where B_{lj} is the matrix tabulated by Penfold and Leiss

$$=\sum_{l=1}^{n-1} E_l \sum_{j=1}^{l} B_{lj}y_j + \tfrac{1}{2}E_n \sum_j^n B_{nj}y_j$$

$$=\sum_{l=1}^{n} E_l \sum_j^n B'_{lj}y_j$$

$$=\sum_j^n T_{nj}y_j.$$

Thus

$$T_{nj}=\sum_{l=1}^{n} E_l B'_{lj}. \tag{A.12}$$

(A.12) is an explicit relation between the required matrix T and the tabulated matrix B.

Error in the smoothed cross sections $\bar{\sigma}$.—From (A.10), we have

$$\begin{aligned}\bar{\sigma}_m &= \sum_j C_{mj}S_j \\ &= \sum_j \sum_k C_{mj}T_{jk}y_k \\ &= \sum_k \sum_j C_{mj}T_{jk}y_k,\end{aligned} \tag{A.13}$$

$$\Delta^2\overline{(\sigma_m)} = \sum_k |\sum_j C_{mj}T_{jk}|^2 \Delta^2 y_k.$$

(A.13) gives the error of σ for given weighting factors C_{mj}.

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Relativistic Corrections to p-p Scattering*

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Effects of wave function distortion by nuclear forces of nonelectromagnetic origin are qualitatively considered. It is found that the relativistic corrections to the Coulomb wave contain effects of wave function distortion which may affect these corrections by reasonably large fractional amounts. The spin-orbit interactions arising from the action of the electric field are found to be affected by wave function distortion. Since these interactions affect the polarization of proton beams in double and triple scattering, the analysis of high-energy data is affected. The theory of spin-orbit interactions is brought into relation with that of atomic spectra. The unreliability of contact terms contained in the relativistic corrections is brought out. A concise proof of the vanishing of first-order tensor force effects on the polarization applying independently of the origin of the tensor force effects is supplied in an appendix.

I. INTRODUCTION AND NOTATION

RELATIVISTIC corrections for p-p scattering have been discussed by Garren,[1] Breit,[2] Ebel and Hull,[3] and again by Garren.[4] The corrections worked out in these papers apply to Coulomb scattering. The viewpoint taken[2] was that specifically nuclear forces introduce phase shifts of their own which can be defined in the center-of-mass system and which require no additional consideration regarding relativistic effects. Two procedures were considered[2] for carrying through the rigorous solution of the problem. Their discussion is contained between Eq. (16.4) and Eq. (17) of the above reference. As an approximation to "procedure (a)" the distortion of the wave function by specifically nuclear interaction effects was neglected and the relativistic corrections to the undistorted Coulomb wave were calculated. Garren's[1,4] approach is equivalent to this approximation. In some of the applications[4] it is tacitly assumed that the approximation is good enough although the question was left open for future consideration in the other work referred to.[2] Further examination shows that specifically nuclear interactions may affect the relativistic corrections to Coulomb scattering to an appreciable degree. This applies in particular to the corrections which matter most for polarization. The quantities involved are large enough to make the application of these corrections to the polarization questionable in any but a qualitative sense. Improvements on the corrections can be made, as will be described below, but a definite value even in the first order of e^2 will be seen to require knowledge of wave functions in the presence of nuclear interactions. It will also be seen that the terms[4] caused by the anomalous part of the proton magnetic moment which have their origin in the divergence of the electric field can be expected to be especially seriously modified. Some of the most frequently occurring symbols used

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[1] A. Garren, Phys. Rev. **96**, 1709 (1954).
[2] G. Breit, Phys. Rev. **99**, 1581 (1955).
[3] M. E. Ebel and M. H. Hull, Jr., Phys. Rev. **99**, 1596 (1955).
[4] A. Garren, Phys. Rev. **101**, 419 (1956).

below are as follows:

e = proton charge,
M = nucleon mass,
θ = scattering angle in the center-of-mass system,
$\boldsymbol{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ = vector formed by Pauli's spin matrices,
v = absolute value of relative velocity of the two protons in the laboratory system,
$k=Mv/(2\hbar)$,
$\rho=kr$,
$\eta=e^2/(\hbar v)$,
$\sigma_0=\arg\Gamma(1+i\eta)$.

