Catalysis of Nuclear Reactions between Hydrogen Isotopes by μ^- Mesons

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The mechanism by which negative μ mesons catalyze nuclear reactions between hydrogen isotopes is studied in detail. The reaction rate for the process $(p+d+\mu^-\rightarrow He^3+\mu^-+5.5$ Mev), observed recently by Alvarez et al., is calculated and found to be in accord with the available data. The μ^- meson binds two hydrogen nuclei together in the μ-mesonic analog of the ordinary H_2^+ molecular ion. In their vibrational motion the nuclei have a finite, although small, probability of penetrating the Coulomb barrier to zero separation where they may undergo a nuclear reaction. The intrinsic reaction rates for other, more probable, reactions are also estimated. The results are $\sim 0.3\times 10^6$ sec^{-1} for the observed $p-d$ reaction, $\sim 0.7\times 10^{11}$ sec^{-1} for the $d-d$ reaction, and $\sim 0.4\times 10^{13}$ sec^{-1} for the $d-t$ reaction. For the reaction observed by Alvarez rough estimates are made of the partial widths for nonradiative and radiative decay of the excited He^3 nucleus. The ejection of the μ^- meson by "internal conversion" seems somewhat less likely. Speculations are made on the release of useful amounts of nuclear energy by these catalyzed reactions. The governing factors are not the intrinsic reaction rate once the molecule is formed, but rather the time spent ($\sim 10^{-8}$ sec) by the μ^- meson between the breakup of one molecule and the formation of another and the loss of μ^- mesons in "dead-end" processes. These factors are such that practical power production is unlikely. In liquid deuterium, each μ^- meson will catalyze only ~ 10 reactions in its lifetime, while for the $d-t$ process it will induce ~ 100 disintegrations. A longer lived particle will not be able to catalyze appreciably more reactions.

1. INTRODUCTION

IT has been found recently[1] at Berkeley that negative μ mesons incident on a hydrogen bubble chamber containing both light hydrogen and deuterium act as a catalyst for the nuclear reaction

$$p+d+\mu^-\rightarrow He^3+\mu^-+5.5 \text{ Mev}, \qquad (1)$$
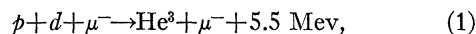
with the μ meson carrying off the available energy. The lifetime for the reaction is comparable to the mean life of the meson $(2.1\times 10^{-6}$ sec).

The catalytic process can be envisioned in the following way.[2] In the first step, the negative μ meson rapidly comes to rest[3] in the chamber and is captured into a stable, though highly excited, orbit around a hydrogen molecule. In the further process of de-excitation, most frequently a μ-mesonic hydrogen atom is formed,[3] but at high densities subsequent molecule formation takes place in a time short compared to the μ-meson lifetime (see Sec. 4 and Appendix D). We are interested in those mesons which form a bound molecule consisting of a proton, a deuteron, and a μ^- meson. The largeness of the meson mass compared to the

electronic mass means that the bound $(p+d+\mu^-)$ system will be much smaller than electronic orbital distances. Consequently any electrons that may be present can be ignored. The bound system will be the μ-mesonic analog of the $(H^1H^2)^+$ electronic molecular ion.

Once the meson has formed the μ-mesonic molecular ion the second stage of the process begins. This consists of the existence of a well-defined molecular system, with its typical rotational and vibrational motions, for a very long time compared to the characteristic molecular periods. For a μ-mesonic hydrogen molecule the vibrational period is of the order of 5×10^{-18} sec, and the "electronic" period about 10 times shorter. This is to be compared with the mean life of process (1) of $\sim 10^{-6}$ sec. Even for much more likely processes with other hydrogen isotopes (see Sec. 2) the molecular periods are always very small compared to the characteristic reaction time.

The third stage occurs when the two hydrogen nuclei in their vibrational motion penetrate the classically forbidden Coulomb barrier and come within a nuclear interaction distance of each other. They then have a certain probability of forming a compound nuclear system which subsequently de-excites with the liberation of energy in the form of kinetic energy of particles or radiation. For reaction (1) the excited He^3 nucleus de-excites by electromagnetic interaction with the μ^- meson (internal conversion), or, perhaps more probably, by the emission of a gamma ray (see Appendix C). In other reactions the energy may be carried off mainly by nuclear particles, with the μ^- meson receiving only a small energy (see Sec. 2). The net result is the occurrence of a nuclear reaction through the intermediary of a μ^- meson, the meson being set free unaffected in the process.[4]

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[1] L. W. Alvarez et al., post-deadline paper, American Physical Society Meeting, Monterey, California, December 28, 1956; Phys. Rev. 105, 1127 (1957).

[2] After the completion of this paper, the author's attention was called to a ten year old paper by F. C. Frank, Nature 160, 525 (1947), in which, in the course of examining and rejecting all possible alternative explanations of the $\pi-\mu$ decay events observed in emulsions by Powell and co-workers, he discussed the catalysis mechanism in the same physical terms as presented here, although in rather less detail. See also Ya. B. Zel'dovitch, Doklady Akad. Nauk U.S.S.R. 95, 493 (1954).

[3] A. S. Wightman [Phys. Rev. 77, 521 (1950) and Ph. D. thesis, Princeton University, 1949 (unpublished)], has estimated that a 10-Mev μ^- meson in liquid hydrogen at 20°K and 1 atmosphere (density 0.071 g/cm³) will take of the order of 10^{-9} sec to reach 100 kev, and will take only a few times 10^{-12} sec more to reach a speed at which it is captured by a hydrogen molecule. Once bound around the molecule the meson takes about 10^{-10} sec to cascade down to its lowest atomic orbit. For higher densities the time scale is correspondingly reduced.

[4] One may ask whether molecule formation is necessary for the catalytic action in view of the fact that the μ^- meson is quite

The μ^- meson serves as a *catalyst* for the nuclear reaction in the usual sense of the word, being an added agent which accelerates the reaction rate without being consumed itself. The efficacy of a μ^- meson as a catalyst compared to an electron is due to its much larger mass which causes the nuclei bound together in the μ-mesonic molecule to be about 200 times closer together than in the normal molecule. The "leakage" through the barrier to zero separation is exponentially dependent on the linear dimension. Consequently the catalytic action is a very rapidly increasing function of the mass of the binding particle.[5]

The molecular potential energy curve as a function of the separation r between the nuclei, and the lowest vibrational wave function are shown schematically in Fig. 1. The drawing is not to scale, and is deliberately simplified to illustrate the important features of the process. The molecular system exists in the lowest vibrational energy level,[6] with binding energy ϵ and equilibrium separation r_e. The radii r_1 and r_2 are the classical turning points. Inside r_1 the vibrational wave function falls off rapidly but still has a finite, although very small, value at zero separation. The reaction rate can be written

$$1/T = A |\Psi(0)|^2 \qquad (2)$$

where A is a constant of the nuclear reaction (dimensions of cm^3/sec) which can be determined from available reaction data (see Sec. 2), and $\Psi(0)$ is the value of the molecular wave function at zero separation, or more correctly, at the interaction distance between the nuclei. It is found (Sec. 3) that $|\Psi(0)|^2$ can be approximated by

$$|\Psi(0)|^2 \simeq \text{const}(M/\mu)^{\frac{3}{2}} a^{-3} \exp(-\lambda), \qquad (3)$$

where M is the reduced mass of the nuclei in the molecule, μ is the mass of the binding particle (μ^- meson), $a = \hbar^2/\mu e^2$ is the Bohr radius of the binding particle, and λ is the barrier penetration factor. For μ-mesonic hydrogen molecules, the factors are such that

