General Relativistic Red Shift and the Artificial Satellite

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Singer's formula for the general relativistic red shift on an earth satellite is modified to take account of the diurnal rotation of the earth and the lack of spherical symmetry of its gravitational field. It is shown that the Singer rates of the earth and satellite clocks need slight modifications, but that these modifications tend to cancel each other except at large distances from the earth, so that when one uses a mean radius of the earth in Singer's formula the formula is adequate for present purposes.

SINGER[1] has shown that, according to the general theory of relativity, there should be a measurable difference between the rate of a clock on the earth and the rate of a similar clock on an artificial satellite. Denoting the times indicated by these clocks respectively by t_E, t_{sat}, the mass and radius of the earth by M_E, R_E, the Newtonian gravitational constant by G, the radius of the satellite orbit by R_E+h, and the velocity of light by c, we may write his result as

$$\begin{aligned}\Delta &\equiv (dt_{\text{sat}}-dt_E)/dt_E \\ &= (GM_E/c^2R_E)\{1.5(1+h/R_E)^{-1}-1\} \\ &\sim 6.96\times 10^{-10}\{1.5(1+h/R_E)^{-1}-1\}. \quad (1)\end{aligned}$$

Singer points out[2] that atomic clocks are said to be accurate to better than 1 part in 10^{12}. In his derivation he does not take account of the diurnal rotation of the earth. Because this involves a term of the magnitude 1.24×10^{-12} in Δ, it seemed worth while to study the situation more closely.

The Schwarzschild line element for the gravitational field of the earth is

$$ds^2=(1-2GM_E/c^2r)c^2dt^2-dr^2/(1-2GM_E/c^2r) -r^2(d\theta^2+\sin^2\theta d\varphi^2). \quad (2)$$

For large r this is not only flat but "nonrotating." Observationally this means that relative to the coordinate system involved in (2) the stars will exhibit no significant resultant angular momentum. Thus in using (2) one must regard the earth as rotating at the approximate rate of 2π radians per day. So, for a clock on the earth at colatitude θ, we have

$$c^{-2}(ds/dt_E)^2=1-c^{-2}R_E^2\sin^2\theta(d\varphi_E/dt)^2 -2GM_E/c^2R_E. \quad (3)$$

Singer omits the second term on the right of (3). When it is taken into account, one obtains, for a satellite in the plane[3] $\theta=\frac{1}{2}\pi$

$$\Delta=(GM_E/c^2R_E)\{1.5(1+h/R_E)^{-1}-1\} -\tfrac{1}{2}c^{-2}R_E^2\sin^2\theta(d\varphi_E/dt)^2. \quad (4)$$

The first term on the right coincides with the Δ obtained by Singer. The second term is numerically $-1.24\times 10^{-12}\times\sin^2\theta$, and this is -1.24×10^{-12} when the clock is at the equator.

However, the situation is not as simple as this. Where such small effects are concerned, the oblateness of the earth may not be neglected. It shows itself in two ways: (a) the value of R_E depends on latitude, and (b) the gravitational potential of the earth is not spherically symmetric. Because of (b), the Schwarzschild line element is, strictly speaking, inapplicable; but the deviation from spherical symmetry is so slight that it can be treated as a linear perturbation despite the nonlinearity of the field equations. We may therefore replace GM_E/c^2r in (3) by c^{-2} times the Newtonian expression[4] for the gravitational potential of the oblate earth, namely

$$GM_E/c^2r+(KGM_E/2c^2r^3)(1-3\cos^2\theta)+\cdots, \quad (5)$$

where K is a constant having the value 4.47×10^{14}. At the equator the second term is of magnitude 3.8×10^{-13}; at a pole it is -7.7×10^{-13}. The difference is of the order 10^{-12}.

For an earth clock at the equator we now obtain, instead of (3), taking the approximate square root,

$$c^{-1}ds/dt_E=1-\tfrac{1}{2}c^{-2}R_e^2(d\varphi_E/dt)^2-GM_E/c^2R_e -KGM_E/2c^2R_e^3, \quad (6)$$

where R_e is the equatorial radius of the earth.

