Theory of High-Energy Bremsstrahlung and Pair Production in a Screened Field

HAAKON OLSEN,* † *Laboratory of Nuclear Studies, Cornell University, Ithaca, New York*
L. C. MAXIMON,‡ † *Division of Applied Mathematics, Brown University, Providence, Rhode Island*

AND

HARALD WERGELAND, *Fysisk Institutt, Norges Tekniske Høgskole, Trondheim, Norway*

(Received November 15, 1956)

The differential cross sections for high-energy bremsstrahlung and pair production in a screened Coulomb field are calculated without the use of the Born approximation. It is shown that for pair production the correction to the Born approximation occurs only for momentum transfers q of order mc, for any amount of screening. For bremsstrahlung, however, the correction is only important for q's of order $(mc^2/E)mc$, where E is the energy of the electron. As in the case of no screening, the correction to the differential cross section for bremsstrahlung is found to be given by a factor multiplying the Bethe-Heitler cross section. It is then shown that the bremsstrahlung cross section integrated over the angles of the final electron is additive, just as in the case of pair production: one part is the Bethe-Heitler cross section including screening; to this is then added the Coulomb correction which is independent of screening.

The cross sections are evaluated by using wave functions which are accurate in the region in space which contributes significantly to the matrix element. This region is determined by the order of magnitude of q, and different wave functions must be used in the regions (I) corresponding to q's of order mc and (II) corresponding to q's of order $(mc^2/E)mc$. In (I) the wave functions are obtained by an expansion in partial waves and use of a WKB method on the radial wave equation. In (II) we use a WKB technique on the three-dimensional wave equation itself. Corrections due to the use of Sommerfeld-Maue wave functions, which are solutions to the second-order Dirac equation, are shown to be negligible. Finally, the method is used to obtain the cross section for small-angle elastic scattering.

1. INTRODUCTION

THE differential cross section for high-energy bremsstrahlung and pair production taking into account the Coulomb field of the nucleus exactly has been derived by Bethe and Maximon[1] in the case of an unscreened atom. It was also shown by them that the extension to the screened case could readily be made in the case of pair production. It was then shown by Olsen[2] that the cross section for bremsstrahlung integrated over the direction of motion of the final electron can be inferred from the corresponding expression for pair production for any amount of screening.

The present work was then intended to give the remaining unknown quantity, the differential cross section for bremsstrahlung in the screened case (differential both with respect to the photon and electron momenta). It soon became clear, however, that it was possible to solve the entire problem in a way which pointed out clearly the significant factors which determine the matrix element. In particular very simple accurate high-energy wave functions for an electron in an arbitrarily screened potential have been constructed. These wave functions give the exact results for the matrix elements in the case which may be checked, namely the case of a pure Coulomb potential treated by B-M.

At the same time, the present calculation, being so much more transparent than the exact one, should be able to give a little insight into the mechanism of the Coulomb correction at these high energies. It was especially intended to study more closely the effects caused by the difference in the spatial part of the states for pair production and bremsstrahlung; namely that in pair production both particles are represented by wave functions with asymptotic behavior plane wave plus ingoing spherical waves, while in bremsstrahlung the initial electron is represented by a normal scattering state (plane wave plus *outgoing* spherical waves) and the final electron by a wave function with asymptotic behavior plane wave plus *ingoing* spherical waves. It has been shown[1] that this causes a big difference between the differential cross sections for the two processes. It has, however, also been shown[2] that this effect disappears when the cross sections are integrated over the motion of the final electron, and that consequently the Coulomb correction to this integrated cross section is the same for both processes. As a consequence of this, the remark about shower theory in B-M should be corrected: Conventional shower theory is *not* changed by the Coulomb effect.

It is interesting to note that the essential calculations for the differential cross section for high-energy bremsstrahlung were given by Sommerfeld[3] for the case of

* Supported by a grant from the International Cooperation Administration and the National Academy of Sciences. On leave of absence from Fysisk Institutt, Norges Tekniske Høgskole, Trondheim, Norway.

† Present address: Fysisk Institutt, Norges Tekniske Høgskole, Trondheim, Norway.

‡ Supported by a grant from the National Science Foundation.

[1] H. A. Bethe and L. C. Maximon, Phys. Rev. 93, 768 (1954). In the following referred to as B-M.

[2] Haakon Olsen, Phys. Rev. 99, 1335 (1955).

[3] A. Sommerfeld, *Atombau und Spektrallinien* (Vieweg und Sohn, Braunschweig, 1939), pp. 501–505 and p. 551. The cross section for bremsstrahlung is given by Eq. (VII, 7.28) and the preceding equations for Q_1, Q_2, and Q_3. This is exactly the expression for bremsstrahlung corresponding to B-M (reference 1) Eq. (6.26)

an unscreened atom. These calculations are, however, only used to obtain the Born approximation result.

Finally, the electron wave functions studied here may be of interest for other high-energy electron processes.

2. GENERAL DISCUSSION

We consider first the general approach used in obtaining wave functions which are accurate in the regions of space from which there is a significant contribution to the matrix elements for bremsstrahlung and pair production. This is followed by a discussion of the salient features of the wave functions and the related cross sections.

