and D in Eq. (11), the result:

$$\left(\frac{dW}{dp}\right)_{p=1} \approx \frac{2}{\alpha^2 r_s^2}+\frac{2}{\pi\alpha r_s}\int_{-1}^{1}\frac{xdx}{2(1-x)} \times\left[1+\frac{4\alpha r_s}{\pi}\frac{1}{2(1-x)}\right]^{-1}. \quad (19)$$

or

$$\left(\frac{dW}{dp}\right)_{p=1} = \frac{2}{\alpha^2 r_s^2}+\frac{1}{\pi\alpha r_s}\{\ln(\pi/\alpha r_s)-2\}+\cdots. \quad (20)$$

At low temperature, then, the specific heat of a free electron gas is modified through Coulomb interactions by the factor

$$C/C_F=\left[1+\frac{\alpha r_s}{2\pi}[-\ln r_s+\ln(\pi/\alpha)-2]+\cdots\right]^{-1}, \quad (21)$$

where the expansion is valid at high densities. Inserting numerical values, we have

$$C/C_F=[1+0.083r_s(-\ln r_s-0.203)+\cdots]^{-1}. \quad (22)$$

This is to be compared with the approximate result of Pines,[2] who finds

$$C/C_F=[1+0.083r_s(-\ln r_s+1.47)+\cdots]^{-1}. \quad (23)$$

The method given here permits the computation of higher terms in the series, and the next correction is now being calculated. Applications to the specific heats of metals are also being studied.

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Correlation Energy of an Electron Gas at High Density

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The correlation energy of an electron gas at high density is evaluated up to terms of orders $(r_s)^0$ and $\log r_s$. It is shown that the correlation energy to this order can be evaluated without using perturbation treatment. The result obtained coincides with the result of formal summation of apparently divergent series arising from small momentum transfer processes which has been discussed fully by Gell-Mann and Brueckner. The method to treat the small momentum transfer effect exactly is given by following the analogy of processes with well-known treatment of systems with Hamiltonians which are bilinear in creation and annihilation operators such as the neutral scalar pair theory. Some simple interpretations of the correlation energy to this order are also given.

RECENTLY, it was shown by Gell-Mann and Brueckner[1] that the correlation energy of an electron gas at high density can be evaluated exactly to order of the constant term $(r_s)^0$ and the term of $\log r_s$, where r_s is the ratio of the radius of a volume within which one electron exists to the Bohr radius and energy is measured in terms of Rydbergs. (The kinetic energy is $2.21/r_s^2$, and the exchange energy is $-0.916/r_s$ per electron.) Their result was obtained by a selected summation of the formally divergent power series expansion of Rayleigh-Schrödinger perturbation theory. This procedure introduces some uncertainties into the final result; in fact, it can be seen that the constant term in the energy contains very curious divergences when the sum is taken in a straightforward manner. Regarding these terms as unphysical, the aforementioned authors discarded them and were able to obtain a unique result.

The simple structure of the diagrams summed by G-B. suggests that it may be possible to find a more rigorous way to get their answer without using a perturbation procedure. In Fig. 1 we give a typical diagram which contributes to the terms of order $(r_s)^0$ and $\ln r_s$. Formally, only the second-order iterated Coulomb

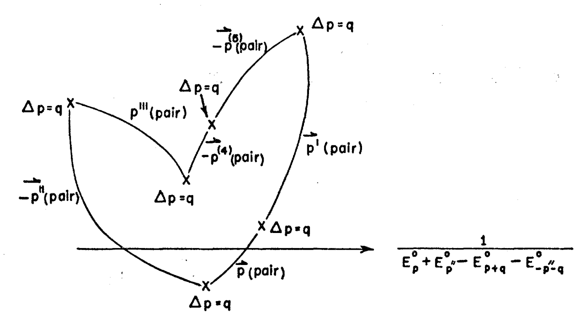

FIG. 1. Typical diagram of a process contributing to the correlation energy terms of order $(r_s)^0$ and $\ln r_s$.

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[1] M. Gell-Mann and K. A. Brueckner, this issue Phys. Rev. **106**, 364 (1957), hereafter referred to as G-B.

interaction energy is of e^4 and hence of order $(r_s)^0$ when measured in rydbergs. Namely, in the notations given in reference 1, the e^4-correlation energy becomes

$$\frac{32\pi^2\hbar^4e^4}{\Omega^2}\sum_q\frac{1}{q^2}\sum_p\sum_{p'}\frac{1}{q^2}\frac{1}{\dfrac{p^2}{2m}+\dfrac{p'^2}{2m}-\dfrac{(\mathbf{p}+\mathbf{q})^2}{2m}-\dfrac{(\mathbf{p}+\mathbf{q})^2}{2m}}$$

$$-\frac{16\pi^2\hbar^4e^4}{\Omega^2}\sum_q\frac{1}{q^2}\sum_p\sum_{p'}\frac{1}{(\mathbf{q}+\mathbf{p}+\mathbf{p}')^2}$$

$$\times\frac{1}{\dfrac{p^2}{2m}+\dfrac{p'^2}{2m}-\dfrac{(\mathbf{p}+\mathbf{q})^2}{2m}-\dfrac{(\mathbf{p}'+\mathbf{q})^2}{2m}}=\epsilon_{(a)}{}^{(2)}+\epsilon_{(b)}{}^{(2)},$$

corresponding to the processes in which electrons in the Fermi sea with momentum $\mathbf{p}$ and $-\mathbf{p}'(|\mathbf{p}|, |\mathbf{p}'| <$Fermi momentum P) are excited by mutual Coulomb interaction into states with momentum $\mathbf{p}+\mathbf{q}$ and $-\mathbf{p}'-\mathbf{q}(|\mathbf{p}+\mathbf{q}|, |\mathbf{p}'+\mathbf{q}|>P)$, and then return to their original positions in the sea ($\epsilon_{(a)}{}^{(2)}$, first term) or exchanged positions ($\epsilon_{(b)}{}^{(2)}$, second term) on the second action of the mutual Coulomb interaction ($\sum_p,\sum_{p'}$ above do not include a spin sum). However, since $\epsilon_{(a)}{}^{(2)}$ contains an infrared divergence arising from low momentum transfers, terms of formally higher order in e^2 also must be taken into account because they remove the divergence (since the theory must give a finite energy of correlation). The necessary subset of higher order contributions were selected by G-B from the following considerations. First, the work of Bohm and Pines[2] on the plasma oscillation of an electron cloud suggests that the Coulomb field of an electron is effectively screened at the range $\sim r_0^{\frac{1}{2}}a^{\frac{1}{2}}$ (where $a=$Bohr radius) which in turn means the momentum transfer between electrons is cut off at the order of a minimum momentum $q_{\min}\sim r_s^{2}$. The terms in a given perturbation order which can contribute to this cutoff can easily be selected by inspection of the perturbation series. In each order the term which diverges most strongly for small momentum transfers (measured in Rydbergs) is proportional to

$$(r_s/q_{\min}{}^2)^n,$$

and hence, if $q_{\min}{}^2\sim r_s$, will contribute to the constant term in the energy. In the same order, terms appear proportional to higher powers of r_s, which will not contribute to the energy in the desired approximation. These remarks are, of course, based on the assumed character of the low-momentum cutoff; this will be verified in the following.

The processes which contribute to the energy in this approximation all depend on the same factor e^2/q^2 in each interaction, and so come from processes with the same momentum transfer q in each interaction. Other terms which are less singular in each power of e^2 need not be considered in higher order than e^4. For example, when an exchange interaction takes place between electrons with momentum $\mathbf{p}$ and $-\mathbf{p}'$, there appears instead of e^2/q^2

$$e^2/(\mathbf{q}+\mathbf{p}+\mathbf{p}')^2,$$

as a matrix element, and since this factor is regular as $q\rightarrow 0$ (then $|\mathbf{p}|$, $|\mathbf{p}'|\approx$Fermi momentum) and is of order r_s, the term which contains one exchange interaction leads to a contribution of higher power in r_s. Hence, we need not consider exchange interaction, or other processes which do not carry momentum transfer e^2/q^2 as a matrix element in the higher orders e^6, e^8, $\cdots$.