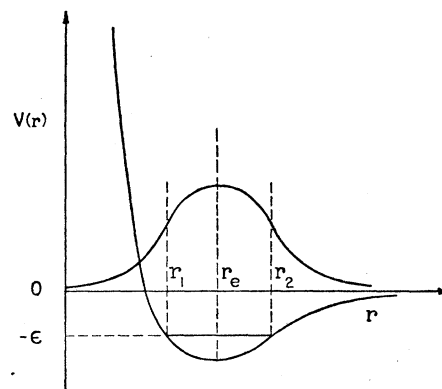$$|\Psi(0)|^2 \sim 10^{30} \exp(-\lambda) \text{ cm}^{-3},$$

Fig. 1. Nuclear potential energy curve in μ-mesonic hydrogen molecule, and ground-state vibrational wave function for the relative motion of the two nuclei. r_e is the equilibrium separation, while r_1 and r_2 are the classical turning points. The bound state energy level is $-\epsilon$.

and for the $(p+d+\mu^-)$ system in particular, $\lambda \simeq 5.5$. For process (1) the reaction constant is $A \sim 10^{-22}$ cm^3/sec [Eq. (9)], so that the reaction time is estimated at $T \sim 2.5 \times 10^{-6}$ sec, the observed order of magnitude.[7] More careful estimates of the reaction rate will be given in Table II.

A word needs to be said about the use of molecular arguments with a particle as heavy as the μ^- meson. The expansion parameter in the usual separation of the "electronic" and vibrational-rotational motions is is $(\mu/M)^{\frac{1}{4}}$. For electronic hydrogen molecules this parameter is of order 0.15, whereas for μ-mesonic hydrogen molecules it is of the order of 0.58. This would seem to cast doubt on the validity of such a separation in the present problem. A counter argument seems possible, however. First of all, the molecular approximation to electronic molecules is much better than the value of $(\mu/M)^{\frac{1}{4}}$ would first lead one to expect. It is perhaps easier to speak of the relative periods of the various types of motion. In an electronic hydrogen molecule the vibrational periods are of the order of 100 times the electronic periods ($\sim 10^{-14}$ sec *versus* $\sim 10^{-16}$ sec), and the separation of the two types of motion can obviously be expected to be a good approximation. In μ-mesonic molecules the vibrational periods are about 10 times the "electronic" periods ($\sim 5 \times 10^{-18}$ sec *versus* $\sim 5 \times 10^{-19}$ sec). While this is not as favorable as the electronic case, it may be expected that the errors involved will not be too large, especially as only semiquantitative estimates are being made.

In the next section we discuss the determination of the reaction constant A for the process (1), as well as other possible reactions between various hydrogen isotopes. Section 3 is devoted to the molecular vibra-

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effective in lowering the Coulomb barrier between two nuclei by virtue of its very small orbit around one of them. Even if the screening effect were perfect and the μ-mesonic atom acted as a neutron in its penetration up to another nucleus, the collision reaction rate would be of the order of 10^{-6} times the intrinsic rate once a molecule is formed. A more realistic estimate is obtained by saying that once a nucleus gets inside the μ-mesonic atom the barrier penetration will be similar to that in the molecule. Then the relevant ratio is the cube of the ratio of the average separations between an atom and an ordinary gas molecule and between the two nuclei in the μ-mesonic molecule. With $\sim 10^{22}$ molecules/cm^3, this ratio is $\sim 10^{-9}$.

[5] Particles heavier than the μ^- meson would be more efficient catalysts, but the particles must interact appreciably with nuclei only by electromagnetic forces. π^- mesons or K^- mesons would in general disappear or be transformed by nuclear interaction, and so violate the requirement of survival after the reaction occurs.

[6] Even at room temperature higher vibrational states are not populated appreciably; rotational levels are suppressed for simplicity.

[7] For electrons we can estimate the reaction rate very crudely by noting that λ is roughly proportional to $(M/\mu)^{\frac{1}{2}}$ (see Appendix B). Thus the electronic rate relative to the μ-mesonic rate will be
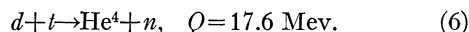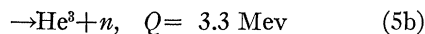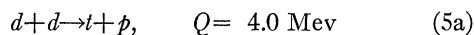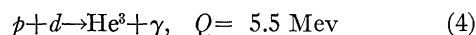
$$(m/\mu)^{5/2} \exp\{-[(\mu/m)^{\frac{1}{2}}-1]\lambda_\mu\} \cong 10^{-38},$$

a completely negligible value.

tional motion and the determination of the barrier penetration factor λ. The intrinsic reaction rates for several reactions are estimated in Sec. 4, and the actual observable rates are discussed. Some speculations on the release of useful amounts of nuclear energy are presented in Sec. 5. Section 6 contains a brief summary. Four appendices contain details of the molecular potential energy curve, a WKB treatment of the barrier penetration, some rough estimates of the competing modes of decay for process (1), and a Born approximation treatment of molecular formation problem.

2. POSSIBLE REACTIONS AND REACTION CONSTANTS

In addition to the process (1), there are several other reactions between isotopes of hydrogen that immediately come to mind as ones that can also be catalyzed by μ^- mesons. Listed below are some exoergic reactions between hydrogen isotopes for which there are detailed data at low energies:

$$p+d \rightarrow He^3+\gamma, \quad Q= 5.5 \text{ Mev} \quad (4)$$

$$d+d \rightarrow t+p, \quad Q= 4.0 \text{ Mev} \quad (5a)$$

$$\rightarrow He^3+n, \quad Q= 3.3 \text{ Mev} \quad (5b)$$

$$d+t \rightarrow He^4+n, \quad Q=17.6 \text{ Mev}. \quad (6)$$

Reaction (4) is the normal version of process (1), and is discussed by Salpeter[8] in connection with energy production in stars. The $d-d$ and $d-t$ reactions have been studied in detail, the latest work at low energies being that of Arnold *et al.*[9] We shall imagine all these reactions to occur at low energies through the formation of a compound nucleus and its subsequent de-excitation. At very low energies the probability of re-emission of the incident particle will be negligible, and the observed cross section will be the same as the cross section for compound nucleus formation.

