PHYSICAL REVIEW VOLUME 106, NUMBER 1 APRIL 1, 1957

Rotation of Liquid Helium II

S. M. Bhagat and R. K. Pathria
Department of Physics, University of Delhi, Delhi, India

(Received October 16, 1956; revised manuscript received January 15, 1957)

Experiments concerned with the measurement of viscous resistance in rotating liquid helium II have been analyzed on the basis of the stratification model for the superfluid motion. The onset of nonlinear dissipation is attributed to the setting in of large-scale vorticity in the superfluid and it is proposed that this would happen at a certain "critical" velocity corresponding to which the thickness of the layers becomes of the order of the "correlation length." The magnitude of this characteristic length turns out to be about 8×10^{-5} cm. Further, the concept of stratified motion has also been applied to linear flow through the supersurface film, leading thereby to a satisfactory explanation of the observations of Chandrasekhar and Mendelssohn.

I. INTRODUCTION

It is well known[1] that liquid helium II, that is, liquid helium below the lambda temperature, exhibits a peculiar type of superfluid flow. Also, it has been found that as the velocity of flow exceeds a certain critical value the flow appears to become highly dissipative. Whereas the behavior of liquid helium II in linear motion has been investigated in great detail both theoretically and experimentally, it is only recently that attention has been drawn to the behavior of the liquid when in rotation. Experimental studies by Hollis-Hallett and his collaborators[2] (and more recently by Kolm and Herlin[3]) show that the rotating liquid exhibits properties which indicate the onset of nonlinear dissipative forces at certain well-defined angular velocities. However, no consistent theoretical study seems to have emerged so far to account for these results.

At first sight, one may be tempted to suggest that these effects may be due to the frictional forces between the normal and the superfluid components or due to the onset of turbulence in the normal (viscous) part of the liquid helium II. However, firstly, as has also been pointed out earlier by Atkins,[4] in a case of steady rotation, the contribution due to the mutual frictional forces obviously vanishes; and secondly, the anomalies appear at such small velocities that the Reynolds number corresponding to the viscosity of the normal component is not high enough to account for the setting in of turbulence (see also the detailed discussion in Sec. II). In this context, we feel that these abnormalities in the motion may be explained if one considers the setting up of vorticity in the superfluid component. Such a suggestion also seems to have been recently made by Feynman.[5] Consequently, in the present paper we have developed a model for the *rotation* of the superfluid along the lines of the stratification model proposed by London[6] combined with the concept of a correlation distance[7]—a distance within which particles exhibit strong momentum correlations. It turns out that the appearance of the observed abnormalities is closely associated with the development of a "macroscopic" turbulence field in the superfluid. It is interesting to note that the model, though essentially crude, provides a fairly clear picture of the role of the two components in liquid helium II, especially with respect to the experiments under consideration. Also, it provides a more or less direct method for fixing the magnitude of the correlation length, a concept of great significance.

At this stage one may point out that in our considerations two types of critical velocities seem to appear. The first marks the beginning of circulation in the superfluid (see London[6]) and does not seem to be of much practical importance since it has an exceedingly small value. The second, on the other hand, is the velocity at which dissipative effects may be expected to appear and is hence of importance when comparison with experiment is desired. The second velocity, it is proposed, would correspond to the situation in which the thickness of the layers (which progressively reduces as the speed of rotation is increased) becomes of the order of the correlation length insofar as the layer boundaries, that is, the vortex sheets, come so close to each other that the system acquires a macroscopic turbulence field.