A typical diagram of higher order of the kind required is drawn in Fig. 1, where × indicates a Coulomb interaction, and the diagram is to be read in time sequence from the bottom upwards, $\mathbf{p}$ $(-\mathbf{p}'')$ pair, etc. represents an excited electron with momentum $\mathbf{p}+\mathbf{q}$ $(-\mathbf{p}''-\mathbf{q})$ and a hole in the Fermi sea with momentum $-\mathbf{p}$ $(\mathbf{p}'')$. From the consideration stated above, at each interaction points the momentum transfer is $\Delta p=q$. (The cross point has a factor $4\pi\hbar^2e^2/\Omega q^2$, Ω being the normalization volume, except for the one point lying lowest which carries $2\pi\hbar^2e^2/\Omega q^2$, because we count the diagrams which are mirror conjugate as to the vertical axis as different diagrams.) Reading the diagram from the bottom up, supplying a factor $4\pi\hbar^2e^2/\Omega q^2$ at each cross-point and the energy denominator between each cross-point (as indicated in Fig. 1 for one case, $E_p{}^0=p^2/2m$), we can get a part of the interaction energy from this one diagram (of order e^{12}), then arranging the cross-points in all possible time orders, summing all of these contributions, and performing the summation over the momentum transfer q, we get the correlation energy to this order which contributes to the $(r_s)^0$ and $\ln r_s$ terms.

From the close correspondence of this diagram to the Feynman diagram in field theory, G-B showed that we can use a propagator for pairs between interactions. The pair created with momentum transfer q propagates from one interaction to the next, interacting with momentum transfer $\Delta p=q$ and changing itself into another pair (but with the same momentum difference between excited electron and hole). In other words, the pair propagates as if it were a single particle.

Observing these facts, we can compare our interaction diagrams with that of the well-known and exactly solved problem of the interaction energy of an infinitely heavy particle interacting with, for example, a neutral scalar meson field, through the product potential (core interaction), which can be represented by the following typical Hamiltonian:

$$H_{\text{total}}=H_0+H_{\text{int}},\quad H_0=\sum_k\omega_k\phi_k{}^*\phi_k,$$

$$H_{\text{int}}=\frac{\eta}{2\Omega}\sum_k\frac{1}{(2\omega_k)^{\frac{1}{2}}}(\phi_k{}^*+\phi_k)\cdot\sum_{k'}\frac{1}{(2\omega_{k'})^{\frac{1}{2}}}(\phi_{k'}{}^*+\phi_{k'}),\quad\text{(C)}$$

[2] D. Bohm and D. Pines, Phys. Rev. 92, 626 (1953).

where ϕ_k is the annihilation operator of meson with momentum k, $\omega_k=(\mu^2+k^2)^{\frac{1}{2}}$ is the energy of the meson, η is the coupling constant, and Ω the normalization volume. If we evaluate the energy of this system for the ground state (no meson present as $\eta\rightarrow 0$) in perturbation series, we encounter the same type of diagrams as in Fig. 1. Figure 2 gives the analogous diagram for this interaction. In this figure each cross-point carries a factor

$$\frac{\eta}{\Omega}\frac{1}{(2\omega_k)^{\frac{1}{2}}}\frac{1}{(2\omega_{k'})^{\frac{1}{2}}}$$

(for the $\mathbf{k}\rightarrow\mathbf{k}'$ cross-point), and the construction of the perturbation series can be made completely in the same way as in Fig. 1. We can also speak of propagation of created mesons. Thus, the summation of G-B has the same character as the summation of perturbation series in this core interaction case. Since the latter problem can be solved exactly without a power series expansion by using a normal coordinate transformation [because H_{total} of (C) is a bilinear form] and was solved by Wentzel,[3] our problem concerning the interaction selected as in Fig. 1 must also be soluble exactly without power series expansion. In our case, however, a complication arises in the definition of the normal coordinates, because the Hamiltonian of our system is not bilinear form, and so we shall develop an alternative method to obtain effectively the same answer as one gets using the normal coordinate transformation in the case of a neutral scalar meson interaction. The result which we finally obtain is identical with the result of formal summation of G-B, showing that the summing of series of powers in e^2 is allowed and that the divergences are in fact spurious and may be neglected.

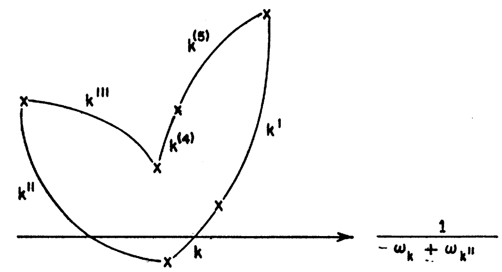

FIG. 2. Typical diagram of the energy of the ground state in the core interaction of neutral scalar mesons. (The denominator of the fraction should read $-\omega_k-\omega_{k''}$.)

1. FUNDAMENTAL FORMULAS

To evaluate the contribution of small-momentum-transfer effects, we first write the total Hamiltonian of the system by means of second-quantized operators as follows:

$$H_{\text{total}}=H_0+H_C,\quad H_0=\sum_{|\mathbf{p}|>P}\frac{p^2}{2m}a_p^*a_p-\sum_{|\mathbf{p}|<P}\frac{p^2}{2m}b_p^*b_p+\sum_{|\mathbf{p}|<P}\frac{p^2}{2m},$$

$$H_C\equiv H_{\text{Coulomb}}=\frac{2\pi\hbar^2e^2}{\Omega}\sum_q\frac{1}{q^2}\sum_p\sum_{p'}\times(a_{p+q}^*b_{p'-q}^*a_{p'}b_p^*+b_{p+q}a_{p'-q}^*b_{p'}^*a_p+a_{p+q}^*a_{p'-q}^*b_{p'}^*b_p^*+b_{p+q}b_{p'-q}a_{p'}a_p \tag{1}$$

+ exchange interaction of electrons excited and that of holes, and terms of odd number of creation operator),

[3] G. Wentzel, Helv. Phys. Acta 15, 111 (1942).

where a_p, b_p are the annihilation operators of excited particle and hole in the Fermi sea with momentum $\mathbf{p}$ and $-\mathbf{p}$, respectively, and P is the Fermi momentum.

Now to get the interaction energy whose power series expansion is represented by the series in G-B and schematically drawn in Fig. 1, it is sufficient to treat the problem with an interaction Hamiltonian which does not contain exchange interactions of excited electrons, exchange interactions of holes, or interactions with odd numbers of creation operators. These do not occur in our diagrams, where there appear only interactions which create or annihilate two pairs simultaneously or represent scattering of one pair to another pair. Hence, in (1) we can omit these terms from the Coulomb interaction energy.

We can further rewrite this simplified Coulomb interaction to emphasize the suggested analogy with the case of the neutral scalar meson interacting through a product potential with an infinitely heavy particle (C):

$$H_C=\frac{2\pi\hbar^2e^2}{\Omega}\sum_q\frac{1}{q^2}\Big[\sum_p(a_{p+q}^*b_p^*+b_{p+q}a_p)\cdot\sum_{p'}(a_{p'-q}^*b_{p'}^*+b_{p'-q}a_{p'})-\sum_{|\mathbf{p}-\mathbf{q}|<P}a_p^*a_p+\sum_{|\mathbf{p}-\mathbf{q}|>P}b_p^*b_p-\sum_{|\mathbf{p}-\mathbf{q}|>P;|\mathbf{p}|<P}\cdots\Big], \tag{2}$$

where the last three terms in this expression may be neglected, because the first two of these represent the difference of the numbers of excited particles and holes and is a constant of the motion in our case, its eigenvalue being zero, and the last term is merely a constant (which we need to take into account only when finally computing expectation value of H_C). All the sums $\sum_p$, $\sum_{p'}$ include spin sums. The simplified interaction (2) still contains interaction leading to $\epsilon_{(b)}^{(2)}$ and higher exchange interactions between holes and excited electrons, but since we need only maintain $\epsilon_{(b)}^{(2)}$ among these exchange interactions, we take account of these interactions only in determining the final total correlation energy.