If a μ^- meson is present in the vicinity, the method of de-excitation of the compound nucleus may be altered. Because the μ^- meson interacts with light nuclei only by electromagnetic forces, de-excitation by μ-mesonic "internal conversion" will compete favorably only if the normal mode of decay is by gamma emission. Hence, for example, reaction (4) can occur either as indicated by Eq. (1), or in the manner

$$\mu^-+p+d \rightarrow He^3+\gamma+\mu^-, \quad (1')$$
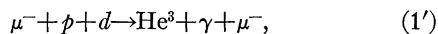
where Eq. (1) is the "internal conversion" mode and Eq. (1') is the radiative mode. Rough estimates of these rates of decay (see Appendix C) show that the "internal conversion" mode is somewhat less likely. On the other hand, for reactions (5a) and (5b) the compound nucleus will presumably de-excite by the emission of a nucleon, even in the presence of a μ^-

[8] E. E. Salpeter, Phys. Rev. 88, 547 (1952).

[9] Arnold, Phillips, Sawyer, Stovall, and Tuck, Phys. Rev. 93, 483 (1954).

meson, the competing "internal conversion" mode $[(He^4)^*+\mu^- \rightarrow He^4+\mu^-]$ being unimportant in spite of the large Q value (23.8 Mev). Similarly, reaction (6) will behave the same way whether μ^- mesons are present or not.

The concept of compound nucleus formation is of questionable validity for light nuclei such as are dealt with here. A perturbation theory viewpoint is perhaps closer to the truth. It would then be argued that the presence of the μ^- meson around the reacting nuclei gives an added mode of decay and the transition rate will be increased correspondingly. From the previous remarks it is clear that this will be of importance only in the $(p+d)$ reaction. For this process, the effective reaction constant in Eq. (2) would be the value deduced below multiplied by $(1+\alpha)$, where $\alpha=N_\mu/N_\gamma$ is the usual internal conversion coefficient. The rates in Table II for the $(p+d)$ process will be altered by the same factor. For our semiquantitative discussion we will ignore this possible correction.

At very low energies the cross section for the foregoing reactions can be written in the form,

$$\sigma= (A/v)C_0^2, \quad (7)$$

where A is the reaction constant (in cm³/sec) of Eq. (2), v is the relative velocity of the incident particle, and C_0^2 is the s-wave Coulomb penetration factor. If the finite size of the interaction volume is neglected, the Coulomb factor at low energies is

$$C_0^2 \simeq 2\pi\eta \exp(-2\pi\eta), \quad (8)$$

where $\eta=e^2/\hbar v$. The combination of Eqs. (7) and (8) gives the familiar Gamow formula, $\sigma=\text{const}E^{-1} \times \exp(-\text{const}E^{-\frac{1}{2}})$.

For reaction (4) there are no data at low energies, but Salpeter[8] has argued that the mirror reaction $[d(n,\gamma)t]$ should have essentially the same cross section, except for the Coulomb barrier factor. The low-energy neutron capture in deuteron does show a $(1/v)$-law behavior, and the value of the thermal cross section is 0.57×10^{-27} cm². This corresponds to a reaction constant,

$$A_4= (1.25 \times 10^{-22}) \text{ cm}^3/\text{sec}. \quad (9)$$

Following Salpeter, we will assume that this value holds for its mirror reaction (4).

Reactions (5) and (6) are much more probable than (4). Below 100 kev, the cross section[9] for (5) follows the form of Eq. (7) with a reaction constant,

$$A_5=2.01 \times 10^{-16} \text{ cm}^3/\text{sec}. \quad (10)$$

The two modes of decay (5a) and (5b) are very nearly equally abundant. It is notable that the reaction constant A_5 is over 10^6 times as large as A_4. The $d-t$ reaction shows a resonance at around 80 kev in addition to the barrier effects implied in Eq. (7). At very low energies, however, the cross section appears to follow the Gamow form, the cross section having an absolute

magnitude of about 100 times the $d-d$ cross section.[9] Thus the reaction constant for reaction (6) is approximately

$$A_6 \simeq 2.0 \times 10^{-14} \text{ cm}^3/\text{sec}. \tag{11}$$

It is clear from Eqs. (9), (10), and (11) that the reaction rate in the μ-mesonic $d-d$ or $d-t$ molecule will be of the order of 10^6 times the observed rate of $\sim 10^6$ sec^{-1} for the $p-d$ molecule. Thus, in pure deuterium or a deuterium-tritium mixture, the μ^- meson should be able to act as a catalyst a very large number of times during its lifetime. Quantitative discussion will be reserved for Sec. 4, where the question of the time spent by the μ^- meson between molecular formation will be considered, as well as an important loss mechanism.

3. MOLECULAR ION AND BARRIER PENETRATION

The problem of two hydrogen atoms bound together by an electron to form a molecular ion is one of the earliest and simplest problems to be treated quantum mechanically.[10] The transcription to a μ-mesonic (or other particle) type of molecule is elementary. Fine effects, such as the spins of the particles and the statistics of the nuclei, are ignored since they have little influence on the problem at hand.

The ground state molecular potential energy curve has the form shown in Fig. 1. If lengths are expressed in units of the Bohr radius a of the binding particle and energies in units of (e^2/a), the potential is independent of the mass of the binding particle. The details of the potential curve, the equilibrium separation of the nuclei, etc., are described in Appendix A. The radial wave function for the lowest vibrational state is just the harmonic oscillator eigenfunction:

$$u_{\text{vib}}(x) = (\alpha/\pi)^{\frac{1}{4}} \exp[-\tfrac{1}{2}\alpha(x-x_0)^2], \tag{12}$$

where x_0 is the equilibrium separation in units of a, and $\alpha = M\hbar\omega/\mu$ is the usual oscillator parameter in units of a^{-2}. The wave function (12) is only an approximate description of the vibrational motion in a μ-mesonic molecule, even in the region between the turning points, since the energy level [Eq. (A3)] is raised up sufficiently from the potential minimum that anharmonic terms in the potential begin to be important. For our rough estimates, however, Eq. (12)

TABLE I. Barrier penetration factor $\lambda(x_N)$ for μ-mesonic hydrogen molecules.

x_N	$p+d$	$d+d$	$d+t$
0	6.0	8.3	9.5
0.02	5.6	7.6	8.8
0.05	5.0	6.9	7.9

[10] For detailed references, see for example, Eyring, Walter, Kimball, *Quantum Chemistry* (John Wiley and Sons, Inc., New York, 1944), p. 192 ff.

TABLE II. Nuclear reaction rates $1/T$ (sec^{-1}) in μ-mesonic hydrogen molecules.

x_N	$p+d$	$d+d$	$d+t$
0	0.18×10^6	0.35×10^{11}	0.11×10^{13}
0.02	0.26×10^6	0.70×10^{11}	0.23×10^{13}
0.05	0.48×10^6	1.41×10^{11}	0.57×10^{13}

will be adequate, except that it must be extended into the Coulomb barrier region towards zero separation.