The method which has been used for calculating matrix elements is briefly this. The Born approximation is assumed to be nearly correct, that is, the regions in space which contribute significantly to the more accurate matrix element are assumed to be determined by the Born approximation matrix element. For bremsstrahlung and pair production the important regions in the Born approximation are determined by the factor $\exp(i\mathbf{q}\cdot\mathbf{r})$, where $\mathbf{q}$ is the momentum transfer to the nucleus. The most important contributions come, therefore, from regions where $\mathbf{q}\cdot\mathbf{r}\sim 1$. Again we use a result from the Born approximation, that for high energies the cross section is only significant for $q\sim 1$ and $q\sim 1/\epsilon$, and these regions are of equal importance. Further, it can easily be seen from energy and momentum conservation that q_z, the component of momentum transfer in the direction of $\mathbf{k}$, the quantum, is always of order $1/\epsilon$. Thus the signicant regions of $q_\perp$, the component of $\mathbf{q}$ perpendicular to $\mathbf{k}$, are $q_\perp\sim 1$ and $q_\perp\sim 1/\epsilon$. Now in evaluating the matrix element we will use cylindrical coordinates ρ, z, φ with z axis along $\mathbf{k}$. Thus if we evaluate the matrix element for q's of order 1, the big contribution to the matrix element comes from the region in space near the z axis, where the "impact parameter" ρ is of order 1, and $|z|\sim\epsilon$. On the other hand, if we evaluate the matrix element for q's of order $1/\epsilon$, the big contributions to the matrix element comes from the region in space far from the z axis, where $\rho\sim\epsilon$ and $|z|\sim\epsilon$. Thus, with $q\sim 1$ we need, when evaluating the matrix element, wave functions which are accurate only in region (I), ($\rho\sim 1$, $|z|\sim\epsilon$) whereas with $q\sim 1/\epsilon$ we need wave functions which are accurate only in region (II) ($\rho\sim\epsilon$, $|z|\sim\epsilon$).

It is clear that the form of a particular wave function will be simplest when described in a coordinate system with z axis along the momentum, $\mathbf{p}$, of the particle to which it refers.

Let (ρ_1,z_1) and (ρ_2,z_2) be the cylindrical coordinates with z axis along $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively. Because of the small angles $[O(1/\epsilon)]$ between the fast particles, ρ_1 and ρ_2 are both of order 1, and $|z_1|$ and $|z_2|$ are of order ϵ in region (I). Similarly, in region (II) $\rho_1\sim\rho_2\sim\epsilon$ and $|z_1|\sim|z_2|\sim\epsilon$. We therefore solve the wave equation in the region $\rho\sim 1$, $|z|\sim\epsilon$, where the z axis is now along $\mathbf{p}$, and use such solutions for both the initial and final states in the matrix element when $q\sim 1$. Similarly, for $q\sim 1/\epsilon$ the wave functions in the matrix element are given by the solutions to the wave equation in the region $\rho\sim\epsilon$, $|z|\sim\epsilon$. The method of solution is different in the two regions. In region (I), the region of small "impact parameters" ($\rho\sim 1$), we use an expansion in partial waves and apply a WKB technique to the radial wave equation. In region (II), where $\rho\sim\epsilon$, we apply a WKB method directly to the three-dimensional wave equation.

Such methods have been applied to the scattering problem by Molière,[4] by Landau and Lifschitz,[5] and more recently by others.[6] In that case, only the phase shifts are required, i.e., the *asymptotic* form of the wave function. Here, however, we need the wave function at finite distances (of order ϵ) from the nucleus.

The scattering problem does not demand that a distinction be made between the wave functions pertaining to the two regions. Indeed, as is explicitly shown in Sec. 9, the wave functions pertaining to either region lead to the same scattering amplitude.

However, for bremsstrahlung and pair production we find that neither of the wave functions may be used for both $q\sim 1$ and $q\sim 1/\epsilon$. Specifically, in the case of bremsstrahlung, the wave functions pertaining to $q\sim 1$ give the Born approximation cross section in this region (although the wave functions themselves are not those given by the Born approximation). Thus if these wave functions were applied to the bremsstrahlung problem in the region $q\sim 1/\epsilon$ as well, we would obtain the incorrect result that there is no Coulomb correction to the Born approximation for all q's. Similarly, in the case of pair production, the wave functions pertaining to $q\sim 1/\epsilon$ give the Born approximation cross section in the region $q\sim 1/\epsilon$ (although the wave functions are not those given by the Born approximation) and thus if these wave functions were applied to the pair production problem in the region $q\sim 1$, we would obtain the same incorrect result that there is no Coulomb correction to the Born approximation cross section for pair production.

As we have just noted, the cross sections for pair production and bremsstrahlung behave differently in the two regions $q\sim 1$ and $q\sim 1/\epsilon$. In order to understand this, let us consider the character of the wave functions appropriate to each of these regions.

In the case of $q\sim 1/\epsilon$, the wave function for the normal scattering state (asymptotically a plane wave plus outgoing spherical waves) is, apart from spin-

for pair production. The expressions for J, $\mathbf{J}_1$, and $\mathbf{J}_2$ (which are substantially equal to B-M I_1, $\mathbf{I}_2$, and $\mathbf{I}_3$) are given by Eqs. (VII, 7.8 and 7.9) in terms of X (B-M: I_0). The integral X is evaluated in Eqs. (VII, 2.19 and 2.19a).