Then, looking at the effect of interaction (2) on the motion of a pair, we see from the diagram drawn by

G-B that the motion of the pair is affected only by other pairs with the same momentum transfer (a **p** pair transmutes into a **p**″ pair with momentum transfer $\Delta p=q$ in Fig. 1). Hence, putting aside the exchange interaction of order $(r_s)^0$, $\epsilon_{(b)}{}^{(2)}$, we only consider the equation of motion of pairs interacting through interactions which lead to our interaction diagram Fig. 1. The commutator between $a_{p-q}{}^*b_p{}^*$, is

$$[a_{p-q}{}^*b_p{}^*,H_C]_-=\frac{2\pi\hbar^2e^2}{\Omega}\sum_{q'}\frac{1}{q'^2}$$
$$\times[(a_{p-q}{}^*a_{p-q'}-b_{p+q'-q}b_p{}^*)$$
$$\times\sum_{p'}(a_{p'-q'}{}^*b_{p'}{}^*+b_{p'-q'}a_{p'})$$
$$+\sum_{p'}(a_{p'+q'}{}^*b_{p'}{}^*+b_{p'+q'}a_{p'})$$
$$\times(a_{p-q}{}^*a_{p+q'}-b_{p-q-q'}b_p{}^*)].\quad(3)$$

To the desired approximation this may be taken to be

$$[a_{p-q}{}^*b_p{}^*,H_C]_-=-\frac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p'}(a_{p'-q}{}^*b_{p'}{}^*+b_{p'-q}a_{p'}),$$

where we have omitted the influence of pairs with different momentum transfers $q'\neq q$. To get (3), we have also omitted the number operators referring to a particular particle, $N_{p-q}{}^a(=a_{p-q}{}^*a_{p-q})$ and $N_p{}^b$, because these lead to one factor $1/\Omega$ higher in the energy (and hence independent of the number of electrons). This can be seen, for example, by operating the second expression of (3) on the ground state of H_0. (One summation $\sum_{p'}$ which carries Ω in the form $\sum_{p'}=[\Omega/(2\pi\hbar)^3]\times\int d\mathbf{p}'$ drops out for the term containing $N_{p-q}{}^a$ or $N_p{}^b$.)

Similarly, in our approximation,

$$[b_{p-q}a_p,H_C]_-=\frac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p'}(a_{p'-q}{}^*b_{p'}{}^*+b_{p'-q}a_{p'}).\quad(4)$$

[Of course, to get the correlation energy to order $(r_s)^0$ and $\ln(r_s)$, we must add $\epsilon_{(b)}{}^{(2)}$ to our correlation energy obtained from our commutators (3) and (4).]

These simple commutators derived from our consideration based upon the effect of interaction in our diagram [together with our simplified interaction energy (2)], just correspond to the case of neutral scalar meson interaction with product potential. In the latter case, the commutator between $\phi_k{}^*$ or ϕ_k with H_{int} can be seen from (C) to be

$$[\phi_k{}^*,H_{\text{int}}]=-\frac{\eta}{\Omega}\frac{1}{(2\omega_k)^{\frac{1}{2}}}\sum_{k'}\frac{1}{(2\omega_{k'})^{\frac{1}{2}}}(\phi_{k'}{}^*+\phi_{k'}),$$
$$[\phi_k,H_{\text{int}}]=\frac{\eta}{\Omega}\frac{1}{(2\omega_k)^{\frac{1}{2}}}\sum_{k'}\frac{1}{(2\omega_{k'})^{\frac{1}{2}}}(\phi_{k'}{}^*+\phi_{k'}),$$

and thus to be of the same form as (3) and (4). We can now proceed in the same way as in the core interaction case by using our Hamiltonian and commutation relations. But, one difference is that since our Hamiltonian is not of bilinear form it is hard to define normal coordinates. Nevertheless, from the interaction diagram, we expect the energy of these two systems to have the same structure, and hence we may obtain some clue to the solution by proceeding in a parallel way.

2. EIGENVALUE OF ENERGY OF ONE-PAIR STATE

Corresponding to the equation which determines the normal frequencies in the neutral meson case, we construct an eigenvalue equation for the one-pair state. To construct the eigenvalue equation, we write the equation of motion of the variable representing a pair, $a_{p-q}{}^*b_p{}^*$ and $b_{p-q}a_p$, in the Schrödinger representation. We take for Ψ some eigenstate of the total Hamiltonian $H_T(=H_0+H_C{}^{(2)})$ of our simplified problem with eigenvalue E, and for Ψ_0 the exact ground state (no pairs when the interaction vanishes) with energy E_0. Then, using our commutators (3) and (4), we get

$$(\Psi,\,(E-H_T)a_{p-q}{}^*b_p{}^*\Psi_0)=(E-E_0-E_{p-q}{}^0+E_p{}^0)(\Psi,a_{p-q}{}^*b_p{}^*\Psi_0)$$
$$-\frac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p'}(\Psi,\,(a_{p'-q}{}^*b_{p'}{}^*+b_{p'-q}a_{p'})\Psi_0)=0,\quad|\mathbf{p}|<P,\ |\mathbf{p}-\mathbf{q}|>P,\ p^2/2m=E_p{}^0,\quad(5)$$

and

$$(\Psi,\,(E-H_T)b_{p-q}a_p\Psi_0)=(E-E_0-E_{p-q}{}^0+E_p{}^0)(\Psi,b_{p-q}a_p\Psi_0)$$
$$+\frac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p'}(\Psi,\,(a_{p'-q}{}^*b_{p'}{}^*+b_{p'-q}a_{p'})\Psi_0)=0,\quad|\mathbf{p}|>P,\ |\mathbf{p}-\mathbf{q}|<P.\quad(6)$$

Eliminating the amplitude $(\Psi,b_{p'-q}a_{p'}\Psi_0)$ from (5) by using (6), we reach the eigenvalue equation†:

$$\left(1+\frac{4\pi\hbar^2e^2}{\Omega q^2}\Big(\sum_{\substack{p\\|\mathbf{p}|<P\\|\mathbf{p}-\mathbf{q}|>P}}-\sum_{\substack{p\\|\mathbf{p}|>P\\|\mathbf{p}-\mathbf{q}|<P}}\Big)\frac{1}{E_0-E+E_{p-q}{}^0-E_p{}^0}\right)\sum_{p'}(\Psi,\,(a_{p'-q}{}^*b_{p'}{}^*+b_{p'-q}a_{p'})\Psi_0)=0.\quad(7)$$

† *Note added in proof.*—It has been pointed out by Dr. R. Brout that the eigenvalue equation (7) has a bound state solution corresponding to the plasma oscillation discussed by Bohm and Pines.[2] A detailed discussion of the effects of this state on the wave function and energy will be published shortly by Brueckner, Fukuda, and the author. The inclusion of the plasma mode decreases the large negative constant term in (24), and preliminary evaluation shows that the correlation energy due to plasma mode is constant and about +0.13 rydberg per electron at high density limit, and so the final answer seems to coincide with the value obtained by G-B.