II. SPIN-ORBIT INTERACTION

The understanding of the effects is helped by a consideration of the relationship of the problem to that of the ordinary spin-orbit interaction between two charged particles.[5,6] For Diracian particles the interaction energy, discarding the relatively small spin-spin interaction, may be expressed as

$$H'=-\left(\frac{\hbar e}{2Mc}\right)\frac{e}{c}\left\{\frac{1}{2}\left[\mathcal{E}_1\times\frac{\mathbf{p}_1}{M}\right]\cdot\boldsymbol{\sigma}_1+\frac{1}{2}\left[\mathcal{E}_2\times\frac{\mathbf{p}_2}{M}\right]\cdot\boldsymbol{\sigma}_2 - e\left[\frac{\mathbf{r}_1-\mathbf{r}_2}{r^3}\times\frac{\mathbf{p}_2}{M}\right]\cdot\boldsymbol{\sigma}_1-e\left[\frac{\mathbf{r}_2-\mathbf{r}_1}{r^3}\times\frac{\mathbf{p}_1}{M}\right]\cdot\boldsymbol{\sigma}_2\right\}. \quad (1)$$

Here $\mathbf{r}_1$ and $\mathbf{r}_2$ are the displacement vectors to the positions of the charges from the origin; e is the charge on each particle; $\mathbf{p}_1$ and $\mathbf{p}_2$ are the momenta; $r=|\mathbf{r}_1-\mathbf{r}_2|$ is the distance between the charges; $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are Pauli's spin matrix vectors; M is the mass of each particle; $\mathcal{E}_1$ and $\mathcal{E}_2$ are the electric fields at particles 1 and 2, respectively. The first two terms in Eq. (1) contribute each the negative of the Thomas term to the Hamiltonian. Their form applies whether $\mathcal{E}_1$ and $\mathcal{E}_2$ have their origin exclusively in the field caused by the two particles or not. They may be thought of as arising through the combined effect of the energy of the electric dipole caused by the motion of the magnetic moments in the electric field and of the Thomas term. The electric dipole effect is -2 times the Thomas term. The third and fourth terms represent the interactions of the spin magnetic moments with the magnetic fields which are produced by the motion of the charge of each particle. If the electric fields are caused entirely by the two charges,

$$\mathcal{E}_1=e(\mathbf{r}_1-\mathbf{r}_2)/r^3,\quad \mathcal{E}_2=e(\mathbf{r}_2-\mathbf{r}_1)/r^3, \quad (1.1)$$

then

$$H'=(\mu_0^2/\hbar r^3)\{[(\mathbf{r}_1-\mathbf{r}_2)\times(2\mathbf{p}_2-\mathbf{p}_1)]\cdot\boldsymbol{\sigma}_1+[(\mathbf{r}_2-\mathbf{r}_1)\times(2\mathbf{p}_1-\mathbf{p}_2)]\cdot\boldsymbol{\sigma}_2\}, \quad (1.2)$$

$$\mu_0=e\hbar/(2Mc). \quad (1.3)$$

It will be noted that in the combination $2\mathbf{p}_2-\mathbf{p}_1$ the $2\mathbf{p}_2$ arises from the magnetic effect while $-\mathbf{p}_1$ comes about partly through the dipole and partly through the Thomas acceleration effects. This association of terms will be helpful later on in obtaining the anomalous moment effects.

Since the two particles are now supposed to interact only with each other,

$$(\mathbf{p}_1+\mathbf{p}_2)\psi=0, \quad (1.4)$$

and hence from (1.2),

$$H'\psi=-(3\mu_0^2/\hbar r^3)[(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{p}_1]\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\psi. \quad (2)$$

Employing the method of Coulomb phase shifts[2] and the consideration of Sec. 4 of the above-mentioned paper, one justifies the inclusion of the effect of H' in terms of its matrix element in momentum space. A brief calculation gives then, on including the effects of the e^2/r in the potential energy, the combination

$$\frac{e^2}{k^2}\left\{1-\frac{3\mu_0^2}{e^2 i}[\mathbf{k}'\times\mathbf{k}'']\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\right\} \quad (2.1)$$

in the expression for the momentum space matrix element. Here $\mathbf{k}'$ and $\mathbf{k}''$ are respectively the final and initial values of $\mathbf{p}/\hbar$, while $\mathbf{p}$ may be identified as $\mathbf{p}_1$; $k=|\mathbf{k}'|=|\mathbf{k}''|=Mv/2\hbar$, the formula being meant for the center-of-mass system. Corrections to the main term e^2/k^2 are not discussed here since they are available elsewhere.[2–4] The effect of the spin-orbit term in (2.1) is to add to S^c, the scattering matrix of the Coulomb field,[7] a correction term, so that it becomes

$$S^{c'}=-\frac{\eta}{2k\mathbf{s}^2}\exp[i(\Phi-\eta\ln\mathbf{s}^2)]\times\left\{1+3i\frac{\hbar^2}{4M^2c^2}[\mathbf{k}_f\times\mathbf{k}_i]\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\right\}, \quad (2.2)$$

where

$$\Phi=\rho-\eta\ln 2\rho+\sigma_0. \quad (2.3)$$

The replacements $\mathbf{k}'=\mathbf{k}_f$, $\mathbf{k}''=\mathbf{k}_i$ have been made here so as to indicate more clearly initial and final states, and the abbreviation

$$\mathbf{s}=\sin(\theta/2) \quad (2.4)$$

is used. For low energies the spin-orbit term of (2.2) is the same as the effect of $\Delta'\alpha_1=\Delta'\alpha_4$ of reference 2 and it is essentially the same as the Dirac part of the spin-orbit correction in.[1] That Eq. (18.2) of reference 2 represents a spin-orbit interaction is clear from the fact that it arose from J_5 of that reference which shows directly its spin-orbit origin in Eqs. (12.8) and (18).

