this problem, and for his constant encouragement and valuable advice throughout all phases of this work.

The author also wishes to express his gratitude to William S. Porter and John P. Lazarus for help in checking many of the calculations, and to Mrs. Maureen Berry for help with numerical calculations.

APPENDIX A. COULOMB INTEGRALS

$$I_c[\ ]\equiv\int\exp\{-(ar_1^2+br_2^2)\}[\ ]d\mathbf{r}_1d\mathbf{r}_2/r_{12}, \quad (A\text{-}1)$$

$$F\equiv a+b,$$

$$I_c[r_1^3r_2^3\cos\theta_{12}]=(5\pi^{\frac{5}{2}}/4)a^{-2}b^{-2}(2a^2+7ab+2b^2)F^{-7/2}, \quad (A\text{-}2)$$

$$I_c[r_1^2r_2^2\cos^2\theta_{12}]=(\pi^{\frac{5}{2}}/2)a^{-2}b^{-2}(2a^2+7ab+2b^2)F^{-\frac{5}{2}}, \quad (A\text{-}3)$$

$$I_c[r_1^2r_2^2\sin^2\theta_{12}]=(2\pi^{\frac{5}{2}})a^{-2}b^{-2}F^{-\frac{1}{2}}, \quad (A\text{-}4)$$

$$\int \exp[-a(r_1^2+r_2^2)]\frac{r_1^3r_2^3\cos\theta_{12}\sin^2\theta_{12}}{r_{12}}d\mathbf{r}_1d\mathbf{r}_2 = 16\pi^{\frac{5}{2}}(2a)^{-11/2}. \quad \text{(A-5)}$$

These integrals (and other simpler ones) were evaluated by differentiating

$$I'=\int \frac{\exp\{-(ar_1^2+br_2^2+cr_{12}^2)\}}{r_{12}}d\mathbf{r}_1d\mathbf{r}_2$$

with respect to the parameters a and b, and by iterations of the operator $\mathcal{O}\equiv\frac{1}{2}(\partial/\partial a+\partial/\partial b-\partial/\partial c)$ which is seen to be equivalent to multiplication of an integrand by $r_1r_2\cos\theta_{12}$. The use of the operator $\mathcal{O}$ was suggested to the author by Professor G. Breit.

Many Gaussian integrals of the form

$$I''[\ ]=\int \exp\{-(ar_1^2+br_2^2+cr_{12}^2)\}[\ ]d\mathbf{r}_1d\mathbf{r}_2$$

were evaluated in a similar manner.

PHYSICAL REVIEW VOLUME 106, NUMBER 3 MAY 1, 1957

Production and Properties of the Nuclides Fermium-250, 251, and 252†

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(Received January 22, 1957)

The nuclides Fm^{250}, Fm^{251}, and Fm^{252} were produced by alpha bombardment of Cf^{249}. The excitation functions for their formation, as well as some of their nuclear properties, were measured.

INTRODUCTION

In a previous paper from this laboratory[1] the production of some einsteinium isotopes by alpha bombardment of a target of Bk^{249} was described. Bk^{249} decays with a half-life of 280 days by beta emission to the 5×10^2-year alpha-emitting Cf^{249}. This paper will describe some studies of reactions of the type (α,xn) brought about by bombarding Cf^{249} with helium ions in the energy region 20 to 40 Mev. The experimental technique, which was fully described earlier,[1] involved catching the reaction products recoiling from the thin target in a separate gold foil. Thus, it is possible to use the same target for several bombardments. The target used in the present experiments was the same one as used in the irradiations of Bk^{249} although it now contained about 10^{13} atoms of Cf^{249} grown in from the original 3×10^{13} atoms of Bk^{249}. In fact, this target has been subjected to about 100 bombardments or a total of roughly 1000 μa-hr.

The chemical purification and separation of the products involved mainly ion exchange techniques and electroplating as described before.[1]

RESULTS

The fermium isotopes produced and studied in these experiments were Fm^{250}, Fm^{251}, and Fm^{252}. Of these, Fm^{250} was produced earlier at Stockholm and later at Berkeley by oxygen bombardment of uranium,[2] and Fm^{252} was produced at Berkeley by several of the above authors by alpha bombardment of targets containing the isotopes Cf^{249}, Cf^{250}, Cf^{251}, and Cf^{252}. However, the mass assignments are not certain on the basis of this work.

The element identification was established by means of a cation exchange column separation using alpha-hydroxy isobutyric acid as eluant.[3] Mass assignments were based on the excitation functions. The properties of these nuclides are summarized in Table I. The half-lives given are good to about $\pm10\%$ and the alpha particle energies to ±0.05 Mev.

The amounts of Fm^{250} produced correspond to about 40 alpha counts per minute at the end of the bombard-

Table I. Nuclear properties of light fermium isotopes.

Isotope	Type of decay	Half-life	Alpha-particle energy	Branching ratio electron capture/alpha
Fm^{250}	α, E.C.?	30 min	7.43	E.C. not observed
Fm^{251}	E.C., α	7 hr	6.89	$\sim$100
Fm^{252}	α	30 hr	7.05	β-stable[a]

[a] Glass, Thompson, and Seaborg, J. Inorg. Nuclear Chem. 1, 3 (1955).

† Work done under the auspices of the U. S. Atomic Energy Commission.

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[3] Choppin, Harvey, and Thompson, J. Inorg. Nuclear Chem. **2**, 66 (1956).