[4] G. Molière, Z. Naturforsch. **2a**, 133 (1947).

[5] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Ogiz, Moscow, 1948) (in Russian), Part I, pp. 184–203, 470–473.

[6] I. I. Gol'dman and A. B Migdal, Soviet Phys. **1**, 304 (1955); R. J. Glauber, Phys. Rev. **91**, 459 (1953); L. I. Schiff, Phys. Rev. **103**, 443 (1956).

dependent factors,

$$\psi_+=\exp\left[i\mathbf{p}\cdot\mathbf{r}-i\int_{-\infty}^{z}V(\rho,\zeta)d\zeta\right],$$

where the z axis is in the direction of $\mathbf{p}$. The wave function with asymptotic form plane wave plus ingoing spherical waves is

$$\psi_-=\exp\left[i\mathbf{p}\cdot\mathbf{r}+i\int_{z}^{\infty}V(\rho,\zeta)d\zeta\right].$$

Now the matrix element for pair production is of the form

$$\int\psi_{\text{el},-}{}^*e^{i\mathbf{k}\cdot\mathbf{r}}\psi_{\text{pos},-}{}^*d^3r,$$

in which $\exp[+i\mathbf{k}\cdot\mathbf{r}]$ appears since the quantum is annihilated, and the complex conjugate of each of the wave functions appears since both particles are created; for the same reason the ingoing type wave function must be chosen for each of the particles.

Thus the matrix element depends upon the potential only through the factor

$$\exp\left[-i\int_{z2}^{\infty}V_{\text{el}}(\rho_2,\zeta)d\zeta-i\int_{z1}^{\infty}V_{\text{pos}}(\rho_1,\zeta)d\zeta\right],$$

where V_{el} and V_{pos} have opposite signs and ρ_1,z_1 and ρ_2,z_2 are cylindrical coordinates that refer to the positron and electron, respectively. Because of the small angle between $\mathbf{p}_{\text{pos}}$ and $\mathbf{p}_{\text{el}}$, $(\rho_1-\rho_2)/\rho_1=O(1/\epsilon)$ and $(z_1-z_2)/z_1=O(1/\epsilon)$, so that the terms in the exponent nearly cancel, leaving terms of order $1/\epsilon$ multiplying $\int_z^\infty V(\rho,\zeta)d\zeta$. The exponent being small, the exponential may be expanded. Thus the matrix element is proportional to the potential, and is the Born approximation result. Thus the Born approximation matrix element results from the cancellation of phases in wave functions which are not themselves Born approximation wave functions.

This is in contrast to the bremsstrahlung process, where the initial state is the normal scattering state and thus the matrix element is

$$\int\psi_{2,-}{}^*e^{-i\mathbf{k}\cdot\mathbf{r}}\psi_{1,+}d^3r.$$

Here the dependence upon the potential is given by the factor

$$\exp\left[-i\int_{z_2}^{\infty}V(\rho_2,\zeta)d\zeta-i\int_{-\infty}^{z_1}V(\rho_1,\zeta)d\zeta\right]$$
$$=\exp\left[-i\int_{-\infty}^{\infty}V(\rho,\zeta)d\zeta+iO(1/\epsilon)\int_{z]}^{\infty}V(\rho,\zeta)d\zeta\right].$$

The exponent therefore contains the extra term $-i\int_{-\infty}^{\infty}V(\rho,\zeta)d\zeta$ in addition to the term which is of order $1/\epsilon$ and which gives the Born approximation. This leads to a finite correction to the Born approximation for bremsstrahlung in the region $q\sim1/\epsilon$ for arbitrarily high energies.

For the region $q\sim1$ we need to know the wave functions accurately close to the z-axis ($\rho\sim1$) but far away from the nucleus ($|z|\sim\epsilon$). Consder first the wave function for the normal scattering state. At high energies it is clear that there are no scattered waves in the region $z\sim-\epsilon$, $\rho\sim1$, which corresponds very nearly to backscattering; therefore the wave function in this region is a plane wave. In the forward region, ($z\sim\epsilon$, $\rho\sim1$), the wave function is given by small-angle scattering theory. Consider next the ingoing-type wave function, which represents a particle *leaving* the nucleus. In this case there cannot be any distortion of the plane wave in the region $z\sim\epsilon$, $\rho\sim1$ at high energies, and scattering occurs only in the region in *front* of the nucleus: $z\sim-\epsilon$, $\rho\sim1$.