The determination of the wave function at small separations of the nuclei is treated by means of the WKB approximation in Appendix B. There it is shown that for very small separations of the nuclei the vibrational wave function can be written approximately as

$$u_{\text{vib}}(x) \simeq (\alpha/2\pi)^{\frac{1}{2}} x \exp[-\tfrac{1}{2}\lambda(x)]. \tag{13}$$

The parameter α is that appearing in Eqs. (12) and (A4), while the factor $\lambda(x)$ is the barrier penetration factor defined by Eq. (B6). The values of λ appropriate for the combinations of hydrogen isotopes appearing in reactions (4), (5), and (6) were computed numerically for μ-mesonic molecules. The results are shown in Table I. Three values are given for each molecule, corresponding to a nuclear interaction distance $x_N = 0$, 0.02, 0.05. These are equivalent to the sum of the nuclear "radii" being 0, 5×10^{-13} cm, and $\sim 1.2 \times 10^{-12}$ cm, respectively. To find very rough values of λ for other binding particles and/or other combinations of hydrogen isotopes, the numbers in Table I can be scaled according to $(M/\mu)^{\frac{1}{2}}$, as implied by the form of $Q^2(x)$ in Eq. (B2').

4. INTRINSIC AND ACTUAL REACTION RATES

The rate of the catalyzed reactions is given by Eq. (2), where the reaction constant A was discussed in Sec. 2. The wave function $\Psi(0)$ in Eq. (2) is given by

$$\Psi(0) = \frac{1}{(4\pi a^3)^{\frac{1}{2}}} \frac{u_{\text{vib}}(x)}{x} \bigg|_{x=x_N}, \tag{14}$$

where the factor 4π comes from the rotational wave function, and the factor a^3 from the introduction of proper units of length. Combination of Eqs. (2), (13), and (14) leads to the basic result for the intrinsic reaction rate,

$$\frac{1}{T} \simeq \frac{A}{4\pi a^3} \left(\frac{\alpha}{2\pi}\right) \exp(-\lambda). \tag{15}$$

With the help of Eq. (A4) and the value $a = 2.53 \times 10^{-11}$ cm, Eq. (15) can be specialized for μ-mesonic hydrogen molecules:

$$1/T \simeq 0.70 \times 10^{30} A (M/M_p)^{\frac{1}{2}} \exp(-\lambda) \text{ sec}^{-1}, \tag{16}$$

where M_p is the proton mass.

The values of λ from Table I can be inserted into Eq. (16) along with the appropriate reaction constants

from Sec. 2 to yield the intrinsic reaction rates for the catalyzed reactions corresponding to Eqs. (4), (5), and (6). These rates are displayed in Table II. For the $p-d$ reaction the most likely value of the interaction distance is $x_N = 0.02$, while for the $d-d$ process it is closer to 0.05. For the $d-t$ reaction an intermediate value is probable.[11] We see from the figures in Table II, first of all, that the rate for process (1) is of the observed order of magnitude. Whether the agreement is quantitative or not cannot be said without further experimental data. Secondly, the rates for the $d-d$ reaction and the $d-t$ reaction are of the order of 10^5 to 10^7 times that of the observed $p-d$ rate.[12] The possibility consequently arises that the μ^- meson may serve as a catalytic agent for these reactions over and over again during its lifetime.

The actual reaction rate that will be observed is smaller than the intrinsic rate of Table II because of the time spent by the μ^- meson between catalyzing events and also due to loss of the μ^- meson to the catalytic chain. Three important effects are:

(1) slowing-down time,

(2) time spent by the μ^- meson bound in a μ-mesonic hydrogen atom rather than a molecule,

(3) loss of μ mesons in "dead-end" processes from which they cannot escape to serve as catalysts.

None of these processes will appreciably affect the rate of reaction (1) since its intrinsic rate is comparatively slow. They are of vital importance, however, in the $d-d$ and $d-t$ reactions. The slowing-down time can be estimated from Wightman's work.[3] From the discussion of Sec. 2 it is evident that in the $d-d$ and $d-t$ processes the μ^- meson will be liberated with only a few kev energy. Consequently the time taken to slow down and form a μ-mesonic hydrogen atom in its ground state will be of the order of 10^{-10} sec. If this were the governing factor the μ^- meson could still catalyze $\sim 10^4$ reactions during its lifetime. Unfortunately, the slowing-down time is short compared

to the time spent by the μ-mesonic atom finding a partner with which to make a μ-mesonic molecular ion.

There are two mechanisms for molecule formation, the first being radiative capture and the second being the ejection of orbital electrons by direct electrostatic interaction. The latter process is the more likely of the two, and it is discussed in Appendix D. The calculation is based on the first Born approximation, but rough estimates have been made of the corrections necessary on account of the distortion of the incident wave of relative motion at very low velocities. When these corrections are included, it is found that the molecular formation time in liquid hydrogen or in hydrogen gas at room temperature and ~ 1000 atmospheres it is about 10^{-8} sec. With this estimate we see that each μ meson will catalyze only ~ 100 reactions in liquid deuterium or deuterium-tritium mixture during its lifetime.

The other limiting factor is the loss of μ^- mesons to the catalytic chain by trapping in bound orbits around the helium nuclei formed in the nuclear reaction.[13] At the moment of breakup of the excited nucleus in all the reactions treated here, the μ^- meson is in a K-shell orbit of a helium atom since it responds adiabatically to the relatively slow motion of the nuclei. The fragments are ejected suddenly, and there is an appreciable probability that the μ^- meson will form a bound state around a charged fragment. The problem is a standard one of "sudden perturbation,"[14] and involves a simple overlap integral. For those reactions in which a helium isotope and a neutron are ejected, the probability that the μ^- meson is bound in the $(1s)$ orbit is

$$P = [1 + (v/4v_0)^2]^{-4}, \qquad (17)$$

where v is the velocity of recoiling He fragment and v_0 is the Bohr orbit velocity in hydrogen $(c/137)$. It is noteworthy that the trapping probability P depends only on the velocity of the fragment and not on the mass of the catalyzing particle. When the small contributions to excited states are included, it is found that when He^3 is emitted in the $d-d$ reaction the μ^- meson is caught about 16% of the time, while in the $d-t$ reaction the μ^- meson is trapped around the He^4 fragment only 1.0% of the time. When a proton and a triton are emitted in the $d-d$ reaction the probability that the μ^- meson is caught by either particle is about 2%. In all $d-d$ reactions, then, the μ^- meson is trapped around the He^3 fragment in about 8% of the events, and around the proton or triton in 1%. This effect is cumulative and means that for the $d-d$ reaction the μ^- meson will catalyze only about 12 reactions on the average before being caught by the He fragment. For the $d-t$ process the corresponding figure is $\sim 10^2$ reactions.

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[11] One might question the significance of any value in Table II other than $x_N = 0$ in view of the fact that the reaction constants A of Sec. 2 were seemingly deduced from the zero-range penetration factor [Eq. (8)]. Actually the experimenters[9] determine the exponent in the penetration factor empirically, and its value reflects in some way the finite interaction radius. In any event the uncertainty in the $(1/T)$ values which may be present from this effect is of no importance in the subsequent discussion.