In the following sections, we have analyzed the experiments of Hollis-Hallett and of Kolm and Herlin on the basis of these concepts. Further, in Sec. IV we have tried to apply the concept of the layer structure to the flow of liquid helium through the supersurface film. In this case we find that the rate of transfer would become nearly constant as the heat input exceeds a certain value. These results, at least qualitatively, are in excellent agreement with the experiments of Chandrasekhar and Mendelssohn.[8]

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[2] A. C. Hollis-Hallett, Proc. Cambridge Phil. Soc. **49**, 717 (1953); W. J. Heikkila and A. C. Hollis-Hallett, Can. J. Phys. **33**, 420 (1955).
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[4] K. R. Atkins, in *Advances in Physics* (Taylor and Francis, Ltd., London, 1952), Vol. 1, p. 169.
[5] R. P. Feynman, in *Progress in Low Temperature Physics* (North-Holland Publishing Company, Amsterdam, 1955), Vol. 1, Chap. II.
[6] F. London, *Superfluids* (John Wiley and Sons, Inc., New York, 1954), Vol. 2, Sec. 23.
[7] Blatt, Butler, and Schafroth, Phys. Rev. **100**, 481 (1955).
[8] B. S. Chandrasekhar and K. Mendelssohn, Proc. Phys. Soc. (London) **A64**, 512 (1951).

II. ONSET OF NONLINEAR DISSIPATION (EXPERIMENTAL)

In their experiments with the rotating cylinder viscometer, Heikkila and Hollis-Hallett[2] observed that the viscosity of liquid helium II determined from the measurements of the torque, imparted to the inner nonrotating cylinder because of the uniform rotation of the outer one, remained constant up to a certain value v_c of the velocity of rotation of the latter. Beyond this critical value, the coefficient of viscosity was observed to increase with velocity, that is, the torque departed from its usual linear dependence on velocity, marking thereby the appearance of some sort of nonlinear effects. In the experiment of Kolm and Herlin,[3] on the other hand, the outer cylinder was kept fixed and the deceleration of the inner cylinder, coasting freely, was observed. In this case too it was found that the deceleration of the rotor exhibited an abrupt change at a certain critical speed of rotation. So once again it appears that some new dissipative forces come into operation at a well-defined velocity of the rotating wall.

First of all, one should note, as has been emphasized by Hollis-Hallett (1953), that the presence of the mutual frictional force between the normal and the superfluid components of liquid helium II is unable to explain the observed dependence of the torque on the speed of rotation. The reason is simple—under the steady conditions of the experiments the mutual friction terms disappear from the equations of motion of the two components,[4] and one again expects the torque to vary linearly with velocity, a conclusion which is not upheld by observation.

The second possibility, which at first sight seems to be somewhat plausible, is the setting in of turbulence in the normal component. Taylor[9] has studied in detail, both mathematically and experimentally, the problem of the stability of a viscous liquid contained in concentric rotating cylinders. He concluded that against disturbances symmetrical about the axis and periodic along it, the flow was stable at all speeds of rotation when the inner cylinder was at rest and the outer was uniformly rotated (hereafter referred to as case A).

On the other hand, when the outer cylinder was at rest and the inner was in uniform rotation (case B), the instability set in at some definite speed of rotation determined by the geometry of the apparatus and the kinematic viscosity ν of the liquid. This sort of instability, which we shall call the Taylor type, manifests itself in the form of well-defined, stable, and reproducible vortex patterns (Taylor[9] and Lewis[10]), and is certainly different from the usual Reynolds type of instability which marks the transition from laminar to turbulent flow.

It would, consequently, be unjustified to conclude from Taylor's analysis, as has been done by Hollis-Hallett (1953), that the arrangement corresponding to case A would be stable to fluctuations due to turbulence at *all* speeds of rotation, however large. In fact, one should expect that at some fairly high value of the Reynolds number the flow *will* become turbulent even though it is stable against the disturbances considered by Taylor. This is precisely what Taylor observed in his later experiments[11] on the measurements of the torque reaction between two concentric cylinders, of which one was stationary and the other was in uniform rotation. In case B the torque showed a departure from its linear dependence on velocity at almost the same speed of rotation at which the instability was expected by Taylor's analysis. In case A too, this sort of departure invariably occurred at a fairly well-defined speed of the rotating cylinder. Taylor has given a plot of the critical Reynolds number $N_c(=\omega_c at/\nu)$ as a function of t/a, where a is the radius of the outer cylinder and the width of the gap, t, is taken as the characteristic length.