Because of the variation of R_E with latitude, ds/dt_E turns out to be substantially independent of latitude. The reason for this is that the second term on the right of (6), which can be thought of as a second-order relativistic Doppler term, can also be regarded as the contribution of the centrifugal acceleration to $1/c^2$ times the gravitational potential, this contribution, by the principle of equivalence, being equivalent to $1/c^2$ times a gravitational contribution. The geoid is an equipotential surface of this combined gravitational and centrifugal potential; and it is the facts that the geoid

[1] S. F. Singer, Phys. Rev. **104**, 11 (1956).
[2] See footnote 6 of reference 1.
[3] This is here the equatorial plane of the earth, though in Singer's case it could be any plane through the earth's center. We consider this special case here for simplicity and easy comparison, though the orbit of the actual satellite will not lie in the earth's equatorial plane.
[4] The fact that r is not a geodetic interval but a radial coordinate makes negligible difference here.

is a very close approximation to the figure of the earth, and local effects such as surface irregularities and isostasy are small, that make (6) substantially independent of latitude.

We therefore may use (6) for all latitudes. If we wish to avoid the term involving $d\varphi_E/dt$, we may take a clock at a pole. In this case, because of the $\cos\theta$ in (5), we have

$$c^{-1}ds/dt_E = 1 - GM_E/c^2R_p + KGM_E/c^2R_p^3, \quad (7)$$

where R_p is the polar radius of the earth. Here the third term on the right is of magnitude 7.7×10^{-13}.

The values of GM_E/c^2R_E at a pole and at the equator differ by approximately 2.35×10^{-12}. But the value of $c^{-1}ds/dt_E$ can be approximated to within 8.4×10^{-13} by the Singer expression $1-GM_E/c^2R_E$, despite the absence of the $d\varphi_E/dt$ term, if one takes R_E to be a mean radius of the earth, as Singer does.

From now on we shall use the equatorial expression (6), and express the radius of the orbit of the satellite as R_e+h'. Since we are taking this orbit in the equatorial plane of the earth, we may still assume a circular orbit. If μ is the mass of the satellite, we have, using the negative derivative of the expression (5) with respect to r,

$$\{G\mu M_E/(R_e+h')^2\}\{1+3K/2(R_e+h')^2\} = \mu(R_e+h')(d\varphi_{\rm sat}/dt)^2. \quad (8)$$

From this and the modified form of (2) we find that

$$c^{-1}ds/dt_{\rm sat} = 1 - 3GM_E/2c^2(R_e+h') - 5KGM_E/4c^2(R_e+h')^3, \quad (9)$$

which, apart from the use of R_e instead of R_E and h' instead of h, differs from the corresponding Singer expression in the presence of the last term, which is numerically $-9.6\times10^{-13}(1+h'/R_e)^{-3}$.

The value of Δ comes out to be

$$\begin{aligned}\Delta &= (GM_E/c^2R_e)\{1.5(1+h'/R_e)^{-1}-1\} - (R_e^2/2c^2)(d\varphi_E/dt)^2 + (KGM_E/2c^2R_e^3) \times\{(2.5(1+h'/R_e)^{-3}-1\}\\ &= 6.9535\times10^{-10}\{1.5(1+h'/R_e)^{-1}-1\} - 1.24\times10^{-12} + 3.8\times10^{-13}\{2.5(1+h'/R_e)^{-3}-1\}\\ &= 6.9535\times10^{-10}\{1.5(1+h'/R_e)^{-1}\} + 9.5\times10^{-13}(1+h'/R_e)^{-3} - 6.9697\times10^{-10}. \quad (10)\end{aligned}$$

If one uses the mean radius $R_E = 6.3712\times10^8$ cm in Singer's formula, one obtains

$$\begin{aligned}\Delta &= 6.9613\times10^{-10}\{1.5(1+h/R_E)^{-1}-1\}\\ &= 6.9535\times10^{-10}\{1.5(1+h'/R_e)^{-1}\} - 6.9613\times10^{-10}. \quad (11)\end{aligned}$$

We have used five significant figures chiefly for purposes of comparison.

The difference between these two formulas for Δ is

$$9.5\times10^{-13}(1+h'/R_e)^{-3} - 8.4\times10^{-13}, \quad (12)$$

and while each term is close to 10^{-12}, the two terms pull in opposite directions, so that, except for large h', their combined effect is negligible. For a satellite in an elongated orbit, such as Singer recommends, the first term in (12) would become less important, but even for infinite h' the quantity (12) does not quite reach the order of magnitude 10^{-12}. Thus the Singer formula is adequate for present purposes when one takes R_E to be a mean radius.

Although the deviations from the Singer formula are numerically unimportant at present, they do have theoretical significance, and they show the need for defining the height of the satellite above the earth in terms of an appropriate radius of the earth. Measurements of Δ to within even 1 part in 10^{12} would be impossible under the foreseeable conditions of early satellite experiments. But greater experience with satellites coupled with an improvement in the accuracy of clocks might bring deviations of the sort discussed above within the range of measurement.

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