[12] The rates in Table II do not take into account the nuclear spins. The most important effect of the spins occurs for the $d-t$ process. The $d-t$ molecule will be formed in its lowest state with a combined nuclear spin equal to $\frac{1}{2}$ one-third of the time, and equal to $\frac{3}{2}$ two-thirds of the time. The large reaction constant [Eq. (11)] at zero energy is due to the resonance at ~ 80 kev, and this resonance is known [Argo, Taschek, Agnew, Hemmendinger, and Leland, Phys. Rev. 87, 612 (1952)] to have $J = \frac{3}{2}^+$. Consequently the rate in Table II actually applies only for two thirds of the $d-t$ molecules formed. The other one-third will react at a much slower rate, perhaps comparable to the $d-d$ rate. This is of no importance for the observed rate of reaction because these intrinsic reaction times are very small compared to the other relevant times discussed below.

[13] This mode of loss was pointed out to the author by Professor E. P. Wigner.

[14] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), p. 211.

If the μ^- meson cannot be stripped off the helium fragment it is lost to the catalytic chain since the μ-mesonic helium ion cannot form a stable molecular ion with hydrogen. In actual fact the $(\mu^-+\text{He})$ system will behave like a hydrogen isotope as far as the electrons are concerned, and will capture an electron to form a pseudo-hydrogen atom when it slows down to velocities of order v_0. Then a μ-mesonic molecular ion could not be formed even if it were stable. The only question is whether the μ^- meson can be stripped off during the slowing-down process. Bohr[15] has discussed the problem of loss of electrons by partially ionized fragments passing through matter, and his arguments can be applied to the present problem by simple scaling from electronic sizes to μ-mesonic dimensions. For a helium fragment with a μ^- meson bound to it and passing through hydrogen, Bohr's result [his Eq. (4.2.4)] becomes

$$\sigma_{\text{loss}}=2\pi a^2(v_0/v)^2, \qquad (18)$$

where a is the μ-mesonic Bohr radius. Equation (18) is expected to be valid for $v\gg v_0$, and to give considerably too large values for the loss cross section for $v\sim v_0$.

The probability that the μ^- meson will be stripped off during the slowing down will be given by

$$p=\int N\sigma_{\text{loss}}dx,$$

where N is the number of atoms/cm^3 and the integral is taken over the range of the fragment. This can be converted into an integral over energy:

$$p=\int_{E_{\text{min}}}^{E} N\sigma_{\text{loss}}(E)\left(\frac{-dx}{dE}\right)dE. \qquad (19)$$

The $(\mu^-+\text{He})$ system loses energy as if it were a hydrogen isotope, so that in hydrogen gas or liquid the energy loss is approximately

$$\frac{-dE}{dx}\simeq 4\pi N\frac{e^4}{mv^2}\ln\left(\frac{2mv^2}{I}\right), \qquad (20)$$

where $I\simeq 16$ ev is the effective ionization potential in hydrogen. In Eq. (19) the lower limit E_{min} is roughly the energy where $v=v_0$ since, on the one hand, the loss cross section falls rapidly for lower velocities, and on the other hand, the energy loss departs from the simple form (20). Using Eqs. (18) and (20) in (19), we find that for the He^3 ion in the $d-d$ process the probability of loss of the μ^- meson during slowing down is about 13%. For the He^4 fragment in the $d-t$ reaction this stripping probability is about 40%. These values are, however, definitely too high because of the overestimation of Eq. (18) at low velocities. If one takes the experimental data[16] for the equivalent electronic loss cross section

[15] N. Bohr, Kgl. Danske Videnskab. Selskabs, Mat.-fys. Medd. **18**, No. 8 (1948), Chap. 4.
[16] Allison, Cuevas, and Murphy, Phys. Rev. **102**, 1041 (1956).

and scales it down in the ratio of the squares of the electron and μ-meson masses, one finds loss cross sections much smaller than Eq. (18). If these data are extrapolated in such a way that they approach (18) at very high velocities, and are substituted into (19), we find that the stripping probabilities are reduced to about 4% for the $d-d$ reaction and about 22% for the $d-t$ reaction.

From the above analysis it appears that stripping is not prevalent enough to affect the "dead-end" loss appreciably, even for the $d-t$ process.

It is noteworthy that at liquid hydrogen densities the "dead-end" loss process and the molecular formation times lead to roughly the same limit for the number of $d-t$ reactions catalyzed by each μ^- meson. An increase in the density can decrease the molecular formation time, but cannot affect the mean number of reactions before loss in "dead-end" capture. Similarly a longer lifetime for the μ^- meson (or some other particle) is of no avail against this particular form of loss.

5. SPECULATIONS ON USEFUL ENERGY RELEASE

Consideration of these reactions which are catalyzed by μ^- mesons leads naturally to speculation concerning the release of useful amounts of nuclear energy by this means. The process is especially attractive because the energy is released at low temperatures and the confinement problems are less severe than for thermonuclear reactions. We propose to make a few tentative remarks on the energy release problem, without having attempted to study the matter exhaustively.

One possible line of pseudo-realistic reasoning goes as follows: At the present state of the art, in order to have appreciable numbers of μ^- mesons one must first generate π^- mesons, and in order to generate π^- mesons one must first produce high-energy nucleons in some sort of accelerator. For purposes of argument, let us consider an accelerator which produces ~ 1-Bev protons. Higher energies can be attained only at the expense of intensity; lower energies are not as useful for π^--meson production. In order to get a high absolute power, as well as high efficiency, a fixed-frequency machine of the FFAG type or a linear accelerator is probably desirable. It is perhaps not beyond hope to expect an ultimate efficiency of 20% in conversion of power from the ac mains to proton beam power. Thus one 1-Bev proton corresponds to an input of 5-Bev energy from the ac mains. One 1-Bev proton incident on a suitable target will produce at least one π meson, but probably not two. For definiteness we suppose 0.5 *negative* π mesons per 1-Bev proton. If these π^- mesons are suitably collimated, they can be allowed to travel a few meters and decay in flight (stopping them in some absorber immediately after production would lead to their disappearance by nuclear interaction). The resulting μ^- mesons can then be allowed to stop in liquid hydrogen, or better, in hydrogen gas at room

temperature or above and 10^3 atmospheres (technical questions of hydrogen vessels, entrance windows, etc., will be ignored). If the engineer has been clever enough, we may suppose that there are very nearly 0.5 μ mesons in the hydrogen per 1-Bev proton initially produced. Let x be the number of reactions catalyzed by each μ^- meson during its lifetime, and y be the energy release (in Mev) in each reaction. Then $0.5xy$ Mev nuclear energy is produced per 5-Bev input energy to the accelerator. The ratio of energy release to energy input is thus $\sim 10^{-4}xy$. For the $d-d$ reactions, $x\sim 10$ and $y\sim 4$, so that this ratio is $\sim 4\times 10^{-3}$, far smaller than unity. For the $d-t$ process, $x\sim 10^2$ and $y\sim 18$, and the ratio of nuclear energy release to input energy is ~ 0.2. The $d-d$ reaction is hopeless as a power source, while the $d-t$ reaction misses by a factor of 5, if one disregards further losses in the conversion of nuclear heat energy to electrical energy. It seems unlikely that useful power production is possible with these reactions, even though one may question individual estimates of the various factors involved, unless an energetically cheaper way of producing μ^- mesons can be found.