The connection with the formulas of previous work having been described, the effect of the anomalous part of the proton magnetic moment will be considered, making use of the interpretation of effects in (1.2)

[5] W. Heisenberg, Z. Physik **39**, 499 (1926).
[6] G. Breit, Phys. Rev. **34**, 553 (1929); **36**, 383 (1930); **39**, 616 (1932).
[7] G. Breit and M. H. Hull, Jr., Phys. Rev. **97**, 1047 (1955).

which has been mentioned immediately after that equation. The anomalous part of the proton's moment will be written as

$$(e\hbar/2Mc)\mu_a, \tag{3}$$

so that μ_a is the anomalous moment expressed in units of the nuclear Bohr magneton. Neglecting relativistic corrections, the μ_a part of the moment is acted on by the magnetic field of the other particle in the same way as the Diracian part. The motion of the proton produces an electric doublet associated with μ_a also in the same way, this effect being a purely kinematical one. There is no additional Thomas term effect to take into account, however, because this effect is already included in (1). Since in Eq. (1.2) the combination $2\mathbf{p}_2-\mathbf{p}_1$ contains the part $-\mathbf{p}_1$ which was $-2\mathbf{p}_1$ before the Thomas correction reduced it to $-\mathbf{p}_1$, the anomalous moment contributes

$$H''=2\mu_a(\mu_0^2/\hbar r^3)\{[(\mathbf{r}_1-\mathbf{r}_2)\times(\mathbf{p}_2-\mathbf{p}_1)]\cdot\boldsymbol{\sigma}_1 +[(\mathbf{r}_2-\mathbf{r}_1)\times(\mathbf{p}_1-\mathbf{p}_2)]\cdot\boldsymbol{\sigma}_2\}. \tag{3.1}$$

Consequently

$$H''\psi=-4\mu_a(\mu_0^2/\hbar r^3)[(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{p}_1]\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\psi. \tag{3.2}$$

Comparing with Eq. (2), it is apparent that the anomalous moment contributes

$$4\mu_a/3 \tag{3.3}$$

times as much as the Dirac moment and that the origin of the factor $4/3=(2+2)/(2+1)$ is the absence of the Thomas correction for the anomalous moment part. Garren's results[1,4] are in agreement with the above in the low-energy limit. As a matter of completeness, the Coulomb part of the scattering matrix including the effect of H'' may be written down:

$$S^{c''}=-\frac{\eta}{2k\mathbf{s}^2}\exp[i(\Phi-\eta\ln\mathbf{s}^2)] \times\left\{1+(3+4\mu_a)i\frac{\hbar^2}{4M^2c^2}[\mathbf{k}_f\times\mathbf{k}_i]\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\right\}. \tag{3.4}$$

The spin-orbit terms in this formula are the only significant ones giving direct effects on the polarization for high-energy scattering. The spin-spin terms are much smaller and the tensor-like terms described in references 2–4 cannot give rise to polarization directly although some modifications of the polarization effects caused by the purely nuclear phase shifts are, of course, possible. Since one of the primary points of interest in connection with relativistic corrections to the Coulomb wave is the effect of the spin-orbit terms on polarization, the question of their reliability is a natural one. In the present note the v^2/c^2 approximation is used throughout, all the formulas becoming simplified as a result.

Spin-orbit effects are well understood in atomic spectra. As is well known they have to be calculated by employing the actual wave functions rather than free-particle approximations. One may suspect therefore that in p-p scattering, insufficiently accurate results will be obtained by neglecting the distortion of the wave function by nuclear forces of nonelectromagnetic origin. The fact that Coulomb scattering is important only for small-angle deflections does not settle the question, because the spin-orbit effects do not give the major part of Coulomb scattering so that a relatively small error in Coulomb scattering as a whole can conceivably amount to a non-negligible effect on the spin-orbit energy. It becomes necessary therefore to examine the effect of wave function distortion on the relativistic corrections.