Applying the principle of dynamic similitude, we may now utilize the above results in order to study the possibility of turbulence in the normal component of liquid helium II in the experiments under consideration. The design of the Hollis-Hallett experiments corresponds to case A, with $\log_{10}(t/a)=-1.3$. The corresponding critical Reynolds number N_c is about 2500. In actual experiments, however, the departure from linear dissipation occurred at considerably different values of the Reynolds number. For instance, at $T=2°$K, the observed value of $N_c(=v_c t\rho_n/\eta_n)$ was about 75, decreasing with temperature to about unity at $T=1.25°$K. It is very significant to note that not only are these values far below that required for the onset of turbulence in a normal viscous liquid but also they themselves extend over a fairly wide range. On the other hand, the velocity at which the abnormality sets in is, over a wide range, almost independent of temperature, and consequently of ρ_n (or of ρ_n/η_n). It is evident, therefore, that something happens to the liquid at a definite velocity of rotation and not at a definite Reynolds number. Thus we conclude that the observed behavior of liquid helium II in this case *cannot* be attributed to the setting in of turbulence in the normal component.

On the other hand, in the experiment of Kolm and Herlin, which corresponds to the case B, one would expect, from the Taylor criterion, that instability would appear when the Reynolds number is about 80. In the actual experiment, however, the Reynolds numbers are, almost throughout, higher than this value. Hence, in case the normal component were to exhibit instability this would have been apparent throughout the range of observation. No such behavior has, however, been observed. It would not therefore be advisable to attribute the abrupt change observed by them to the

[9] G. I. Taylor, Trans. Roy. Soc. (London) **223**, 289 (1923).
[10] J. W. Lewis, Proc. Roy. Soc. (London) **117**, 388 (1928).
[11] G. I. Taylor, Proc. Roy. Soc. (London) **157**, 546 (1936).

appearance of turbulence in the normal component, provided one is allowed to treat the normal component as any ordinary viscous liquid.

Consequently, it seems reasonable to state that neither the force of mutual friction between the two components of liquid helium II nor the appearance of turbulence in the normal component may be invoked in order to account for the aforesaid observations. Obviously, therefore, the superfluid is the proper mode into which one should look for the cause of the reported anomalies: probably the appearance of vorticity, on a macroscopic scale, in the superfluid would provide the requisite basis for the expected nonlinear dissipative forces.

Being a fluid of apparently zero viscosity, the superfluid might be expected to be highly susceptible to turbulence. London,[6] on the other hand, has suggested, as emphasized by Landau[12] also, that the behavior of the superfluid component is described by the equation of potential motion $\mathrm{curl}\,\mathbf{v}_s=0$, which suggests reluctance to become turbulent. It is natural, therefore, to look for a model of the rotating superfluid consistent with the requirement of irrotational flow for velocities less than a certain critical value. The motion would remain nondissipative in this range of velocities and above that would suffer a transition accompanied by a rupture of the potential motion and the onset of nonlinear dissipation.

III. THE "ROTATING" SUPERFLUID

We have based our considerations on the stratification model proposed by London,[6] and recently adopted by Landau and Lifshitz,[13] for the motion of the rotating superfluid. According to their picture,[14] the superfluid remains in a state of complete rest until the angular speed of rotation exceeds a certain minimum value ω_1. For speeds higher than ω_1, the superfluid is looked upon as stratified into a number of coaxial layers characterized by their bounding radii $r_1, r_2, \cdots, r_k, r_{k+1}, \cdots, r_n$, in such a way that within each layer there is a curl-free circulation with the velocity undergoing quantum jumps at the interfaces (that is, the vortex sheets). The quantized potential motion in the region r_k-r_{k+1}, designated as the kth layer, is given by the tangential velocity $kh/(2\pi mr)$ and the angular momentum $kh/2\pi$ per particle. As the speed of rotation increases, the number of layers also increases. The layers go on becoming thinner and thinner and the vortex sheets come closer and closer. It is expected that a critical stage will be reached when the layers become so thin that the consecutive vortex sheets come too close to remain independent of each other. The layer structure then becomes virtually disrupted and the superfluid acquires a macroscopic turbulence field. At this stage the system experiences a macroscopic breach of superfluidity and nonlinear dissipation sets in.