It is now possible to show that in the present region, $q\sim1$, the cross section for bremsstrahlung is proportional to the cross section for elastic scattering, provided only that the definition for $\mathbf{q}$, the momentum transfer, is changed appropriately. Since in bremsstrahlung the initial state is a normal scattering state while the final state is described by an ingoing type wave function, then, as discussed above, the initial state is a plane wave for $z<0$ while the final state is a plane wave for $z>0$. Thus in the matrix element one of the wave functions will always be a plane wave. The radiation interaction $e^{-i\mathbf{k}\cdot\mathbf{r}}$ may therefore be combined with the initial state $e^{i\mathbf{p}_1\cdot\mathbf{r}}$ for $z<0$, changing the momentum of that state from $\mathbf{p}_1$ to $\mathbf{p}_1-\mathbf{k}$. In the same way, for $z>0$, the final state $e^{-i\mathbf{p}_2\cdot\mathbf{r}}$ is changed into $e^{-i(\mathbf{p}_2+\mathbf{k})\cdot\mathbf{r}}$. A typical term in the spatial part of the matrix element is thus

$$I_1=\int d^3r\varphi_{2,-}{}^*e^{-i\mathbf{k}\cdot\mathbf{r}}\varphi_{1,+}$$
$$=\int_{z<0}d^3r\varphi_{2,-}{}^*e^{i(\mathbf{p}_1-\mathbf{k})\cdot\mathbf{r}}+\int_{z>0}d^3re^{-i(\mathbf{p}_2+\mathbf{k})\cdot\mathbf{r}}\varphi_{1,+}.$$

Again, since $\varphi_{2,-}{}^*=e^{-i\mathbf{p}_2\cdot\mathbf{r}}$ for $z>0$, and $\varphi_{1,+}=e^{i\mathbf{p}_1\cdot\mathbf{r}}$ for $z<0$, we may write

$$I_1=-(2\pi)^3\delta(\mathbf{p}_1-\mathbf{p}_2-\mathbf{k})+\int d^3r\varphi_{2,-}{}^*e^{i(\mathbf{p}_1-\mathbf{k})\cdot\mathbf{r}}$$
$$+\int d^3re^{-i(\mathbf{p}_2+\mathbf{k})\cdot\mathbf{r}}\varphi_{1,+}.$$

Since $\mathbf{p}_1-\mathbf{p}_2-\mathbf{k}$ is never zero, the delta function is always zero. Finally, using the wave equation for φ, *viz.*

$$(\nabla^2+p^2-2\epsilon V)\varphi=0,$$

I_1 may be written in the form

$$I_1=\frac{2\epsilon_2}{p_2^2-(\mathbf{p}_1-\mathbf{k})^2}\int d^3r\,\varphi_{2,-}{}^*Ve^{i(\mathbf{p}_1-\mathbf{k})\cdot\mathbf{r}}+\frac{2\epsilon_1}{p_1^2-(\mathbf{p}_2+\mathbf{k})^2}\int d^3r e^{-i(\mathbf{p}_2+\mathbf{k})\cdot\mathbf{r}}V\varphi_{1,+}.$$

Each of the above integrals may now be recognized as the T matrix for elastic scattering. Moreover, since the energies of the initial and final states in the T matrix are almost equal [in the first integral $p_2-|\mathbf{p}_1-\mathbf{k}|=O(1/\epsilon)$ and in the second $p_1-|\mathbf{p}_2+\mathbf{k}|=O(1/\epsilon)$], the value of the T matrix will be very close to that which it takes on the energy shell. T is then only a function of the momentum transfer $\mathbf{q}=(\mathbf{p}_1-\mathbf{k})-\mathbf{p}_2=\mathbf{p}_1-(\mathbf{p}_2+\mathbf{k})$. The expression for I_1 thus simplifies to

$$I_1=\left[\frac{2\epsilon_2}{p_2^2-(\mathbf{p}_1-\mathbf{k})^2}+\frac{2\epsilon_1}{p_1^2-(\mathbf{p}_2+\mathbf{k})^2}\right]T(\mathbf{q}).$$

In the same way one finds that $\mathbf{I}_2$ and $\mathbf{I}_3$ are proportional to $T(\mathbf{q})$. [See (6a.2,3) in text.] Since the cross section is a homogeneous quadratic function of the I's (8.1) it follows that the cross section for bremsstrahlung is proportional to the cross section for elastic scattering in the region $q\sim1$.

It may be observed that the initial state wave function in the region $z\sim-\epsilon$ and the final state wave function in the region $z\sim\epsilon$ have been approximated by plane waves. Actually, as is shown explicitly in Sec. 5a, the plane waves in those regions should be modified slightly by a factor which for a pure Coulomb field is the familiar term $e^{\pm ia\log 2r}$. As the detailed calculation shows, this factor may be neglected throughout, since it is independent of the energy of the particular state and independent of whether we choose ingoing or outgoing waves.

What has been said here about the proportionality between the bremsstrahlung matrix element and the scattering T matrix may also be seen from the explicit calculations of Sec. 6a and Sec. 9. In particular see (6a.8) for bremsstrahlung and (9.2) for elastic scattering. The second term in (9.2), which is $\delta(\mathbf{q})$, is always zero when q_{brems} is substituted for q_{scat} since q_{brems} is never zero.