III. WAVE FUNCTION DISTORTION

The general viewpoint will be the same as that used previously.[2] The particles are viewed in the center-of-mass system. The electromagnetic effects are considered as a perturbation and the Coulomb phase shifts are used to the first nonvanishing order in the electromagnetic interaction. Knowledge of phase shifts enables the construction of the wave function outside the range of nuclear forces. As has been shown,[2] the results of such a wave construction agree in the nonrelativistic limit, and to first order in e^2 with those of the standard method for nonrelativistic problems. In the latter, one starts with the Coulomb wave and calculates the modification caused by phase shifts superposed on the phases of the Coulomb radial wave functions. The procedure of starting with the wave modified by nuclear interactions is a natural one because the distortion of the wave function caused by these effects is larger than that caused by the Coulomb field. Experience[8] indicates that even at low energies, in calculations of the $^1S_0$ effects, the Coulomb effects inside potential wells are quite well represented by first order approximations. At low energies (i.e., at $\sim$7 Mev) it would be a poor approximation to use first order formulas outside the potential wells, but the parameter η is in this case much larger than at energies above 100 Mev. The very small values of η make first-order formulas for $\Gamma(L+1+i\eta)$ and related quantities reasonably accurate and the proposed method[2] of calculation is thus satisfactory. It is also relevant that the spin-orbit effects can be considered for their own sake independently of the order in which the e^2/r and the nuclear potentials are taken into account. In either case it is the phase shift introduced by the spin-orbit interaction that matters and the distortion of the wave function caused by the nuclear potential cannot be neglected in estimates of this phase shift.

Before considering the reliability of estimates of spin-orbit effects with neglect of wave function distortions, the analogous problem will be considered for the main part of Coulomb scattering. The usual non-

[8] Breit, Condon, and Present, Phys. Rev. **50**, 825 (1936); Breit, Thaxton, and Eisenbud, Phys. Rev. **55**, 1018 (1939).

relativistic expression for the Coulomb-scattered wave is

$$\psi_{sc}{}^c \sim -\frac{\eta}{k(r-z)} \exp\{i[kr-\eta \ln(2\rho \mathbf{s}^2)+2\sigma_0]\}$$
$$= -\frac{\eta}{2\mathbf{s}^2} \exp(-i\eta \ln \mathbf{s}^2)\frac{e^{i\Phi}}{kr}. \quad (4)$$

The angle-dependent factor in this expression may be expressed as

$$-\frac{\eta}{2\mathbf{s}^2} \exp(-i\eta \ln \mathbf{s}^2)$$
$$= \sum_L (2L+1) P_L(\cos\theta) \frac{[e^{2i(\sigma_L-\sigma_0)}-1]}{(2i)}. \quad (4.1)$$

This formula is essentially the same as Eq. (21) of reference 2 with the difference that

$$(1/i)\delta(1-\cos\theta) = \sum_L (2L+1) P_L(\cos\theta)/(2i) \quad (4.2)$$

has been subtracted from the right side. Here $\delta(1-\mu)$ is used in the convention of giving unity rather than $\frac{1}{2}$ on integration over μ in the limits from -1 to $+1$. The subtraction does not affect the value at $\theta \neq 0$ and is permissible therefore. It is convenient to make the subtraction because the right-hand side is made thereby to contain the factor η if the exponentials are expanded in Taylor series. This expansion gives

$$-\frac{\eta}{2\mathbf{s}^2} \exp(-i\eta \ln \mathbf{s}^2)$$
$$\cong \eta[3P_1+5(1+\tfrac{1}{2})P_2+7(1+\tfrac{1}{2}+\tfrac{1}{3})P_3+\cdots]. \quad (4.3)$$

The approximation to individual terms in the series breaks down when $2\eta(1+\frac{1}{2}+\cdots+1/n) \cong 2\eta \ln n \cong 1$. For $E=150$ Mev the value of L at which this occurs is $L \sim 10^{17}$. The relative contributions of the P_L to the series may thus be estimated by means of (4.3). If $E=150$ Mev then $\eta=0.0129$ and for $\theta=10°$, $\eta \ln \mathbf{s}^2 = -0.063$, so that $\exp(-i\eta \ln \mathbf{s}^2) \cong 1$, $1/(2\mathbf{s}^2) \cong 65$, while $3P_1+(15/2)P_2+7\times 1.83P_3 \cong 23$. The sum of the first three terms is about 30% of the whole and the sum of the first two terms is about half this amount. The main nuclear force effects on the wave function are probably confined to $L \leq 3$ at this energy. Since η occurs on both sides of (4.3), the estimate is affected by going to higher energies mainly because of larger distortions at higher L. For $\theta=20°$, $1/(2\mathbf{s}^2)=16.6$. The successive contributions to the square bracket in (4.3) are $2.8+6.2+8.5+\cdots$. For the larger L the P_L change sign, making the relatively small value of the sum possible. For $\theta=20°$ the contribution of $L=1$ and 2 amount to $\sim$50% of the whole. An appreciable distortion effect thus has a good opportunity of affecting the Coulomb scattering amplitude.