To determine the eigenvalue E, we consider a finite (but very large) normalization volume Ω. The energies $E_{p-q}{}^0$, $E_p{}^0$ then form a discrete set of eigenvalues with a level spacing of order $1/\Omega^{\frac{1}{3}}$. Then, the root of (7) exists between $E_1=E_0+E_{p_0-q}{}^0-E_{p_0}{}^0$ (where p_0 is some discrete momentum restricted by $|\mathbf{p}_0|<P$, $|\mathbf{p}_0-\mathbf{q}|>P$) and the neighboring discrete energy $E_2=E_0+E_{p_0'-q}{}^0-E_{p_0'}{}^0$, because if one takes E between E_1 and E_2 and if one makes E approach E_1 one obtains $+\infty$ (or $-\infty$) for the quantity in the bracket of (7) and by making E approach E_2 one obtains $-\infty$ (or $+\infty$). Hence a zero point of (7) must exist between E_1 and E_2. Namely, the energy eigenvalue is (with p_0 an arbitrary momentum restricted by $|\mathbf{p}_0|<P$, $|\mathbf{p}_0-\mathbf{q}|>P$)

$$E=E_0+E_{p_0-q}{}^0-E_{p_0}{}^0+O\left(\frac{1}{\Omega^{\frac{1}{3}}}\right). \tag{8}$$

Then, if, and only if, E is given by (8), $\sum_{p'}(\Psi_E, (a_{p'-q}{}^*b_{p'}{}^*+b_{p'-q}a_{p'}), \Psi_0)$ does not vanish. The energy (8) goes over to the energy of the one-pair state if the interaction vanishes, and so the state Ψ_E [E given by (8)] belongs to the one-pair exact solution.

We thus know from Eq. (7) that the energy eigenvalue of the one-pair state (excited electron with momentum $\mathbf{p}_0-\mathbf{q}$ and hole $-\mathbf{p}_0$) is $E_0+E_{p_0-q}{}^0-E_{p_0}{}^0$ in the limit of infinite normalization volume and so our pairs have no self-energies in our approximation. Moreover the following expansion is possible:

$$\sum_p(a_{p-q}{}^*b_p{}^*+b_{p-q}a_p)\Psi_0=\sum_p C_p\Psi_{p-q,\,p}, \tag{9}$$

where $\Psi_{p-q,\,p}$ indicates the exact eigenstate with pair $\mathbf{p}-\mathbf{q}$ (a) and $\mathbf{p}$ (b). Equation (9) follows from the fact that if we take the inner product with an eigenstate which does not belong to the one-pair solution, then, because the energy eigenvalue of such a state does not make the quantity inside the square brackets of (7) vanish, the left hand side of (9) vanishes. The expansion of the "packet" $\sum_p(a_{p-q}{}^*b_p{}^*+b_{p-q}a_p)\Psi_0$ into an eigenstate of H_T, thus, contains only one-pair states.

The expansion (9) enables us to evaluate $(\Psi_0,H_C\Psi_0)$[H_C being given by Eq. (2)] immediately if we know the expansion coefficient C_p, because $(\Psi_0,H_C\Psi_0)$ is the overlap integral of the "packet." To evaluate C_p, we first write the state with one pair $\Psi_{p-q,\,p}$ with operators operating on Ψ_0. Since we now know that our pairs have no self-energies, we can write [we add the superscript (+) to indicate the outgoing-wave solution][4]

$$\Psi_{p-q,\,p}{}^{(+)}=a_{p-q}{}^*b_p{}^*\Psi_0+X_{p-q,\,p}{}^{(+)}. \tag{10}$$

Turning to the continuous spectrum treatment, and using

$$(E_0+E_{p-q}-E_p{}^0-H_T)\Psi_{p-q,\,p}{}^{(+)}=0$$

and (3), we find the equation determining $X^{(+)}$ to be

$$(E_0+E_{p-q}{}^0-E_p{}^0-H_T)\Psi_{p-q,\,p}{}^{(+)}=-\frac{4\pi\hbar^2e^2}{\Omega}\frac{1}{q^2}\sum_{p'}(a_{p'-q}{}^*b_{p'}{}^*+b_{p'-q}a_{p'})\Psi_0+(E_0+E_{p-q}{}^0-E_p{}^0-H_T)X_{p-q,\,p}{}^{(+)}=0.$$

Thus, taking the outgoing-wave solution for X,

$$\Psi_{p-q,\,p}{}^{(+)}=a_{p-q}{}^*b_p{}^*\Psi_0+\frac{1}{E_0+E_{p-q}{}^0-E_p{}^0-H_T+i\epsilon}\frac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p'}(a_{p'-q}{}^*b_{p'}{}^*+b_{p'-q}a_{p'})\Psi_0, \tag{11}$$

where ϵ is positive infinitesimal number. To get the coefficient C_p, we must add to (11) a term $b_{p-q}a_p\Psi_0$, but this can be represented by the analogous equation to (11);

$$0=b_{p-q}a_p\Psi_0-\frac{1}{E_0+E_{p-q}{}^0-E_p{}^0-H_T}\frac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p'}(a_{p'-q}{}^*b_{p'}{}^*+b_{p'-q}a_{p'})\Psi_0, \tag{12}$$

[which is an identity, because the denominator has no pole since $|\mathbf{p}|>P$, $|\mathbf{p}-\mathbf{q}|<P$, and because of the commutator (4)]. Combining (11) and (12), remembering in (11) that $|\mathbf{p}|<P$, $|\mathbf{p}-\mathbf{q}|>P$ and in (12) that $|\mathbf{p}|>P$, $|\mathbf{p}-\mathbf{q}|<P$, we obtain for the expansion (10):

$$\sum_{p'}(a_{p'-q}{}^*b_{p'}{}^*+b_{p'-q}a_{p'})\Psi_0$$
$$=\left\{1-\frac{4\pi\hbar^2e^2}{\Omega q^2}\left[\sum_{\substack{|\mathbf{p}|>P\\|\mathbf{p}-\mathbf{q}|<P}}\frac{1}{E_0+E_{p-q}{}^0-E_p{}^0-H_T}-\sum_{\substack{|\mathbf{p}|<P\\|\mathbf{p}-\mathbf{q}|>P}}\frac{1}{E_0+E_{p-q}{}^0-E_p{}^0-H_T+i\epsilon}\right]\right\}^{-1}\sum_{\substack{p\\|\mathbf{p}|<P\\|\mathbf{p}-\mathbf{q}|>P}}\Psi_{p-q,\,p}{}^{(+)}, \tag{13}$$

[4] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956); G. C. Wick, Revs. Modern Phys. **27**, 339 (1955).

or, in view of (8) ($\Omega\rightarrow\infty$),

$$\sum_{p'}(a_{p'-q}{}^{*}b_{p'}{}^{*}+b_{p'-q}a_{p'})\Psi_0=\sum_{\substack{|\mathbf{p}|<P\\|\mathbf{p}-\mathbf{q}|>P}}\Psi_{p-q,p}{}^{(+)}$$
$$\times\left\{1-\frac{4\pi\hbar^2e^2}{\Omega q^2}\left[\sum_{\substack{|\mathbf{p}'|>P\\|\mathbf{p}'-\mathbf{q}|<P}}\frac{1}{E_{p'-q}{}^0-E_{p'}{}^0-E_{p-q}{}^0+E_p{}^0}-\sum_{\substack{|\mathbf{p}'|<P\\|\mathbf{p}'-\mathbf{q}|>P}}\frac{1}{E_{p'-q}{}^0-E_{p'}{}^0-E_{p-q}{}^0+E_p{}^0+C\epsilon}\right]\right\}^{-1}. \quad (14)$$

Using expression (14), we can evaluate $(\Psi_0,H_c\Psi_0)$ immediately.