Even if the process had an efficiency greater than unity, it is questionable whether really useful *absolute* amounts of power could be produced. The limit here is the input power to the accelerator. This can probably not be more than a few megawatts. Even with a ratio of output energy to input energy of 10, this would mean only $\sim 10^4$ kw power. When this is compared to the capacities of steam generating plants or nuclear power plants ($\sim 10^4$ to 2×10^5 kw), it is seen that the absolute power is small.

6. SUMMARY AND CONCLUSIONS

The mechanism whereby μ^- mesons serve as catalysts for reactions between hydrogen isotopes has been discussed in detail. The reaction rate for process (1), observed by Alvarez and co-workers,[1] has been calculated and found to agree approximately with experiment. Intrinsic reaction rates for catalyzed $d-d$ and $d-t$ reactions have been estimated (Table II) and are found to be 10^5 and 10^7 times the rate for process (1). The actual rate of release of energy by the μ^--mesons is limited, however, by the time spent in molecule formation and in loss of μ^- mesons in "dead-end" processes. For liquid deuterium (or gas at equivalent density) it is estimated that each μ^- meson catalyzes only ~ 10 nuclear reactions before its loss into a bound orbit around the He^3 nucleus produced in the reaction. For the $d-t$ reaction, the μ^- meson can catalyze $\sim 10^2$ reactions in its lifetime. The question of liberation of useful amounts of power by these reactions is considered briefly. It is very unlikely that the system can be made exoergic, even with the $d-t$ reaction.

Only reactions between the different hydrogen isotopes have been considered in the present paper. It is possible that other reactions such as $(d+\text{Li}^6)$ could be catalyzed by μ^- mesons. It would appear, however, that even if a molecular system were formed with dimensions comparable to the μ-mesonic hydrogen molecule, the Coulomb barrier for the larger Z values would be sufficiently large that penetration would be negligible and the reaction rate small.

APPENDIX A

The ground state molecular potential energy curve for H_2^+ will be approximated by the following analytic form for calculational purposes:

$$V(x) = \frac{1}{x} - \frac{1.5}{(1+0.127x)^4}. \tag{A1}$$

The energy is expressed in units of (e^2/a) and the distance in units of a. This analytic form is a reasonable approximation over the range of interest $(0 \leq x < 3)$. The equilibrium radius is $x_0 = 2.03$, and the minimum value of the potential is $V(x_0) = -0.106$. The vibrational energy is related to the reduced mass M of the nuclei and the curvature of the potential at $x = x_0$. For $V(x)$ given by Eq. (A1), the vibrational quantum is

$$\hbar\omega \sim 0.302(\mu/M)^{\frac{1}{2}} \tag{A2}$$

in units of (e^2/a), where μ is the mass of the binding particle. The lowest energy level of the system, shown in Fig. 1, is therefore

$$-\epsilon \sim -0.106 + 0.151(\mu/M)^{\frac{1}{2}}. \tag{A3}$$

The radial vibrational wave function $u(x)$ is given by Eq. (12), where the oscillator parameter α is

$$\alpha = 0.302(M/\mu)^{\frac{1}{2}}. \tag{A4}$$

APPENDIX B

The radial wave function, Eq. (12), is an approximate representation of the vibrational motion only between the classical turning points. The potential departs from a simple quadratic behavior rapidly outside these points, and the wave function differs accordingly. The WKB approximation will be employed to extend the wave function in towards the origin from the inner turning point in Fig. 1.

In our dimensionless units the radial wave equation for nuclear motion takes the form

$$\left[\frac{d^2}{dx^2}+Q_0^2(x)\right]u(x)=0. \qquad (B1)$$

where

$$Q_0^2(x)=\frac{2M}{\mu}[-\epsilon-V(x)]. \qquad (B2)$$

The binding energy ϵ is given by Eq. (A3), and the potential by Eq. (A1). Langer[17] has shown that the ordinary WKB connection formula,

$$|Q(x)|^{-\frac{1}{2}}\exp\left(-\int_x^{x_1}|Q|dx\right)\rightarrow2[Q(x)]^{-\frac{1}{2}}$$

$$\times\cos\left(\int_{x_1}^x Qdx-\frac{\pi}{4}\right), \qquad (B3)$$

can be applied to the radial wave equation (with the boundary condition $u(0)=0$) only if the function $Q_0^2(x)$ in the original wave equation is modified by the addition of the term $(-1/4x^2)$. Thus Eq. (B2) must be replaced by

$$Q^2(x)=\frac{2M}{\mu}[-\epsilon-V(x)]-\frac{1}{4x^2}. \qquad (B2')$$

The problem of normalization of the WKB solution can be handled approximately in several ways. Furry[18] has discussed WKB normalization, with special attention to the harmonic oscillator. He finds that the normalization constant by which both sides of Eq. (B3) need be multiplied for the ground state oscillator wave function is $(\alpha/2\pi)^{\frac{1}{2}}=0.40\alpha^{\frac{1}{2}}$. Another more direct way is to make the WKB solution at the potential minimum have the same value as the exact solution Eq. (12). This leads to the normalization constant $0.38\alpha^{\frac{1}{2}}$, essentially equal to that of Furry. Furry showed that with his normalization the WKB wave function equalled the exact wave function within 4% in the far wings of the curve. Consequently we can assume that with Furry's oscillator normalization we can connect the solution Eq. (12) in the classically allowed region to the WKB solution [left side of Eq. (B3)] in the classically forbidden region without appreciable error.

The WKB wave function can therefore be written, in the region inside the inner turning point:

$$u_{\text{WKB}}(x)\simeq\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}|Q(x)|^{-\frac{1}{2}}\exp\left(-\int_x^{x_1}|Q|dx\right), \qquad (B4)$$

where $Q(x)$ is given by Eq. (B2'). The turning point $x=x_1$ is defined as the smallest root of Eq. (B2'). We are interested in the value of $u(x)$ for x very small. In

that case, $|Q(x)|\simeq1/2x$, and it can be shown that $u_{\text{WKB}}(x)$ varies linearly with x for small enough x. To exhibit this linear dependence explicitly Eq. (B4) can be rewritten in the following form, valid only for very small x:

$$u_{\text{WKB}}(x)\simeq\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}x\exp[-\lambda(x)/2], \qquad (B5)$$

where

$$\lambda(x)=\int_x^{x_1}\left(2|Q|-\frac{1}{x}\right)dx+\ln(x_1/2). \qquad (B6)$$

The integrand in Eq. (B6) is finite at $x=0$; thus $\lambda(x)$ approaches a finite limit at $x=0$. The form of $V(x)$ in Eq. (A1) is such that the integrand in Eq. (B6) is not amenable to simple analytic integration. The integration was performed numerically for the μ-mesonic molecules appropriate to the reactions (4), (5), and (6). The numerical values appear in Table I.