In the nonrelativistic approximation these distortion effects are taken into account by the usual procedure of calculating phase shifts which have to be superposed on the Coulomb phase to obtain the complete asymptotic phase. It clearly does not matter whether the latter phase is obtained by starting with the nuclear or the purely Coulombian phase shifts. The calculation of the relativistic corrections to Coulomb scattering is, on the other hand, affected by the nuclear forces in an additional way. That this should be the case is clear intuitively from the fact that if the nuclear forces increase the relative velocity of the two protons, the relativistic corrections should be larger. In the calculations this effect enters through a change in the factors connecting χ_{I}, χ_{II}, and φ with Ψ in the derivation[2] of the relativistic effects. Since the calculation of the value of the relativistic η brings in the successive L in about the same proportions as obtain for the nonrelativistic effect, the value of the relativistic correction to η is seen to be somewhat uncertain. The effect of wave function distortion on this correction depends on the assumed nuclear potential.

For the spin-orbit terms, the interaction has a shorter range and closer collisions are therefore relatively more important. The effect of wave distortion is accordingly more serious. Both terms in (3.4) correspond according to (2) to an interaction energy which is a multiple of

$$(\mathbf{L}\cdot\mathbf{s})/r^3, \quad \mathbf{L}=[(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{p}_1], \quad (5)$$

so that $\mathbf{L}$ is the orbital angular momentum. The values of $(\mathbf{L}\cdot\mathbf{s})$ for the triplet system are $(L, -1, -L-1)$. On the other hand, for non-Coulombian functions one has[9]

$$\int_0^\infty (F_L{}^2/\rho^3)d\rho = 1/[2L(L+1)]. \quad (5.1)$$

The contributions of the phase shifts for different L to the scattered amplitude are therefore as in the series

$$\sum_L (2L+1) P_L(\mu)(L, -1, -L-1)/[L(L+1)], \quad (5.2)$$
$$\mu=\cos\theta. \quad (5.3)$$

In the three cases the coefficients of P_L have the form $2-1/(L+1)$, $(1/L)+1/(L+1)$, $-2-1/L$. The least rapid decrease of the contributions corresponds to the 2 and -2 in these coefficients. In these contributions the smaller L may be expected to be emphasized the least, and the correction for spin-orbit interaction arising from such terms is likely to be the least sensitive to wave function distortion. These contributions can be discussed therefore by considering

$$\sum_{L=1}^{\infty} P_L(\mu) = (2-2\mu)^{-\frac{1}{2}}-1 = \left(\frac{1}{2\mathbf{s}}\right)-1. \quad (5.4)$$

[9] Gluckstern, Lazarus, and Breit, Phys. Rev. **101**, 175 (1956). Equation (6Q) of this reference gives the desired integral.

The term in $L=0$ is not included because for $L=0$ there is no spin-orbit interaction, but its inclusion does not make much difference in the present qualitative considerations.

For $\theta=10°$, $[1/(2\mathbf{s})]-1=4.7$ while $P_1+P_2=1.94$. Thus the values $L=1$ and 2 contribute 0.41 of the whole sum. For $\theta=20°$, $[1/(2\mathbf{s})]-1=1.88$ while $P_1+P_2=1.76$. If there is sufficient wave function distortion to affect the value of $\int_0^\infty F_L^2 d\rho/\rho^3$, the spin-orbit effects will be in error by roughly the same percentage as the integral. The magnitude of the spin-orbit effect arising from the Coulomb wave is therefore questionable whenever the first few values of L are subject to appreciable wave distortion.

The degree to which this is the case depends on the energy. As the energy increases, the shorter wavelength makes the first maximum of F_L move toward shorter distances and higher values of L become affected by nuclear distortion. At $E=150$ Mev the first maxima of F_L fall at $r=2.0\times10^{-13}$ cm for $L=1$, at $r=2.9\times10^{-13}$ cm for $L=2$, and at 3.7×10^{-13} cm for $L=3$. For the lower L the first maximum of F_L is seen to be well within the range of nuclear forces and the wave function distortion effects may be expected to be appreciable. There is a chance that some compensation of effects within the range of nuclear forces by effects outside will take place, but no general reason for expecting such a compensation. Examples to the contrary can be found and wave function distortion is well known to matter for spin-orbit interaction in optical spectra.

The presence of a phase shift indicates wave function distortion. Some idea of the magnitude of the effects involved can be obtained therefore from present indications regarding the phase shifts present. The analysis of experimental data is not sufficiently definite to make a clear-cut assignment possible. It appears probable[10] from polarization studies, however, that there exist ^3F phase shifts at 280 Mev of the order of 15° and the fits obtained by Hull, Ehrman, Hatcher, and Durand[11] indicate similar conditions. Some of the P-wave phase shifts obtained by them are appreciably larger than 15°. The first maximum of F_L^2 is displaced by the phase shift. A crude estimate of the magnitude of the effects dealt with may be obtained from this displacement, which can be approximated by

$$\epsilon_L \cong -\delta_L \Big/ \left\{\left[1-\frac{L(L+1)}{\rho_0^2}\right]F_L^2(\rho_0)\right\}.$$

Here ρ_0 is the value of ρ at the maximum, δ_L is the phase shift, and ϵ_L the displacement in the position of the maximum. For $L=1$ this approximate equation gives $\epsilon_1=-\delta_1/0.83$, so that for $\delta_1=30°$ the first maximum is displaced from 2.75 to $2.75-0.63=2.12$. The value of ρ^3 at the maximum changes by a factor 0.46 and ρ^2 by 0.60. Such a change indicates an unreliability in the value of the radial integral for spin-orbit interaction of perhaps 40%. The δ_1 used is on the large side and there may be compensating effects in the interaction region between nucleons. The error may be somewhat smaller therefore, say 20%, but there appears to be no general reason for expecting the compensation.