3. TOTAL ENERGY

By using (14) and H_C in the form (2), omitting the term $-N^a+N^b$, we have for the expectation value of the Coulomb energy $(\Psi_0,H_c\Psi_0)$ in our approximation:

$$(\Psi_0,H_C\Psi_0)=-\frac{2\pi\hbar^2e^2}{\Omega}\sum_q\frac{1}{q^2}\sum_{\substack{|\mathbf{p}|<P\\|\mathbf{p}-\mathbf{q}|>P}}+\frac{2\pi\hbar^2e^2}{\Omega}\sum_q\frac{1}{q^2}\sum_{\substack{|\mathbf{p}|<P\\|\mathbf{p}-\mathbf{q}|>P}}$$
$$\times\left|1-\frac{4\pi\hbar^2e^2}{\Omega q^2}\left[\sum_{\substack{|\mathbf{p}'|>P\\|\mathbf{p}'-\mathbf{q}|<P}}\frac{1}{E_{p'-q}{}^0-E_{p'}{}^0-E_{p-q}{}^0+E_p{}^0}-\sum_{\substack{|\mathbf{p}'|<P\\|\mathbf{p}'-\mathbf{q}|>P}}\frac{1}{E_{p'-q}{}^0-E_{p'}{}^0-E_{p-q}{}^0+E_p{}^0+i\epsilon}\right]\right|^{-2}. \quad (15)$$

To get the total energy, we use the following procedure. If the system is composed of $H_T=H_0+gH_{\text{int}}$, g being e^2 in our case, the total energy of the ground state is (writing the g dependence explicitly);

$$E_0(g)=(\Psi_0(g),\,(H_0+gH_{\text{int}})\Psi_0(g)).$$

Since

$$\left(\frac{\partial}{\partial g'}(\Psi_0(g'),(H_0+gH_{\text{int}})\Psi_0(g')\right)_{g'=g}=0,$$

owing to the stationary character of the eigenvalue, we find

$$g\frac{\partial}{\partial g}E_0(g)=(\Psi_0(g),gH_{\text{int}}\Psi_0(g)). \quad (16)$$

Hence, integrating

$$E_0(g)=\int_0^g\frac{1}{g'}(\Psi_0(g'),g'H_{\text{int}}\Psi_0(g'))dg'+(\phi_0,H_0\phi_0), \quad (17)$$

ϕ_0 being ground-state wave function without interaction. Then, by using a simple integral,

$$\int_0^g\frac{dg'}{[1+g'(\alpha+i\beta)][1+g'(\alpha-i\beta)]}=\frac{1}{2i\beta}\ln\left(\frac{1+g(\alpha+i\beta)}{1+g(\alpha-i\beta)}\right)=\frac{1}{\beta}\tan^{-1}\left(\frac{g\beta}{1+g\alpha}\right), \quad (18)$$

we can get from (15) and (17), for the correlation energy arising from our simplified problem [small momentum transfer contribution to terms $(r_s)^0$ and $\ln r_s$],

$$\Delta E=-\frac{2\pi\hbar^2e^2}{\Omega}\sum_q\frac{1}{q^2}\sum_p+\frac{2\pi\hbar^2e^2}{\Omega}\sum_q\frac{1}{q^2}\sum_p\frac{1}{\dfrac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p'}\pi\delta(E_{p'}{}^0-E_{p'+q}{}^0+E_{p+q}{}^0-E_p{}^0)}$$
$$\times\tan^{-1}\left\{\frac{(4\pi\hbar^2e^2/\Omega q^2)\sum_{p'}\pi\delta(E_{p'}{}^0-E_{p'+q}{}^0+E_{p+q}{}^0-E_p{}^0)}{1+\dfrac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p'}\left(\dfrac{1}{E_{p'+q}{}^0-E_{p'}{}^0+E_{p+q}{}^0-E_p{}^0}-\dfrac{P}{E_{p'}{}^0-E_{p'+q}{}^0+E_{p+q}{}^0-E_p{}^0}\right)}\right\}, \quad (19)$$

where we have used the relation

$$\lim_{\epsilon\rightarrow 0}\frac{1}{a+i\epsilon}=\frac{P}{a}-i\pi\delta(a),$$

and P stands for the principal value. [We made $\mathbf{q}\to-\mathbf{q}$ in (16), and in the first term of the denominator we further transformed $\mathbf{p}'\to-\mathbf{p}''$ and $\mathbf{p}''\to\mathbf{p}'''+\mathbf{q}$.] All $\sum_p$, $\sum_{p'}$ are restricted by $|\mathbf{p}|$, $|\mathbf{p}'|<P$ and $|\mathbf{p}+\mathbf{q}|$, $|\mathbf{p}'+\mathbf{q}|>P$ and include spin sums.

To get the correlation energy terms of order $(r_s)^0$ and $\ln r_s$, we must add, of course, $\epsilon_{(b)}{}^{(2)}$ arising from the second order exchange interactions. We can interpret (19) as a sum of one-pair to one-pair scattering phase shifts, the proof being given in the Appendix A.[5]

4. COMPARISON WITH PERTURBATION METHOD

To compare our result with the power series expansion in G-B, it is convenient to expand ΔE in powers of e^2 by means of the alternative logarithmic representation. Denoting by $\Delta_q(p)$ the quantity

$$\Delta_q(p)=\frac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p'}\left(\frac{1}{E_{p'+q}{}^0-E_{p'}{}^0+E_{p+q}{}^0-E_p{}^0}-\frac{1}{E_{p'}{}^0-E_{p'+q}{}^0+E_{p+q}{}^0-E_p{}^0+i\epsilon}\right), \tag{20}$$

we can write ΔE in the following form, by using the second expression of (18);

$$\Delta E=-\frac{2\pi\hbar^2e^2}{\Omega}\sum_q\frac{1}{q^2}\sum_p+\frac{2\pi\hbar^2e^2}{\Omega}\sum_q\frac{1}{q^2}\sum_p\frac{1}{[\Delta_q(p)-\Delta_q{}^*(p)]}\ln\left(\frac{1+\Delta_q(p)}{1+\Delta_q{}^*(p)}\right). \tag{21}$$

Since Δ and Δ^* are of order e^2, if we cut off the momentum transfer q at a suitable magnitude, we may expand the logarithm and obtain:

$$\Delta E=\frac{2\pi\hbar^2e^2}{\Omega}\sum_q\frac{1}{q^2}\sum_p\Bigg[-\frac{\Delta_q(p)+\Delta_q{}^*(p)}{2}+\frac{\{\Delta_q(p)\}^2+\Delta_q(p)\Delta_q{}^*(p)+\{\Delta_q{}^*(p)\}^2}{3}$$
$$-\frac{\{\Delta_q(p)\}^3+\{\Delta_q(p)\}^2\Delta_q{}^*(p)+\Delta_q(p)\{\Delta_q{}^*(p)\}^2+\{\Delta_q{}^*(p)\}^3}{4}+\cdots\Bigg]. \tag{22}$$

Owing to the systematic presence of $\pm i\epsilon$ in the denominator, in spite of the presence of integration over p $(\sum_p)$, we can perform a partial fraction decomposition of product denominators without meeting new singularities and can transform them into positive-definite denominators, Consequently, we can demonstrate the correspondence with perturbation theory. We shall show an example in the Appendix B.

Now, regarding the form of the expansion, each expansion term is just that of the perturbation expansion, so that if one uses a plausible lower cutoff of the momentum transfer q, the series summed by G-B has the same value as our sum (21), and if one wants to extend the meaning of the summand to smaller momentum transfers, then it now becomes clear that one should sum them up in the form suggested by (22) to (21).

5. INTEGRATION

First, we transform the expression (19) into the dimensionless form as in G-B. The momenta, $\mathbf{q}$, $\mathbf{p}$, $\mathbf{p}'$ are measured in units of the Fermi-momentum $P=\hbar/\alpha r_0$, $\alpha=(4/9\pi)^{\frac{1}{3}}$, r_0= radius of the volume in which one electron exists, and Ω= normalization volume$=\frac{4}{3}\pi r_0{}^3\times$number of electrons in the Fermi-sea. Energy is measured by rydbergs $=e^4m/2\hbar^2$, and instead of e^2, $r_s=r_0/r_{\mathrm{Bohr}}$ appears (where $r_{\mathrm{Bohr}}=\hbar^2/me^2$).