The critical layer thickness at which this transition takes place must be something in the nature of a correlation distance characteristic of the liquid under investigation. A concept such as that has recently been developed by Blatt *et al.*[7] in connection with their study of the nature of the superfluid state. The correlation distance Λ is a characteristic distance which marks the limit of the range of momentum correlations between two neighboring particles. Their estimate for its magnitude in the case of liquid helium II is $10^{-4}-10^{-5}$ cm.[15] We, therefore, expect that in the case of liquid helium II in rotation the superfluid component does not contribute to the processes of dissipation for those velocities of rotation which keep the layers thicker than Λ. As soon as the layer thickness approaches this limit, dissipative forces come into operation in the superfluid component too. This stage is reached at the critical velocity v^*.

Clearly, in order to evaluate v^* from these considerations, it is required to obtain the relation between the velocity of rotation and the layer thickness. In the state of thermodynamic equilibrium, London obtains the following expression for the radii of the layer interfaces:

$$r_k=\frac{h}{4\pi m\omega}(2k-1), \quad k=1, 2, \cdots, n, \tag{1}$$

where m is the mass of each (superfluid) particle and ω is the angular velocity of rotation of the cylindrical container. Now, first of all, we have to establish the expressions, corresponding to Eq. (1) above, for the different geometrical arrangements of our present interest.

Case A: The inner cylinder of radius b is at rest and the outer cylinder of radius a is rotating with a uniform angular velocity ω. In order to obtain the layer structure in this case, we modify the one appropriate to the case $b=0$, that is, the case represented by Eq. (1), by merely restricting its total extension to the region between the two cyclindrical walls. In this way the individuality of the different layers is left intact except that they get shrunk laterally and elongated axially so that the incompressibility of the fluid is ensured. We denote the new radii by r_k'. Then, the whole of the fluid which was previously occupying the region $r=0$ to $r=a$ is now confined to the region $r=b$ to $r=a$ and further, that part of the fluid which was previously occupying the region $0-r_k$ is now confined to the region $b-r_k'$.

[12] L. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **11**, 592 (1941).

[13] L. D. Landau and E. M. Lifshitz, Doklady Akad. Nauk. S.S.S.R. **100**, 669 (1955).

[14] In his preliminary analysis, London had suggested that, in a more detailed development of the stratification model, account should be taken of the expenditure of surface energy at the layer boundaries. We have shown that if this expenditure is ascribed to some sort of a surface tension at the interfaces, the inclusion of the "surface terms" does not modify the results to any considerable extent (details of these calculations are being reported elsewhere).

[15] S. T. Butler and J. M. Blatt, Phys. Rev. **100**, 495 (1955).

Obviously, one must have

$$(r_k'^2 - b^2)/r_k^2 = (a^2 - b^2)/a^2,$$

each being equal to the ratio of the original height of the fluid to its new height. Whence, using Eq. (1), we have

$$r_k'^2 = b^2 + \left(\frac{a^2 - b^2}{a^2}\right)\left(\frac{h}{4\pi m\omega}\right)(2k-1). \quad (2)$$

Case B: The outer cylinder of radius a is at rest and the inner cylinder of radius b is rotating with a uniform angular velocity ω. The layer structure in this case will evidently be obtained by interchanging the roles of a and b in the previous case, whence

$$r_k''^2 = a^2 - \left(\frac{a^2 - b^2}{b^2}\right)\left(\frac{h}{4\pi m\omega}\right)(2k-1). \quad (3)$$

It may be noted that in the present considerations also (just as in London's treatment, reference 6) no assumption has been made to restrict the velocity of the superfluid in contact with the rotating wall to the velocity of the latter.