The employment of the complete interaction for the anomalous proton magnetic moment includes an effect of the interaction of this moment with div$\mathcal{E}$, where $\mathcal{E}$ is the electric intensity at the proton. This effect is contained in the diagonal terms of matrix L of Garren's second paper.[4] Its diagonal elements do not vanish at low energies and give an angle-independent contribution to the scattering matrix. This effect cannot be reliably estimated without taking into account wave function distortion because of the entrance of $\psi^2(0)$, the square of ψ for $r=0$, in the expectation value of div$\mathcal{E}$. An extreme case of the effect of wave distortion occurs for a hard-core interaction which makes $\psi^2(0)=0$. The value of $\psi^2(0)$ is also affected by the Coulomb field and especially so at low energies.

The fact that Garren's result contains a dominant s-wave effect at low energies has been pointed out to the writer by Mr. Loyal Durand, III, who has also pointed out that in addition to Garren's L his matrix Y contains related terms.[12] The nature of the latter is readily understood since Y enters the scattering matrix with a coefficient μ_a^2. They arise from the interaction of the two anomalous moments with each other. An interaction of this type is well known to given an energy proportional to $\psi^2(0)$, being very similar to that occurring in the theory of hyperfine structure of atomic energy levels.[13] Wave function distortion caused by nuclear forces affects this interaction as well.

Garren's matrix Σ contains a related angle-independent effect on the cross section. Its presence is seen in the trace $\Sigma_{11}{}^1+\Sigma_{00}{}^1+\Sigma_{-1-1}{}^1$, which at low energies becomes $3-16(\epsilon-1)\mathbf{s}^2$, with ϵMc^2 standing for the energy of each proton in the center-of-mass system. The term in $\mathbf{s}^2$ becomes multiplied by $\eta/(2k)$, and since η is proportional to $1/k$ and ϵ to k^2 the contribution of $(\epsilon-1)\mathbf{s}^2$ approaches a constant at low energies. This term is analogous to the $\psi^2(0)$ effect in hyperfine structure and is subject to wave function distortion corrections just as the other two terms. The same applies to the analogous terms of the μ_a-independent

[10] B. D. Fried, Phys. Rev. **95**, 851 (1954); Breit, Ehrman, Saperstein, and Hull, Phys. Rev. **96**, 807 (1954).

[11] Hull, Ehrman, Hatcher, and Durand, Phys. Rev. **103**, 1047 (1952); H. P. Stapp, University of California Radiation Laboratory Report UCRL-3098 (unpublished).

[12] The writer is very grateful to Mr. Durand for having drawn his attention to the features of Garren's results which have been just mentioned.

[13] G. Breit and F. W. Doermann, Phys. Rev. **36**, 1732 (1930). Terms of the type under discussion have been considered in that paper in the discussion of the difference in the interaction of an intrinsic magnetic moment as contrasted with the Diracian moment of the electron when these moments interact with the nuclear magnetic moment. A factor $-\frac{1}{2}$ arises in their comparison. A less formal consideration of this factor is found on pp. 159, 160 of G. Breit, Phys. Rev. **53**, 153 (1938).

contributions of the paper by the writer[2] as well as that of Ebel and Hull.[3] A term of this type appeared[14] in a phenomenologic introduction of the electron's anomalous moment where it entered as an interaction of the electron's moment with the proton. Independently Foldy[15] used this type of contact interaction to explain the major part of the electron-neutron interaction, and the relation of the two ways of calculating the effect has also been discussed.[16] Referring to the latter presentation, the low-energy limit, after the elimination of "small" components, in the absence of an external magnetic field gives a contribution to the Hamiltonian of the form

$$[\mu/(2Mc)]\{-\hbar \operatorname{div}\mathcal{E}+[\mathbf{p}\times\mathcal{E}]\cdot\boldsymbol{\sigma}-[\mathcal{E}\times\mathbf{p}]\cdot\boldsymbol{\sigma}\}, \quad (6)$$

where μ is the anomalous part of the magnetic moment in cgs units. This interaction energy is seen to include the interaction of the electric field with the electric dipole produced by the motion of the magnetic moment in the same order of the calculation as the term in $\operatorname{div}\mathcal{E}$. The inclusion of the latter term employing undistorted wave functions is obviously unreliable at high energies because of the strong short-range interactions in S states. Thus a hard core can keep the protons from making contact and can eliminate the contact-type interaction if the latter is taken literally. At low bombarding energies the effect of the Coulomb field on the probability of the protons making contact needs to be considered also unless a hard core suppresses the effect. The fact that the hard core can produce a serious effect on this term and its related sensitivity to the potential in absence of a hard core may conceivably make the term of interest in comparisons of p-p and n-p data from the viewpoint of charge independence.