In these units, we have for the correlation energy, arising from the interaction represented by the diagram in G-B, per electron [dividing ΔE of Eq. (19) by the number of electrons]:

$$\epsilon=\frac{3}{8\pi^3}\int\frac{d\mathbf{q}}{q^2}\int d\mathbf{p}\frac{1}{\alpha r_s}\left[\frac{1}{2\alpha r_s\dfrac{\mathbf{p}\mathbf{q}}{pq}\dfrac{1}{q^2}}\tan^{-1}\left\{\frac{2\alpha r_s(\mathbf{p}\cdot\mathbf{q}/pq)(1/q^2)}{1+\dfrac{\alpha r_s}{\pi^2}\dfrac{1}{q^2}\displaystyle\int d\mathbf{p}'\left(\frac{1}{q^2+(\mathbf{q}\cdot\mathbf{p}'+\mathbf{p})}+\frac{P}{(\mathbf{q}\cdot\mathbf{p}'-\mathbf{p})}\right)}\right\}-1\right], \tag{23}$$

where $\mathbf{p}$, $\mathbf{p}'$ are restricted by $|\mathbf{p}|$, $|\mathbf{p}'|<1$ and $|\mathbf{p}+\mathbf{q}|$, $|\mathbf{p}'+\mathbf{q}|>1$.

If we denote the term of order $(r_s)^0$ in the (formal) expansion of (23) about r_s with the integration taken over large momentum transfer $q>1$ as $\epsilon_{(c)}$ following G-B (namely, $\epsilon_{(c)}$ is the expression $\epsilon_{(a)}{}^{(2)}$ in the introduction with $q>1$), the contribution of the q integration in the expression $\epsilon-\epsilon_{(c)}$ is seen to come from small momentum transfers

[5] If one uses the procedure given here for the case of a neutral scalar meson, one can furnish a direct proof of the derivation of the many-particle forces from the eigenphase shift of scattering due to many particles; the relation was used to derive Wentzel's result by K. Hasegawa and S. Azuma, Progr. Theoret. Phys. Japan **13**, 360 (1955).

$q \ll 1$. Hence, writing

$$\epsilon = \epsilon_{(c)} + \epsilon', \quad \epsilon_{(c)} = -\frac{3}{8\pi^5}\int_1^\infty \frac{d\mathbf{q}}{q^4}\int d\mathbf{p}\int d\mathbf{p}' \frac{1}{q^2 + (\mathbf{q}\cdot\mathbf{p}' + \mathbf{p})}$$

[in the expansion of (23), the $(r_s)^0$ term do not contain a principal value integral owing to its disappearance due to symmetric integrations over $\mathbf{p}$ and $\mathbf{p}'$], ϵ' can be evaluated in the limit of small q and by introducing a cutoff at $q=1$ as was done in G-B. The restriction $|\mathbf{p}'|<1$, $|\mathbf{p}'+\mathbf{q}|>1$ implies that, if we denote $\cos\theta_{p'q}$ as x', then $0<x'<1$, the integration over p' gives $1-qx' \to 1$, and in the integrand p' may be replaced by 1. Hence in the limit of small q, neglecting q^2, we find

$$\int d\mathbf{p}' \frac{1}{(\mathbf{q}\cdot\mathbf{p}'+\mathbf{p})} = 2\pi\left(1 - \frac{\mathbf{p}\cdot\mathbf{q}}{q}\log\frac{q+\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\mathbf{q}}\right),$$

$$P\int d\mathbf{p}' \frac{1}{(\mathbf{q}\cdot\mathbf{p}'-\mathbf{p})} = 2\pi\left(1 + \frac{\mathbf{p}\cdot\mathbf{q}}{q}\log\left|\frac{q-\mathbf{p}\cdot\mathbf{q}}{\mathbf{p}\cdot\mathbf{q}}\right|\right).$$

Putting for $\int d\mathbf{p}$ also $\int d\mathbf{p} \to 2\pi q \int_0^1 x dx$ ($x = \cos\theta_{pq}$) and $p=1$ in the integrand, we have for ϵ':

$$\epsilon' = \frac{3}{8\pi^3}\int_0^1 \frac{d\mathbf{q}}{q^2} 2\pi q \int_0^1 x dx \frac{1}{\alpha r_s}\left[\frac{q^2}{2\alpha r_s x}\tan^{-1}\left(\frac{2\alpha r_s x}{q^2 + (4\alpha r_s/\pi)[1-\frac{1}{2}x\ln|(1+x)/(1-x)|]}\right) - 1\right].$$

Then, integrating q over the range 0 to 1 and neglecting terms of order r_s and higher, we find

$$\epsilon' = \frac{2}{\pi^2}(1-\ln 2)\left[\ln\frac{4\alpha r_s}{\pi} - \frac{1}{2} + \left\{\int_0^1 x\left(1-\frac{x}{2}\ln\frac{1+x}{1-x}\right)\frac{1}{2}\ln\left(\left(1-\frac{x}{2}\ln\frac{1+x}{1-x}\right)^2 + \frac{\pi^2}{4}x^2\right)dx \right.\right.$$
$$\left.\left. + \int_0^1 \frac{1}{\pi}\left(\left(1-\frac{x}{2}\ln\frac{1+x}{1-x}\right)^2 - \frac{\pi^2}{4}x^2\right)\tan^{-1}\left[\frac{\frac{\pi}{2}x}{1-\frac{x}{2}\ln\frac{1+x}{1-x}}\right]dx\right\}\left\{\int_0^1 x\left(1-\frac{x}{2}\ln\frac{1+x}{1-x}\right)dx\right\}^{-1}\right], \quad (24)$$

where

$$\int_0^1 x\left(1-\frac{x}{2}\ln\left|\frac{1+x}{1-x}\right|\right)dx = \tfrac{1}{3}(1-\ln 2).$$

The correlation energy of order $(r_s)^0$ and $\ln r_s$ is the sum

$$\epsilon_{\text{correl}}((r_s)^0, \ln r_s) = \epsilon' + \epsilon_{(c)} + \epsilon_{(b)}^{(2)}, \quad (25)$$

where $\epsilon_{(b)}^{(2)}$ is the second-order exchange interaction energy as given in the introduction; and in dimensionless units,

$$\epsilon_{(b)}^{(2)} = \frac{3}{16\pi^5}\int\frac{d\mathbf{q}}{q^2}\int d\mathbf{p}\int d\mathbf{p}' \frac{1}{(\mathbf{q}+\mathbf{p}+\mathbf{p}')^2}\,\frac{1}{q^2+(\mathbf{q}-\mathbf{p}'+\mathbf{p})}.$$

The last expression in (24) when combined with the contribution from the plasma zero point energy (see *Note added in proof*) seems to correspond to G-B's $\int_{-\infty}^{\infty} du R^2 \ln R / \int_{-\infty}^{\infty} du R^2$, where $R = 1 - u\tan^{-1}(1/u)$. We have not been able to transform this last term into G-B's term, but the value obtained by numerical integration agrees with their result.

In conclusion, I would like to express my deep gratitude to Professor K. A. Brueckner for his kind introduction and guidance on this problem and for many helpful discussions. I also would like to thank the members of the Department for their hospitality.