Now it is easy to see that the bremsstrahlung cross section in the region $q\sim1$ is given by the Born approximation: Turning to the scattering cross section we notice that in the present range, $q\sim1$, spin effects are unimportant[7] since the scattering angle is small: $\vartheta=q/p\sim1/\epsilon$. Moreover, in this region we may neglect screening. Thus we can use the fact known from scattering theory, that for a spinless particle in a pure Coulomb field, the scattering cross section is given by the Born approximation (Rutherford formula). Having shown above that the bremsstrahlung cross section is proportional to the scattering cross section, it follows that the bremsstrahlung cross section is given by the Born approximation. It may be emphasized that this result is obtained only because, in the case of electrons, $q\sim1$ corresponds to impact parameters of the order of the electron Compton wavelength, which is both considerably smaller than the atomic screening radius and also much larger than the nuclear dimension, and thus we have the pure Coulomb potential $V=-a/r$. On the other hand, for μ mesons, $q\sim1$ corresponds to impact parameters which are of the same magnitude as the nuclear dimension. Therefore the potential will be modified, and we may expect to obtain corrections to the Born approximation in this region. The present result for bremsstrahlung from electrons may thus be contrasted to the case of pair production for $q\sim1/\epsilon$, where we obtain the Born approximation cross section for any potential. Since for $q\sim1$ the cross section is proportional to the scattering cross section, and thus given by the Born approximation, all corrections to the total cross section for bremsstrahlung come from the region $q\sim1/\epsilon$ alone.

The reason that the correction to the total Born approximation bremsstrahlung cross section is so small also appears to follow from the above discussion.

It is well known that in the case of scattering in a pure Coulomb field the correction to the Born approximation cross section is zero instead of being of the order $(Z/137)^2$, as would be expected. As we have seen, this peculiarity of the Coulomb field is also present in the case of bremsstrahlung for the region of momentum transfer of order 1. The correction to the total Born approximation cross section would thus be expected to be somewhat smaller than $(Z/137)^2$. As it has been shown[2] that the total cross sections for bremsstrahlung and pair production are equal except for the usual changes in sign of momentum and energy, and phase space factors, this conclusion also holds for pair production. Here, however, one does not see this effect so clearly, since the entire Coulomb correction appears in the region $q\sim1$.

3. SOMMERFELD-MAUE TYPE WAVE FUNCTIONS

It has been customary to use wave functions pertaining to the iterated (second-order) Dirac equation in calculating matrix elements, such as in the case of bremsstrahlung and pair production.[1,3,8] As, however, the usual perturbation theory for transition amplitudes with "variation of constants" can only be applied directly to wave equations linear in the time derivative, it follows that the matrix elements should be calculated using the wave functions of the linear (first-order) Dirac equation.

However, in the actual case of high-energy bremsstrahlung and pair production it is easy to see that the

[7] I.e., in the Dirac equation (3.2) the spin dependent term $i\boldsymbol{\alpha}\cdot(\boldsymbol{\nabla}V)$ is small compared to $2\epsilon V$.

[8] L. Bess, Phys. Rev. 77, 550 (1950).

solutions to the first- and the second-order equations are identical up to orders $1/\epsilon^2$.

In fact, the iterated equation is obtained from the Dirac equation

$$(-i\boldsymbol{\alpha}\cdot\nabla+\beta+V-\epsilon)\psi=0 \tag{3.1}$$

by putting $\psi=(1/2\epsilon)(-i\boldsymbol{\alpha}\cdot\nabla+\beta-V+\epsilon)\phi$, so that

$$[\nabla^2+p^2-2\epsilon V+V^2-i\boldsymbol{\alpha}\cdot(\nabla V)]\phi=0. \tag{3.2}$$

The solution of this equation up to relative orders $1/r$, which in our case is equivalent to relative orders $1/\epsilon$ as discussed in Sec. 5, is the usual Sommerfeld-Maue type wave function

$$\phi=e^{i\mathbf{p}\cdot\mathbf{r}}\left(1-\frac{i\boldsymbol{\alpha}\cdot\nabla}{2\epsilon}\right)Fu. \tag{3.3}$$

F is the appropriate solution of

$$(\nabla^2+2i\mathbf{p}\cdot\nabla-2\epsilon V)F=0, \tag{3.4}$$

where $V(r)$ is now the screened Coulomb potential. The wave function (3.3) for an unscreened potential is just the wave function used in previous calculations by Sommerfeld,[3] Bess,[8] Bethe and Maximon.[1]

The wave function of the linear equation

$$\psi=\frac{1}{2\epsilon}(\epsilon-V-i\boldsymbol{\alpha}\cdot\nabla+\beta)e^{i\mathbf{p}\cdot\mathbf{r}}\left(1-\frac{i\boldsymbol{\alpha}\cdot\nabla}{2\epsilon}\right)uF,$$

may easily be shown to be

$$\psi=e^{i\mathbf{p}\cdot\mathbf{r}}\left[\left(1-\frac{i\boldsymbol{\alpha}\cdot\nabla}{2\epsilon}\right)-\frac{V}{\epsilon}\left(1-\frac{i\boldsymbol{\alpha}\cdot\nabla}{4\epsilon}\right)\right]uF.$$

Up to relative orders $1/\epsilon^2$ therefore, the high-energy wave functions of the linear and the iterated Dirac equations are identical.