IV. CONCLUDING REMARKS

It is thus seen that the relativistic corrections to p-p scattering made on the basis of corrections to the Coulomb wave as though the Coulomb wave were not affected by the nuclear interactions are of questionable accuracy. It is true that at small angles the Coulomb wave dominates the scattering and that in classical analogy most of the collisions are distant ones. But the Coulombian spin-orbit interaction originates in appreciably closer collisions than the main part of the Coulomb-scattered wave and the contributions of the smaller L form an appreciable fraction of the whole effect. The contributions of different L are not all of the same sign, the Legendre functions for small L and θ being nearly 1 but changing sign if L is large enough. This circumstance contributes to the relatively large importance of the first few L. The alternation of signs is present also for the spin-independent part of the scattered amplitude and the contribution of the first few L was seen not to be negligible. While the relativistic corrections to the undistorted Coulomb wave give an approximation to the desired effects for small scattering angles such as 10° at 150 Mev, the present considerations do not exclude errors such as 20 or 40% in these estimates. For an assumed nucleon-nucleon interaction potential, a treatment substituting the wave function of that interaction potential for the free-particle wave function of previous work[2,3] would provide the answer. In principle such a calculation has to be made in terms of states with definite J and, if there are two participating L, in terms of the two eigenstates for these J. Since the distortion effects are largest for the small L, a practical arrangement would be to correct only the first few L for the distortion effect, thus making it feasible to have the relativistic corrections appear as a sum of the correction for undistorted waves available now and an additional term representing the difference between employing the distorted and undistorted wave functions. With such an arrangement there is no difficulty in summing over the larger L and the problem is largely a computational one.

The procedure just described is "arrangement (a)" of page 1591 of the paper quoted.[2] The existence of the errors discussed in the present note was brought out there and the scattering matrix of that paper was presented as an approximation corresponding to replacing distorted by undistorted wave functions. The Coulomb interference effects obtained[4] from formulas of this type may not be regarded as certain, however. It is also not probable that obvious relativistic covariance at this stage of the calculation is helpful since it properly belongs to the stage in the theory in which nuclear forces are derived.

If the potential energy description is inadequate, the pion field and other fields responsible for the nucleon-nucleon forces have to be brought into the consideration. On account of the likelihood that the potential energy between a pair of nucleons is not a wholly applicable concept, the relativistic corrections may not be sufficiently completely calculable for data analysis without a thorough field-theoretic consideration.

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APPENDIX

A relatively large importance of $(\mathbf{L}\cdot\mathbf{s})$ terms as compared with the tensor force regarding their effect on polarization has been referred to above. Since this matter does not appear to have been clearly brought out in the literature, the following consideration is appended. It is of interest for other aspects of nucleon-nucleon scattering as well. The Hamiltonian with

[14] G. Breit, Phys. Rev. **72**, 984 (1948); **73**, 1410 (1948); **74**, 656 (1948).
[15] L. Foldy, Phys. Rev. **83**, 688 (1951).
[16] G. Breit, Proc. Natl. Acad. Sci. U. S. **37**, 837 (1951).

central forces alone is called H_0. This Hamiltonian is allowed to be quite general except for spherical symmetry and freedom from entrance of the particle spins. The central potential may, e.g., be different for every L. This generality, it will be observed, allows the construction of a Green's function since the latter may be formed as

$$\sum_{L,m} Y_{L,m}(\mathbf{1}')^* Y_{L,m}(\mathbf{1}) G_L(r,r') = G(\mathbf{r},\mathbf{r}').$$

Here the vectors $\mathbf{1}$ and $\mathbf{1}'$ are abbreviations for the polar angles of $\mathbf{r}$ and $\mathbf{r}'$, the $Y_{L,m}$ are normalized spherical harmonics, and the G_L are suitable radial equation Green's functions. The tensor force is supposed to enter through a change in the Hamiltonian H_0 to

$$H = H_0 + H', \quad H' = \lambda(r) S_{12}, \quad S_{12} = 3(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})/r^2 - (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2). \tag{A1}$$

The wave function

$$\psi = \psi^0 + \psi^1 + \cdots,$$

so that

$$(H_0 - E)\psi^1 = -\lambda(r) S_{12}\psi^0.$$

One has

$$\psi^1 = -\int G(\mathbf{r},\mathbf{r}')\lambda(r') S_{12}(\mathbf{r}')\psi^0(\mathbf{r}')d\mathbf{r}'. \tag{A2}$$