Next, the mean flow rate within the kth layer is given by:

Case A:

$$\bar{v}_k(r_{k+1}' - r_k') = \frac{kh}{2\pi m}\ln(r_{k+1}'/r_k'). \quad (4)$$

Substituting for $r_{k+1}'^2$ and $r_k'^2$ from Eq. (2) we obtain after simplification

$$\bar{v}_k(r_{k+1}' - r_k') = \frac{h}{4\pi m}\bigg/\left[1 + \left(\frac{a^2 b^2}{a^2 - b^2}\right)\left(\frac{4\pi m\omega}{h}\right)\frac{1}{2k}\right]. \quad (5)$$

For the layer close to the rotating wall one has from Eq. (2)

$$2k \simeq 4\pi m\omega a^2/h; \quad (\gg 1).$$

Hence, the mean velocity within this layer would be very nearly equal to v, the peripheral velocity. Setting the layer thickness equal to Λ, we obtain, from Eq. (5), for the critical velocity,

$$v_a^* = \frac{h}{4\pi m\Lambda}\left(1 - \frac{b^2}{a^2}\right). \quad (6)$$

Case B: In the same way, we obtain for this case

$$v_b^* = \frac{h}{4\pi m\Lambda}\left(\frac{a^2}{b^2} - 1\right). \quad (7)$$

Next, we investigate how far the conclusions arrived at above are applicable to the experimental results discussed in Sec. II. In the Kolm-Herlin experiment, $a = 0.95$ cm, $b = 0.63$ cm, and the observed critical velocity, $v_b^* = 1.2$ cm/sec. Equation (7) then gives for the correlation length

$$\Lambda = 8 \times 10^{-5} \text{ cm},$$

a magnitude well within the reasonable limits of 10^{-4}–10^{-5} cm proposed by Butler and Blatt.[15] Also one should expect from general considerations that the magnitude of the correlation length would not be very much different from the thickness of the supersurface film.

One may now use this value of the correlation length in Eq. (6) in order to obtain the magnitude of the critical velocity expected in the Hollis-Hallett experiments ($a = 2.097$ cm, $b = 1.991$ cm). One immediately obtains

$$v_a^* = 0.09 \text{ cm/sec},$$

in remarkable agreement with the observed values of 0.08 to 0.09 cm/sec.

At this stage, one may not, however, insist upon the details of the present model. It would be interesting to work out, in greater detail, the actual magnitude of the contribution due to the appearance of large-scale vorticity in the superfluid. Pending the development of such an analysis it seems that the present considerations are a step in the right direction towards a clear understanding of the roles played by the superfluid and the normal components in the rotation of liquid helium II.

IV. FLOW THROUGH THE SUPERSURFACE FILM

Clearly, one may expect that the essential features of the stratification model should also be applicable to the linear flow of the superfluid. Needless to add, the details of such a consideration would be highly complicated owing to the complexity of the macroscopic boundary conditions. However, one may intuitively develop a qualitative argument in order to picture the flow of superfluid helium through the supersurface film. It is interesting to note that this simple consideration leads to quite far-reaching conclusions.

We study here the rate of transfer of the supersurface film under a temperature gradient. Strictly speaking, the stratification model is not applicable insofar as the flow is nonisothermal. However, it still seems interesting to point out some of the simple features of the probable mechanism of the flow phenomenon.