Some confusion seems to exist concerning the general expression of the matrix elements in terms of the iterated Dirac equation wave functions ϕ. From what has been said above, this is evidently

$$\int\psi_f^\dagger H'\psi_i d^3r=\frac{1}{4\epsilon_1\epsilon_2}\int\phi_f^\dagger D_f^\dagger H'D_i\phi_i d^3r, \tag{3.5}$$

where

$$D_i=(-i\boldsymbol{\alpha}\cdot\nabla+\beta-V+\epsilon_i).$$

A formula considerably different from this has been proposed recently by Horton and Phibbs.[9] Their formula was however derived from a different cause, namely from an assumed non-orthogonality of the iterated Dirac wave functions. As, however, any set of wave functions containing a plane wave at large distances constitutes an orthogonal set, no modification of the matrix element is to be expected for this reason.

[9] G. K. Horton and E. Phibbs, Phys. Rev. **94**, 1402 (1954); **96**, 1066 (1954).

4. BORN APPROXIMATION

Before going into details of the actual calculation, it may be of some interest to review briefly the Born approximation calculation.

The solution of (3.4) up to first order in the potential is evidently

$$F_\pm=1-\frac{\epsilon}{2\pi}\int\frac{e^{\pm ipr'+i\mathbf{p}\cdot\mathbf{r}'}}{r'}V(|\mathbf{r}+\mathbf{r}'|)d^3r'. \tag{4.1}$$

The notation $\{\pm\}$ refers to wave functions with asymptotic form: plane wave plus $\left\{\begin{matrix}\text{outgoing}\\ \text{ingoing}\end{matrix}\right\}$ spherical waves. Keeping only terms linear in V in the matrix elements I_1, $\mathbf{I}_2$, and $\mathbf{I}_3$ of B-M, we find the Born approximation expressions for bremsstrahlung

$$I_1=\int F_{2,-}^*e^{i\mathbf{q}\cdot\mathbf{r}}F_{1,+}d^3r=2\left(\frac{\epsilon_1}{D_2}-\frac{\epsilon_2}{D_1}\right)\int e^{i\mathbf{q}\cdot\mathbf{r}}Vd^3r,$$

$$\mathbf{I}_2=\frac{-i}{2\epsilon_1}\int F_{2,-}^*e^{i\mathbf{q}\cdot\mathbf{r}}\nabla F_{1,+}d^3r=\frac{-\mathbf{q}}{D_2}\int e^{i\mathbf{q}\cdot\mathbf{r}}Vd^3r,$$

$$\mathbf{I}_3=\frac{i}{2\epsilon_2}\int(\nabla F_{2,-}^*)e^{i\mathbf{q}\cdot\mathbf{r}}F_{1,+}d^3r=\frac{-\mathbf{q}}{D_1}\int e^{i\mathbf{q}\cdot\mathbf{r}}Vd^3r, \tag{4.2}$$

$$D_1=2\mathbf{p}_2\cdot\mathbf{q}+q^2=\frac{k}{\epsilon_1}(1+p_1^2\theta_1^2),$$

$$D_2=2\mathbf{p}_1\cdot\mathbf{q}-q^2=\frac{k}{\epsilon_2}(1+p_2^2\theta_2^2).$$

The notation is as in B-M. The expressions for pair production are obtained in the usual manner by changing the sign of $\mathbf{p}_1$ and ϵ_1, treating $\mathbf{q}$ as independent of $\mathbf{p}_1$. We shall have opportunity several times to compare our "exact" calculation with these expressions.

Actually, as is well known, it is of no importance in the Born approximation whether we take ingoing or outgoing waves in the final (or initial) state. In fact, it is evident that the expression (4.2) as well as the Born approximation cross sections for pair production and bremsstrahlung[10] are invariant with respect to the simultaneous change of sign and reversal of direction of $\mathbf{p}_2$ (or $\mathbf{p}_1$) ($p\to -p$, $\mathbf{p}/p\to -\mathbf{p}/p$) [see (4.1)]. This property of the Born approximation is perhaps most easily understood considering the usual perturbation theory expression of the form

$$\lim_{\eta\to 0}\sum_n\frac{\langle 2|V|n\rangle\langle n|H_{\mathrm{rad}}'|1\rangle}{E_n-E_1\mp i\eta}$$

$\left(\left\{\begin{matrix}-\\+\end{matrix}\right\}\text{ corresponding to }\left\{\begin{matrix}\text{outgoing}\\ \text{ingoing}\end{matrix}\right\}\text{ spherical waves}\right)$. Since the radiation interaction describes the interaction

[10] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), third edition, pp. 244, 257.

with a plane wave (the photon)

$$\langle n|H_{\text{rad}}'|1\rangle \sim \delta(\mathbf{p}_n - \mathbf{k} + \mathbf{p}_1),$$

the momentum and therefore the energy in the intermediate state is fixed, and the energy denominator $E_n - E_1$ is never zero. Thus the sign of η is immaterial.

As the Coulomb interaction matrix element $\langle m|V|n\rangle$ does not have this property, the sign of η will be relevant in corrections to the Born approximation involving higher orders in the potential, $V(r)$.