APPENDIX C

We wish to make rough estimates of the partial widths for the two modes of decay [Eqs. (1) and (1')] of an excited He[3] nucleus. In the catalyzed reaction (1) the two nuclei bring in no orbital angular momentum. Consequently the compound state formed in He[3] has $J=\frac{1}{2}$ or $\frac{3}{2}$, and even parity. In Russell-Saunders notation the state is most probably $^2S_{\frac{1}{2}}^+$ or $^4S_{\frac{3}{2}}^+$. This state decays to the ground state $(J=\frac{1}{2}$, even; $^2S_{\frac{1}{2}}^+)$, by either gamma emission [Eq. (1')] or "internal conversion," ejecting the μ^- meson [Eq. (1)].

The gamma emission will occur predominantly by magnetic dipole radiation. The radiative width can be written

$$\Gamma_\gamma=(4/3)(\omega/c)^3|\mathfrak{M}|^2, \qquad (C1)$$

where $\mathfrak{M}$ is the matrix element of the magnetic moment operator. Equation (C1) can be written in the convenient form

$$\Gamma_\gamma(\text{ev})=2.77\times10^{-3}|\mathfrak{M}|^2E_\gamma^3(\text{Mev}), \qquad (C2)$$

where the effective magnetic moment is expressed in nuclear magnetons. An upper bound can probably be put on the width by putting $|\mathfrak{M}|^2=1$. With $E_\gamma=5.5$ Mev, this estimate gives

$$\Gamma_\gamma(\text{ev})<0.46. \qquad (C3)$$

Blatt and Weisskopf[19] give estimates of radiative widths based on single-particle transitions. Their value for this $M1$ transition is 6 times larger than Eq. (C3), because of their use of a larger effective magnetic moment. Experimentally the values of $M1$ radiative widths are always considerably smaller than the Weisskopf estimates, even for low-lying states. As a result, the estimate (C3) is probably more reasonable in the present problem. If the $J=\frac{3}{2}$ state were a $^4D_{\frac{3}{2}}^+$

[17] R. E. Langer, Phys. Rev. 51, 669 (1937).
[18] W. H. Furry, Phys. Rev. 71, 360 (1947).

[19] J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics (John Wiley and Sons, Inc., New York, 1952), p. 627.

state, it could make an $E2$ transition as well. The Weisskopf estimate for such a transition is only about 10^{-2} ev, showing that the transition will be mainly $M1$ in any case.

We now turn to the internal conversion problem. In the μ-mesonic molecule, the meson wave function adapts itself adiabatically as the nuclei move back and forth in their vibrational motion. When the nuclei come together to form a compound nucleus, the μ^- meson is in the K-shell orbit around a helium nucleus $(Z=2)$. The conversion process is thus K conversion. The results of the usual theory,[20,21] which treats the nucleus as a point, can be transcribed directly from electronic conversion to μ-mesonic conversion. For the present transition the result is a conversion coefficient of only $\sim 10^{-4}$ for a $M1$ transition, and ~ 0.2 for an $E2$ transition. This would imply that the ejection of the μ^- meson according to Eq. (1) would be very unlikely compared to the radiative decay of Eq. (1'). These results are in error however, because of the neglect of the finite size of the nucleus. In light elements this is allowable for electrons, but not for μ^- mesons because of their closer proximity to the nucleus. We shall now make a rough estimate of the conversion width taking into account the region inside the nucleus. If, for simplicity, we neglect the spin of the μ^- meson, and assume that the nuclear states are $L=0$ states, the discussion follows closely the treatment of $0 \rightarrow 0$ transitions in reference 19, p. 620–621. If the variation of the μ^- meson's final (plane wave) state wave function over the nuclear volume is neglected (even though its wavelength is only $\sim 5.8 \times 10^{-13}$ cm), the perturbation matrix element is approximately

$$H'_{if} \simeq -\frac{4\pi}{3} e^2 \left(\sum_{i=1}^{2} \int \Phi_a{}^* r_i{}^2 \Phi_b dV_N \right) \psi_{1s}(0). \quad (C4)$$

The integral over the nuclear wave functions will be written as βR^2, where R is some sort of nuclear radius $(\sim 3 \times 10^{-13}$ cm$)$, and β is a constant of order unity. The expression for the nonradiative width is

$$\Gamma_\mu = 2\pi |H'_{if}|^2 4\pi \mu p/(2\pi\hbar)^3.$$

With $\psi_{1s}(0) = (1/\pi)^{\frac{1}{2}}(2/a)^{\frac{3}{2}}$, and the estimate Eq. (C4) for H'_{if}, the width becomes

$$\Gamma_\mu \simeq \frac{128}{9} \beta^2 \left(\frac{e^2}{\hbar c}\right) \left(\frac{R}{a}\right)^4 \left(\frac{p}{\mu c}\right) \mu c^2. \quad (C5)$$

With $\beta=1$ and $R=3 \times 10^{-13}$ cm, we obtain the estimate

$$\Gamma_\mu \simeq 0.07 \text{ ev}. \quad (C6)$$

[20] S. M. Dancoff and P. Morrison, Phys. Rev. 55, 122 (1939).
[21] S. D. Drell, Phys. Rev. 75, 132 (1949).

This value is about 6 times smaller than the upper bound [Eq. (C3)] for the radiative width Γ_γ. Since the actual radiative width is likely considerably smaller, we conclude that the "internal conversion" mode of de-excitation is dominant, although the radiative mode need not be negligible in comparison.

The estimate (C6) is valid only for the decay of the $^2S_{\frac{1}{2}}$ state. For the $^4S_{\frac{3}{2}}$ state the μ^- meson must carry off 2 units of orbital angular momentum. In this case the conversion will be very small, and the state will de-excite by gamma radiation.

Note added in proof.—The observed branching ratio of μ-meson ejection to gamma emission is some mixture of the ratios for the $J=1/2$ and $J=3/2$ compound states. On the basis of statistical weights the $J=3/2$ state will occur twice as often as the $J=1/2$ state. Consequently gamma emission will probably be favored over internal conversion, although the numerical value of the conversion coefficient is uncertain.