The expectation value of the spin component in the triplet state caused by S_{12} is

$$\langle s_z \rangle = (\psi^1, s_z\psi^0) + (\psi^0, s_z\psi^1),$$

the inner product applying here to spin coordinates only and s_z standing for $(\sigma_{1z}+\sigma_{2z})/2$. The above expectation value is

$$\langle s_z \rangle = -\int G(\mathbf{r},\mathbf{r}')\lambda(r')\{(s_z S_{12}(\mathbf{r}')\psi^0(\mathbf{r}'),\psi^0(\mathbf{r})) + (\psi^0(\mathbf{r}), s_z S_{12}(\mathbf{r}')\psi^0(\mathbf{r}'))\}d\mathbf{r}'.$$

For a statistical mixture of states

$$\psi_j^0 = \varphi\chi_j,$$

with equal probabilities for each triplet spin function χ_j, the mean over the three j is next obtained from the above s_z. Under the integral one deals therefore with

$$\Sigma_j[(s_z S_{12}(\mathbf{r}')\psi_j^0(\mathbf{r}'),\psi_j^0(\mathbf{r}')) + (\psi_j^0(\mathbf{r}), s_z S_{12}(\mathbf{r}')\psi_j^0(\mathbf{r}'))],$$

so that one is concerned with

$$\mathrm{Tr}\{s_z S_{12}(\mathbf{r}')\} = \mathrm{Tr}\{S_{12}(\mathbf{r}')s_z\}. \tag{A3}$$

Here

$$2\,\mathrm{Tr}\{\sigma_{1z}(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})\} = \mathrm{Tr}[\sigma_{1z},(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})]_+ = \mathrm{Tr}\{2z(\boldsymbol{\sigma}_2\cdot\mathbf{r})\}.$$

Similarly

$$2\,\mathrm{Tr}\sigma_{1z}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) = \mathrm{Tr}[\sigma_{1z},(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)] = 2\,\mathrm{Tr}\sigma_z,$$

and hence

$$\mathrm{Tr}\{s_z S_{12}\} = \mathrm{Tr}\{3z(\mathbf{s}\cdot\mathbf{r}) - s_z r^2\} = 0.$$

The first-order effect of S_{12} on the polarization vanishes therefore. The proof applies independently of the origin of the tensor force.

On the other hand, an addition to the Hamiltonian of the form

$$H'' = \lambda(r)(\mathbf{L}\cdot\mathbf{s})$$

causes the appearance in place of (A3) of the quantity

$$\mathrm{Tr}\{s_z(\mathbf{L}\cdot\mathbf{s})\} = \tfrac{1}{2}\,\mathrm{Tr}\{L_z\}, \tag{A4}$$

which does not vanish. Strong polarization effects are thus more readily accounted for by $(\mathbf{L}\cdot\mathbf{s})$ than by S_{12} forces.

It may be noted that the absence of first-order effects of S_{12} in the polarization may be generalized to a modified S_{12} obtained from the usual one by the replacement of $\mathbf{r}/r$ by $\mathbf{p}/p$. The only essential condition is the structure of S_{12} in terms of $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$.

In isotopic spin space a factor $a+b(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$ may be included with $\lambda(r)$ without affecting the conclusion, such a factor behaving like a constant in all of the operations. Thus $\lambda(r)$ may have one value for ^3P, ^3F, ^3H, $\cdots$ and another for ^3S, ^3D, ^3G, $\cdots$ as is clear from the fact that S_{12} does not couple the two sets of states to each other.

The vanishing of first-order effects of S_{12} is essentially implied by Wolfenstein's analysis of nucleon-nucleon scattering.[17]

[17] L. Wolfenstein, Bull. Am. Phys. Soc. Ser. II, **1**, 284 (1956). The considerations of Wolfenstein [Phys. Rev. **76**, 541 (1949); **82**, 308 (1951)] also come very close to implying the same relation. Wolfenstein's presentation is put in terms of different orders of the Born approximation which is often understood in the sense of first order effects for the whole nucleon potential. In this sense the earlier work of Wolfenstein does not prove the point under discussion. The relations used by him apply, however, to the first order effects of the tensor force to a central nuclear potential. Thus essentially even the earlier work of Wolfenstein has implied the relation referred to here. An independent verification supplementary to Wolfenstein's work and the consideration in the text above has been given by M. S. Wertheim in a part of his Yale dissertation. In this verification first-order effects of S_{12} on the phase shifts are worked out including the case of coupling of states of the same J and different L and the effect on the polarization is calculated by means of known formulas employing Goldfarb-Feldman symbols [L. J. B. Goldfarb and D. Feldman, Phys. Rev. **88**, 1099 (1952)].