5. HIGH-ENERGY WAVE FUNCTIONS

We use throughout the same notation as in B-M. The Sommerfeld-Maue type wave function for a screened potential is, as discussed in Sec. 3,

$$\psi = e^{i\mathbf{p}\cdot\mathbf{r}}\left(1 - \frac{i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{2\epsilon}\right)F_{\pm}u, \tag{5.1}$$

where $F_{\pm}$ is the appropriate solution of

$$(\nabla^2 + 2i\mathbf{p}\cdot\boldsymbol{\nabla} - 2\epsilon V)F = 0. \tag{5.2}$$

The difference in normalization from B-M should be noted. It is more convenient for us to normalize so that

$$e^{i\mathbf{p}\cdot\mathbf{r}}F_{\pm}]_{r\to\infty} \to e^{i\mathbf{p}\cdot\mathbf{r}} + \frac{e^{\pm ipr}}{r}f(\theta). \tag{5.3}$$

As in B-M, the cross sections for pair production and bremsstrahlung are expressible in terms of the matrix elements I_1, $\mathbf{I}_{2\perp}$, and $\mathbf{I}_{3\perp}$ [Eq. (7.11), B-M]. The symbol $\perp$ means perpendicular to $\mathbf{k}$. In the case of no screening, it is possible to solve Eq. (5.2) exactly, and the integrals I_1, $\mathbf{I}_2$, and $\mathbf{I}_3$ can be expressed in terms of hypergeometric functions, as was done by B-M. (See also Sommerfeld, reference 3.)

If screening is of importance one has to find other methods as Eq. (5.2) cannot be solved exactly, particularly since the screened potential is given only numerically. In order to find this new method the following considerations are helpful.

Consider the typical integral

$$I_1 = \int d^3r F_2^* e^{i\mathbf{q}\cdot\mathbf{r}} F_1.$$

In the Born approximation (4.2) it is seen that the order of magnitude is determined by the exponential $e^{i\mathbf{q}\cdot\mathbf{r}}$. Since the exact treatment of the F's may only introduce corrections to the Born approximation, it is clear that the same conclusion holds for the exact F's. (Compare also B-M Sec. IX.) The integrals I_1, $\mathbf{I}_2$, and $\mathbf{I}_3$ are therefore largely determined by the exponential factor. The most important values of $\mathbf{r}$ in the integrals will therefore be those which makes $\mathbf{q}\cdot\mathbf{r}\sim 1$. We now introduce cylindrical coordinates with z axis along $\mathbf{k}$. The contributions to the integrals then come from

$$\rho \sim 1/q_\perp \quad \text{and} \quad |z| \sim 1/q_z. \tag{5.4}$$

At the high energies considered here, the important angles of emission of the fast particles are clearly of order $1/\epsilon$. It then follows from momentum and energy conservation that q_z is of order $1/\epsilon$, while $q_\perp$ generally is of order 1, but may become of order $1/\epsilon$ when the three vectors $\mathbf{k}$, $\mathbf{p}_1$, and $\mathbf{p}_2$ are of the same order of magnitude and very nearly coplanar. The Born approximation then tells us that these two regions of $q_\perp$ are of equal importance, and again we suppose that this is also true for the exact cross sections. For the case of complete screening, $\epsilon \gg 137Z^{-\frac{1}{3}}$, the lower region of $q_\perp$ is $q_\perp \sim Z^{\frac{1}{3}}/137$. For simplicity, however, the lower region will always be referred to as $q_\perp \sim 1/\epsilon$.

We thus conclude that the important regions for the interaction of the electron with the photon field are, in the case of $q\sim 1$:

(I) $\qquad \rho\sim 1, \quad |z|\sim \epsilon$

while in the case $q\sim 1/\epsilon$:

(II) $\qquad \rho\sim \epsilon, \quad |z|\sim \epsilon.$

Again, since the coordinates in the "wave functions" $F_1(\rho_1,z_1)$ and $F_2(\rho_2,z_2)$ refer to only slightly different directions (angles of order $1/\epsilon$) of the z axes which now are along $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, the regions in both these sets of variables are simultaneously exactly either (I) or (II).

We will therefore solve the wave equation (5.2) (see Secs. 5a and 5b) in the two regions (I) and (II), and in Secs. 6 and 7 use these solutions to obtain the matrix elements I_1, $\mathbf{I}_2$ and $\mathbf{I}_3$ for the cases of momentum transfer $q\sim 1$ and $q\sim 1/\epsilon$, respectively.

a. Wave Functions for $\rho\sim 1$, $|z|\sim \epsilon$

These are the regions close to the z axis where ϑ, the polar angle of $\mathbf{r}$, ($\tan\vartheta = \rho/z$) is either of order $1/\epsilon$ or else $\pi - \vartheta$ is of order $1/\epsilon$. Because of these very small forward (and very large backward) angles involved, this may be called the scattering region, and the Rayleigh expansion should accordingly be useful:

$$e^{i\mathbf{p}\cdot\mathbf{r}}F_{\pm} = \frac{1}{pr}\sum_l i^l(2l+1)e^{\pm i\delta_l}u_l(r)P_l(\cos\vartheta),$$

where $\{\pm\}$ refers to $\begin{Bmatrix}\text{outgoing}\\ \text{ingoing}\end{Bmatrix}$ spherical waves. It is specifically because we have here $|z|$ of order ϵ, *but* ρ *of order 1* that $\vartheta\sim 1/\epsilon$ (or $\pi-\vartheta\sim 1/\epsilon$) and thus that l's of order ϵ give significant contribution to this sum. Therefore, in this region we may approximate $P_l(\cos\vartheta)$ by $J_0(l\vartheta)$ and later we shall also replace the sum by an integral over l.