APPENDIX A

To show that ΔE (19) can be represented as a sum of eigenphases of one-pair to one-pair scattering, we first construct the $\mathcal{T}$ matrix for this scattering; just as in (11), we can construct the incoming-wave solution

$$\Psi_{p-q,\,p}^{(-)} = a_{p-q}{}^* b_p{}^* \Psi_0 + \frac{1}{E_0 + E_{p-q}{}^0 - E_p{}^0 - H_T - i\epsilon}\,\frac{4\pi\hbar^2 e^2}{\Omega q^2}\sum_{p''}(a_{p''-q}{}^* b_{p''}{}^* + b_{p''-q} a_{p''})\Psi_0. \quad \text{(A-1)}$$

Then, eliminating $a_{p-q}{}^*b_p{}^*\Psi_0$ from (11) by using (A-1), we can write

$$\Psi_{p-q,\,p}{}^{(+)}=\Psi_{p-q,\,p}{}^{(-)}-2\pi i\delta(E_0+E_{p-q}{}^0-E_p{}^0-H_T)\frac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p''}(a_{p''-q}{}^*b_{p''}{}^*+b_{p''-q}a_{p''})\Psi_0. \qquad \text{(A-2)}$$

Taking the inner product with $\Psi_{p'-q,\,p'}{}^{(-)}$, we obtain the S matrix:

$$S_{p'-q,\,p';\,p-q,\,p}=\delta_{p',\,p}-2\pi i\delta(E_{p-q}{}^0-E_p{}^0-E_{p'-q}{}^0+E_{p'}{}^0)\frac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p''}(\Psi_{p'-q,\,p'}{}^{(-)},\,(a_{p''-q}{}^*b_{p''}{}^*+b_{p''-q}a_{p''})\Psi_0).$$

Further, using the expansion of $\sum_{p''}(a_{p''-q}{}^*b_{p''}{}^*+b_{p''-q}a_{p''})\Psi_0$ into incoming-wave solutions [in (14), $i\epsilon\rightarrow -i\epsilon$ and $\Psi^{(+)}\rightarrow\Psi^{(-)}$], we find

$$S_{p'-q,\,p';\,p-q,\,p}=\delta_{p',\,p}-2\pi i\delta(E_{p-q}{}^0-E_p{}^0-E_{p'-q}{}^0+E_{p'}{}^0)$$
$$\times\frac{4\pi\hbar^2e^2}{\Omega q^2}\left\{1-\frac{4\pi\hbar^2e^2}{\Omega q^2}\left[\sum_{\substack{p''\\|p''|>P\\|p''-q|<P}}\frac{1}{E_{p''-q}{}^0-E_{p''}{}^0-E_{p'-q}{}^0+E_{p'}{}^0}-\sum_{\substack{p''\\|p''|<P\\|p''-q|>P}}\frac{1}{E_{p''-q}{}^0-E_{p''}{}^0-E_{p'-q}{}^0+E_{p'}{}^0-i\epsilon}\right]\right\}^{-1}.$$

Hence, the $\mathcal{T}$ matrix is

$$\mathcal{T}_{p'+q,\,p';\,p+q,\,p}=\frac{4\pi\hbar^2e^2}{\Omega q^2}\left\{1+\frac{4\pi\hbar^2e^2}{\Omega q^2}\sum_{p''}\left(\frac{1}{E_{p''+q}{}^0-E_{p''}{}^0+E_{p'+q}{}^0-E_{p'}{}^0}-\frac{1}{E_{p''}{}^0-E_{p''+q}{}^0+E_{p'+q}{}^0-E_{p'}{}^0+i\epsilon}\right)\right\}^{-1}. \qquad \text{(A-3)}$$

Now, following the notation of Lippmann and Schwinger,[6] we can diagonalize $\mathcal{T}$ by using a suitable transformation f_{aA} (a refers to $p+q,\,p$; A is the diagonal index),

$$\mathcal{T}_{ba}=\sum_A f_{bA}\mathcal{T}_A f_{aA}{}^*.$$

(Note that we fix $\mathbf{q}$ here.) We define the matrix element of $\mathcal{T}^2$ as

$$(\mathcal{T}^2)_{ba}=\sum_c\mathcal{T}_{bc}\delta(E_c-E)\mathcal{T}_{ca}(=\sum_A f_{bA}\mathcal{T}_A{}^2f_{aA}{}^*),\quad (E=E_a=E_b), \qquad \text{(A-4)}$$

and get the following relation (where F means function)

$$\sum_A F(\mathcal{T}_A)=\sum_A\sum_a\sum_b f_{aA}\delta(E_a-E)[F(\mathcal{T})]_{ab}\delta(E_b-E)f_{bA}{}^* \qquad \text{(A-5)}$$
$$=\sum_a\delta(E_a-E)[F(\mathcal{T})]_{aa}.$$

Thus, the sum of the eigenphases becomes [since $\mathcal{S}_A=1-2\pi i\mathcal{T}_A=\exp(-2i\delta_A)$];

$$\int dE\sum_A\delta_A(E)=\int dE\left(\frac{1}{-2i}\right)\sum_A\log(\mathcal{S}_A)=-\frac{1}{2i}\sum_a[\ln(1-2\pi i\mathcal{T})]_{aa},$$

where δ_A are the eigenphases. Now,

$$(\mathcal{T}^2)_{p'+q,\,p';\,p+q,\,p}=\sum_{p''}\mathcal{T}_{p'+q,\,p';\,p''+q,\,p''}\delta(E_{p''+q}{}^0-E_{p''}{}^0-E_{p'+q}{}^0+E_{p'}{}^0)\mathcal{T}_{p''+q,\,p'';\,p+q,\,p};$$

but, from (A-3), $\mathcal{T}_{p'+q,\,p';\,p''+q,\,p''}$ does not depend on p'', and $\mathcal{T}_{p''+q,\,p'';\,p+q,\,p}$ depends on p'' only through $E_{p''+q}{}^0-E_{p''}{}^0$, so that owing to the presence of the δ function, we get

$$(\mathcal{T}^2)_{p'+q,\,p';\,p+q,\,p}=(\mathcal{T}_{p'+q,\,p';\,p+q,\,p})^2\sum_{p''}\delta(E_{p''+q}{}^0-E_{p''}{}^0-E_{p'+q}{}^0+E_{p'}{}^0).$$

In the same way,

$$(\mathcal{T}^n)_{p+q,\,p;\,p+q,\,p}=(\mathcal{T}_{p+q,\,p;\,p+q,\,p})^n(\sum_{p'}\delta(E_{p'+q}{}^0-E_{p'}{}^0-E_{p+q}{}^0+E_p{}^0))^{n-1}. \qquad \text{(A-6)}$$

Since $\mathcal{T}$ is not singular, we can expand $[\ln(1-2\pi i\mathcal{T})]_{aa}$ and use (A-6). The result is

$$\int dE\sum_A\delta_A{}^{(\text{q-fixed})}(E)=-\frac{1}{2i}\sum_p\frac{1}{\sum_{p'}\delta(E_{p'+q}{}^0-E_{p'}{}^0-E_{p+q}{}^0+E_p{}^0)}$$
$$\times\ln[1-2\pi i\mathcal{T}_{p+q,\,p;\,p+q,\,p}\sum_{p'}\delta(E_{p'+q}{}^0-E_{p'}{}^0-E_{p+q}{}^0+E_p{}^0)].$$

[6] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

Here, using (A-3) for $\mathcal{T}$, we finally find,

$$\int dE\,\sum_A \delta_A{}^{(\text{q-fixed})}(E) = \sum_p \frac{1}{\sum_{p'} \delta(E_{p'+q}{}^0 - E_{p'}{}^0 - E_{p+q}{}^0 + E_p{}^0)}$$
$$\times \tan^{-1}\left[\frac{(4\pi\hbar^2 e^2/\Omega q^2)\sum_{p'} \pi\delta(E_{p'+q}{}^0 - E_{p'}{}^0 - E_{p+q}{}^0 + E_p{}^0)}{1 + \frac{4\pi\hbar^2 e^2}{\Omega q^2}\sum_{p'}\left(\frac{1}{E_{p'+q}{}^0 - E_{p'}{}^0 + E_{p+q}{}^0 - E_p{}^0} - \frac{P}{E_{p'}{}^0 - E_{p'+q}{}^0 + E_{p+q}{}^0 - E_p{}^0}\right)}\right], \quad \text{(A-7)}$$

comparing with (19), we find

$$\Delta E = -\frac{2\pi\hbar^2 e^2}{\Omega}\sum_q \frac{1}{q^2}\sum_p + \frac{1}{2\pi}\sum_q \int dE\,\sum_A \delta_A{}^{(\text{q-fixed})}(E). \quad \text{(A-8)}$$

The first term is, so to speak, the diagonal element of the potential, and the second term is due to the one-pair to one-pair scattering eigenphases.