APPENDIX D

In this appendix we consider the problem of formation of a μ-mesonic molecular ion by collision of a μ-mesonic hydrogen atom with an electronic hydrogen molecule. For definiteness we shall assume that the nuclei involved are both deuterons, although the results are insensitive to the actual hydrogen isotopes involved. The μ-mesonic hydrogen atom is such a small structure that it penetrates easily through the electron cloud and can come comparatively close to one or the other of the nuclei in the electronic hydrogen molecule. Consequently we can simplify the discussion by considering the collision between the mesonic atom and an ordinary hydrogen *atom*, without introducing appreciable error. There are two mechanisms of molecular formation, the first being radiative capture and the second being direct ejection of the orbital electrons. The radiative capture will be small, especially for homonuclear molecules, and will not be considered here. The direct ejection of electrons will be treated in the Born approximation. From Appendix A it can be seen that the Q value is ~ 310 ev, so that the outgoing electron can be described reasonably well by a plane wave. The use of a plane wave to describe the relative motion between the mesonic atom and the struck nucleus at low velocities is undoubtedly a poor approximation. Some remarks are made at the end of this appendix on the errors introduced thereby.

If the μ^- meson is initially bound around nucleus 1, and the electron around nucleus 2, the final state perturbation energy is

$$H' = e^2 \left(\frac{1}{r_{\mu e}} - \frac{1}{r_{e1}} - \frac{1}{r_{e2}} \right), \quad (D1)$$

where the subscripts have obvious meanings. The

initial and final state wave functions are

$$\Psi_i \simeq (\pi a_0^3)^{-\frac{1}{2}}(\pi a^3)^{-\frac{1}{2}} \exp\left[-\frac{r_{e2}}{a_0}-\frac{r_{\mu 1}}{a}+i\mathbf{K}\cdot\mathbf{R}\right], \quad (D2)$$

$$\Psi_f \simeq \Psi_{\text{mol}}(\mathbf{r}_\mu,\mathbf{R})\exp(i\mathbf{k}\cdot\mathbf{r}_e), \quad (D3)$$

where a_0 and a are the electronic and μ-mesonic Bohr radii, respectively, $\mathbf{R}$ is the relative coordinate between the two nuclei, $K=Mv/\hbar$ is the wave number of relative motion of the two atoms, k is the wave number of the ejected electron, and Ψ_{mol} is the μ-mesonic molecular ion wave function, including the nuclear vibration as well as the mesonic motion.

For energies of relative motion less than 1 ev, $K^{-1} > 10^{-9}$ cm. Thus in the overlap integral over $\mathbf{R}$, the plane wave of relative motion can be put equal to unity in Eq. (D2). Then the perturbation matrix element becomes

$$H'_{fi} \simeq \frac{e^2}{(\pi a_0^3)^{\frac{1}{2}}}\int d\mathbf{R}\int d\mathbf{r}_\mu \Psi_{\text{mol}}{}^*(\mathbf{r}_\mu,\mathbf{R})\frac{e^{-r_{\mu 1}/a}}{(\pi a^3)^{\frac{1}{2}}}I(\mathbf{r}_\mu,\mathbf{R}), \quad (D4)$$

where

$$I=\int d\mathbf{r}_e e^{-i\mathbf{k}\cdot\mathbf{r}_e}\left(\frac{1}{r_{\mu e}}-\frac{1}{r_{e1}}-\frac{1}{r_{e2}}\right)e^{-r_{e2}/a_0}. \quad (D5)$$

In the integral (D5) the various coordinates $r_{\mu e}$, r_{e1}, r_{e2} differ from each other by magnitudes of order $R\sim 5\times 10^{-11}$ cm. Since $R\ll a_0$ and $kR\ll 1$, the integral I can be written

$$I\simeq -\int d\mathbf{r}_e\frac{e^{-r_e/a_0}}{r_e}e^{-i\mathbf{k}\cdot\mathbf{r}_e}=\frac{-4\pi a_0^2}{1+(ka_0)^2}, \quad (D6)$$

with the neglect of terms of order R/a_0 and kR. Then the matrix element becomes

$$H'_{fi} \simeq \frac{-e^2}{(\pi a_0^3)^{\frac{1}{2}}}\frac{4\pi a_0^2}{(1+k^2 a_0^2)}\int d\mathbf{R}\int d\mathbf{r}_\mu \frac{e^{-r_{\mu 1}/a}}{(\pi a^3)^{\frac{1}{2}}}$$
$$\times \Psi_{\text{mol}}{}^*(\mathbf{r}_\mu,\mathbf{R}). \quad (D7)$$

The remaining integral over $\mathbf{r}_\mu$ is a familiar molecular two-center integral which is then integrated over the rotational and vibrational motion of the nuclei. With the simplest possible molecular wave function,

$$\Psi_{\text{mol}}=\frac{1}{\sqrt{2}}(e^{-r_{\mu 1}/a}+e^{-r_{\mu 2}/a})\frac{\chi_{\text{vib}}}{(\pi a^3)^{\frac{1}{2}}}, \quad (D8)$$

where χ_{vib} is just the ground-state oscillator wave function, we find a cross section times incident velocity of the form

$$\sigma v \simeq 18a^2\left(\frac{e^2}{\hbar}\right)\frac{ka_0}{[1+(ka_0)^2]^2}. \quad (D9)$$

With $ka_0 = 4.80$ for a Q value of 310 ev, this gives $\sigma v \simeq 2.1\times 10^{-14}$ cm^3/sec. With liquid hydrogen densities, $N\simeq 4\times 10^{22}$ atoms/cm^3, so that the rate of molecule formation is

$$N\sigma v \simeq 8\times 10^8 \text{ sec}^{-1}. \quad (D10)$$

As was mentioned at the beginning of the appendix, the most glaring approximation is the use of a plane wave to describe the relative motion of the nuclei at low velocities. To assess the error made, a simple calculation was performed for the $l=0$ part of the incident wave, assuming that the interaction between the mesonic atom and the struck nucleus was a screened Coulomb field. It is found that the result (D10) is an overestimate by about a factor of 6. Consequently in the text we use 10^{-8} sec as a rough value of the molecular formation time at liquid hydrogen densities.

Note added in proof.—The observed saturation of the number of catalyzed reactions as a function of deuterium concentration can be explained[1] by the rapid transfer of the meson from a proton to a deuteron with the release of about 137 ev energy. The transfer process is essentially a charge exchange collision, and standard techniques can be employed [J. D. Jackson and H. Schiff, Phys. Rev. **89**, 359 (1953)]. For low incident velocities, however, it is necessary to use a distorted wave of relative motion. If the time-average hydrogenic ground-state potential $[V(r)=e^2(r^{-1}+a^{-1})\exp(-2r/a)]$ is used to calculate the distorted wave function, the low-energy cross section for transfer of the meson from proton to deuteron is $\sigma v \simeq 9\times 10^{-12}$ cm^3/sec.

The relative number of observed catalyzed reactions as a function of (small) deuterium concentration is given by a constant times $\lambda/(1+\lambda)$, where λ is the ratio of the rate of meson transfer to the rate of $(p+p+\mu^-)$ molecule formation. The constant multiplier reflects the internal conversion—gamma radiation branching ratio, etc. With the estimates from Appendix D on molecule formation and the above charge transfer cross section, it is found that $\lambda \simeq 1.2\times 10^3 N_d/N$, where N_d/N is the relative deuterium concentration. This value of λ gives rough agreement with the data in reference 1.