Consider the change in the flow process as the heat input is gradually increased from a vanishingly small value. In the beginning, the rate of transfer is small and therefore the velocity with which the film slips over the supporting surface is much less than the usually observed critical velocity. As the heat input is increased, a stage is reached when the slip velocity becomes just equal to the critical value. Here, superfluid particles in contact with the surface would start experiencing frictional effects and, in fact, some of the atomic layers may stop moving. Next, since $v_{\text{crit}} \times d \sim \text{const}$, and since the effective thickness of the moving film has been reduced (due to the stoppage of some of the layers), the effective critical velocity for the moving part assumes a higher value. Consequently, the flow breaks up into two parts —a portion of the superfluid (which is in contact with the wall) completely at rest, and the remaining part flowing with a subcritical velocity (pure superflow). On further increasing the heat input two effects are expected to appear. First, the thickness of the stationary layer increases and second, since the superfluid particles can lose energy only in accord with the principles of quantum mechanics, a layer structure is formed. In other words, the moving portion of the superfluid is broken up into layers. The actual velocity distribution may be quite complicated. However, it does not appear to be necessary to go into further details for the purpose of a purely qualitative argument. Suffice it to say that as a result of the frictional effects, the portion of the superfluid which participates in the flow process effectively reduces and hence the rate of transfer would not follow the increase in the heat input. In other words, even though intrinsically the velocity of flow may increase, the observations would not exhibit a larger rate of transfer insofar as the cross section of the moving part goes on shrinking.

Such an effect, it may be noted, has been observed by Chandrasekhar and Mendelssohn.[8] Also, it may be pointed out that so far it has not been found possible to account for their observations on the basis of any one of the well-known models for the flow of superfluid helium. It would be worth while to work out the actual velocity at which the frictional effects first appear, but at present one has to be content with an interesting, though of course sketchy, picture of the flow phenomena.

V. DISCUSSION OF RESULTS

We have seen above that once the stratification concept is accepted as the probable model for the rotation of superfluid helium, it follows that it would exhibit the onset of strong dissipative effects at a well-defined speed of rotation. Since each of the layer boundaries (that is, the vortex sheets) represents a local breach of superfluidity, it is clear that the "merging" of these boundaries when their separation approaches the magnitude of the correlation distance is virtually equivalent to the appearance of a macroscopic turbulence field. One cannot but stress the importance of the model in order to estimate the magnitude of the correlation distance by recourse to well-known experimental results. In this connection, one may utilize the observations of Heikkila and Hollis-Hallett[2] who have measured the values of v_a^* over a wide range of temperatures. It is found that v_a^* varies from 0.08 cm/sec to 0.09 cm/sec as the temperature is varied from 1.3°K to 2.1°K and then increases rather rapidly as the lambda-temperature is approached. This immediately leads to the conclusion that Λ should decrease from 9×10^{-5} cm at 1.3°K to 8×10^{-5} cm at 2.1°K and then show a rapid decrease. Of course, such a behavior of the variation of the correlation length is quite consistent with the concept that it would be of the order of the effective deBroglie wavelength in the system under consideration (see reference 7 for a detailed discussion).

Clearly, in all systems there are a certain number of particles in the lowest state of energy, that is, with large deBroglie wavelength, but in the case of liquid helium, below the condensation point, this number is comparable to the total number of particles in the system. Therefore, the magnitude of the correlation length, which would be large at temperatures fairly below the lambda point, should reduce rapidly as the lambda point is approached. For the number of particles having large deBroglie wavelength becomes vanishingly small in comparison with the total number. In fact, just above the lambda point one should expect the correlation length to be of the order of the deBroglie wavelength of a particle with energy kT, *viz.*, $\sim10^{-7}$ cm. A possible confirmation of these results may be obtained if the Kolm-Herlin experiment is also performed at a number of different temperatures.

Pending the availability of more detailed observations, it is encouraging to note that the present model provides a fairly satisfactory explanation for the anomalies observed with two types of arrangements which are rather widely different from the point of view of the stability of viscous flow. Further, it is hoped that it would be possible to develop the full details of this model within the framework of a general quantum theory of the liquid state.

VI. ACKNOWLEDGMENTS

In conclusion the authors deem it their privilege to express their keen sense of gratitude to Professor D. S. Kothari for several invaluable suggestions. We are also thankful to Professor F. C. Auluck for his kind interest in this work.