APPENDIX B

We take as an example the e^8 term (see Fig. 3),

$$\Delta E^{(8)} = -\frac{2\pi\hbar^2 e^2}{\Omega}\sum_q \frac{1}{q^2}\{\tfrac{1}{4}\sum_p[\Delta_q{}^3(p) + \Delta_q{}^2(p)\Delta_q{}^*(p) + \Delta_q(p)\Delta_q{}^{*2}(p) + \Delta_q{}^{*3}(p)]\}.$$

First, inside the curly brackets there are denominators all with $\pm i\epsilon$ in such combinations as the following:

$$\tfrac{1}{4}\sum_p\left\{\left[\sum_{p'}\frac{1}{W'-W+i\epsilon}\right]^3 + \left[\sum_{p'}\frac{1}{W'-W+i\epsilon}\right]^2\left[\sum_{p'}\frac{1}{W'-W-i\epsilon}\right]\right.$$
$$\left. + \left[\sum_{p'}\frac{1}{W'-W+i\epsilon}\right]\left[\sum_{p'}\frac{1}{W'-W-i\epsilon}\right]^2 + \left[\sum_{p'}\frac{1}{W'-W-i\epsilon}\right]^3\right\}, \quad \text{(B-1)}$$

where $W = E_{p+q}{}^0 - E_p{}^0$, $W' = E_{p+q}{}^{0'} - E_{p'}{}^0$, and all $\epsilon > 0$. The ϵ's should not be of the same magnitude because the limit $\epsilon \to 0$ is arbitrarily taken. However, it is convenient to establish some ordering among the ϵ's. We take the first product in (B-1) as follows:

$$\sum_p\sum_{p'}\sum_{p''}\sum_{p'''}\frac{1}{W'-W+3i\epsilon}\,\frac{1}{W''-W+2i\epsilon}\,\frac{1}{W'''-W+i\epsilon},$$

and performing partial fractions, we find that it is equal to

$$\sum_p\cdots\sum_{p'''}\left[\frac{1}{W'-W+3i\epsilon}\,\frac{1}{W''-W'-i\epsilon}\,\frac{1}{W'''-W'-2i\epsilon} + \frac{1}{W'-W''+i\epsilon}\,\frac{1}{W''-W+2i\epsilon}\,\frac{1}{W'''-W''-i\epsilon}\right.$$
$$\left. + \frac{1}{W'-W'''+2i\epsilon}\,\frac{1}{W''-W'''+i\epsilon}\,\frac{1}{W'''-W+i\epsilon}\right],$$

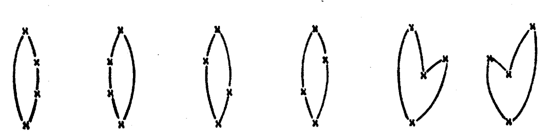

FIG. 3. Diagrams of order e^8.

Relabeling the variables, we have that it is equal to

$$-\sum_p\cdots\sum_{p'''}\left[\frac{1}{W'-W-3i\epsilon}\frac{1}{W''-W-i\epsilon}\frac{1}{W'''-W-2i\epsilon}+\frac{1}{W'-W+i\epsilon}\frac{1}{W''-W-2i\epsilon}\frac{1}{W'''-W-i\epsilon}+\frac{1}{W'-W+2i\epsilon}\frac{1}{W''-W+i\epsilon}\frac{1}{W'''-W-i\epsilon}\right].$$

Hence, in the limit $\epsilon\to 0$, the first term of (B-1) cancels with others and the equation

$$\text{(B-1)}=0$$

follows. The result is independent of our choice of the magnitude of infinitesimal quantities. Next, consider a product denominator with two $\pm i\epsilon$'s and one positive-definite. This term is contained in the curly bracket of $\Delta E^{(8)}$ in the following way:

$$\tfrac{1}{4}\sum_p\left[\sum_{p'}\frac{1}{W'+W}\right]\cdot 4\left\{\left[\sum_{p'}\frac{1}{W'-W+i\epsilon}\right]^2+\left[\sum_{p'}\frac{1}{W'-W+i\epsilon}\right]\left[\sum_{p'}\frac{1}{W'-W-i\epsilon}\right]+\left[\sum_{p'}\frac{1}{W'-W-i\epsilon}\right]^2\right\}. \quad \text{(B-2)}$$

Again, we set up partial fractions for the first $+i\epsilon$ term in (B-2):

$$\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\frac{1}{W''-W+2i\epsilon}\frac{1}{W'''-W+i\epsilon}$$

$$=\sum_p\cdots\sum_{p'''}\left[\frac{1}{W'+W}\frac{1}{W''+W'}\frac{1}{W'''+W'}+\frac{1}{W'+W''}\frac{1}{W''-W+2i\epsilon}\frac{1}{W'''-W''-i\epsilon}+\frac{1}{W'+W'''}\frac{1}{W''-W'''+i\epsilon}\frac{1}{W'''-W+i\epsilon}\right]$$

$$=\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W'''+W}$$

$$-\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\left[\frac{1}{W''-W-2i\epsilon}\frac{1}{W'''-W-i\epsilon}+\frac{1}{W''-W+i\epsilon}\frac{1}{W'''-W-i\epsilon}\right].$$

When we neglect $\pm i\epsilon$ in the positive definite denominators, the last two terms cancel with the remainder in (B-2), leaving a positive definite denominator only:

$$\text{(B-2)}=\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W'''+W},$$

and this contribution is to be added to the term which contains no $\pm i\epsilon$ in the denominator appearing in $\Delta E^{(8)}$.

Finally we consider terms with one $\pm i\epsilon$:

$$\tfrac{1}{4}\sum_p\left[\sum_{p'}\frac{1}{W'+W}\right]^2\cdot 6\left\{\left[\sum_{p'}\frac{1}{W'-W+i\epsilon}\right]+\left[\sum_{p'}\frac{1}{W'-W-i\epsilon}\right]\right\}. \quad \text{(B-3)}$$

The first term can be changed successively as follows:

$$\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W'''-W+i\epsilon}$$

$$=\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\left(\frac{1}{W''+W}+\frac{1}{W'''-W+i\epsilon}\right)\frac{1}{W''+W'''}$$

$$=\sum_p\cdots\sum_{p'''}\left[\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W''+W'''}+\left(\frac{1}{W'+W}+\frac{1}{W'''-W+i\epsilon}\right)\frac{1}{W'+W'''}\frac{1}{W''+W'''}\right]$$

$$=2\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W''+W'''}-\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W'''-W-i\epsilon}.$$

The last term cancels with the remainder in (B-3). Hence

$$(\text{B-3})=3\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W''+W'''}$$

$$=2\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W''+W'''}+\sum_p\cdots\sum_{p'''}\left(\frac{1}{W'+W}+\frac{1}{W''+W'''}\right)\frac{1}{W+W'+W''+W'''}\frac{1}{W''+W}$$

$$=2\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W''+W'''}$$

$$+2\sum_p\cdots\sum_{p'''}\frac{1}{W'+W}\frac{1}{W+W'+W''+W'''}\frac{1}{W''+W}.$$

Collecting (B-1), (B-2), (B-3), and all positive-definite terms of $\Delta E^{(8)}$, we find that

$$\Delta E^{(8)}=-\sum_q\left(\frac{4\pi\hbar^2e^2}{\Omega q^2}\right)^4\sum_p\cdots\sum_{p'''}\left[\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W'''+W}\right.$$

$$\left.+\frac{1}{W'+W}\frac{1}{W''+W}\frac{1}{W''+W'''}+\frac{1}{W'+W}\frac{1}{W+W'+W''+W'''}\frac{1}{W''+W}\right],\quad\text{(B-4)}$$

where the $\sum_p$'s include the sum over spins. These terms correspond to diagrams of the eighth order. We can continue the partial fraction decomposition in an analogous way up to any desired order.