The above sum thus simplifies to

$$e^{i\mathbf{p}\cdot\mathbf{r}}F_{\pm}=\frac{1}{pr}\sum_l i^l(2l+1)e^{\pm i\delta_l}u_l(r)J_0(l\vartheta)\times\begin{cases}1 & (z>0)\\ (-1)^l & (z<0),\end{cases}\tag{5a.1}$$

where from now on ϑ is the angle between $\mathbf{r}$ and $\mathbf{p}$ for $z>0$ and the angle between $\mathbf{r}$ and $-\mathbf{p}$ for $z<0$.

This method in its present form therefore cannot be applied to obtain good approximations to the wave functions in the other region ($|z|\sim\epsilon$, $\rho\sim\epsilon$). The latter case is treated by a different approach in Sec. Vb.

$u_l(r)$ is the solution of the radial equation

$$\left[\frac{d^2}{dr^2}+p^2-2\epsilon V-\frac{l(l+1)}{r^2}\right]u_l(r)=0.\tag{5a.2}$$

Using the WKB method as outlined in Appendix A, one finds, in the region $r\sim\epsilon$,

$$u_l(r)=\sin\left\{pr-\frac{l\pi}{2}+\frac{l^2}{2pr}+\delta_l+\int_r^\infty Vdr'+O(1/\epsilon)\right\}.\tag{5a.3}$$

We have here kept terms including those of order 1. Especially the term $l^2/2pr$ should be noted. This is necessary because the two radial functions from the initial and final states together with $\exp(-i\mathbf{k}\cdot\mathbf{r})$ from the electromagnetic field will give

$$\exp\{i(p_1-p_2)r-i\mathbf{k}\cdot\mathbf{r}\}=\exp\{i(p_1-p_2-k)r+ikr\vartheta^2/2\}$$

(for bremsstrahlung), i.e., terms of order 1 in the exponent. For details see Sec. 6, where the actual calculations are carried out.

The phase shift δ_l is [see Appendix A, (A.27)]

$$\delta_l=-\int_0^\infty V([\zeta^2+(l/p)^2]^{\frac{1}{2}})d\zeta.\tag{5a.4}$$

This expression is valid for electrons. For positrons we have only to change the sign of the charge,

$$\delta_l^{(\mathrm{pos})}=-\delta_l^{(\mathrm{el})}.$$

The wave functions for $z>0$ are, from (5a.1) and (5a.3),

$$e^{i\mathbf{p}\cdot\mathbf{r}}F_{\pm}=\frac{e^{ipr}}{ipr}\sum(l+\tfrac{1}{2})\times\exp\left(\pm i\delta_l+i\delta_l+\frac{il^2}{2pr}+i\int_r^\infty Vdr'\right)J_0(l\vartheta)-\frac{e^{-ipr}}{ipr}\sum(-1)^l(l+\tfrac{1}{2})\times\exp\left(\pm i\delta_l-i\delta_l-\frac{il^2}{2pr}-i\int_r^\infty Vdr'\right)J_0(l\vartheta).\tag{5a.5}$$

In (5a.5) the important difference between the terms in the first sum and those in the second is the additional factor $(-1)^l$ in the latter. In the first sum successive terms add constructively up to l of order ϵ. The individual terms being of order l, this sum is of order ϵ^2. However, in the second sum, because of the factor $(-1)^l$, successive terms will, for l of order ϵ, cancel and leave a contribution of order 1. The second sum is therefore of order ϵ and may be neglected.

Thus for $z>0$ the wave functions simplify to

$$e^{i\mathbf{p}\cdot\mathbf{r}}F_{\pm}=\frac{\exp\left(ipr+i\int_r^\infty Vdr'\right)}{ipr}\int ldl\times\exp[i(\pm\delta_l+\delta_l+l^2/2pr)]J_0(l\vartheta).\tag{5a.6}$$

In the ingoing type wave function, $e^{i\mathbf{p}\cdot\mathbf{r}}F_-$, the phase shifts cancel giving essentially a plane wave

$$e^{i\mathbf{p}\cdot\mathbf{r}}F_-=\frac{\exp\left(ipr+i\int_r^\infty Vdr'\right)}{ipr}\int ldl\times\exp(il^2/2pr)J_0(l\vartheta)=\exp\left[ipr\left(1-\frac{\vartheta^2}{2}\right)+i\int_r^\infty Vdr'\right]=\exp\left(i\mathbf{p}\cdot\mathbf{r}+i\int_r^\infty Vdr'\right).\tag{5a.7}$$

The normal scattering wave function, $e^{i\mathbf{p}\cdot\mathbf{r}}F_+$, is

$$e^{i\mathbf{p}\cdot\mathbf{r}}F_+=\frac{\exp\left(ipr+i\int_r^\infty Vdr'\right)}{ipr}\int ldl\times\exp(2i\delta_l+il^2/2pr)J_0(l\vartheta).\tag{5a.8}$$

(5a.7) shows that the plane wave is given as a superposition of only *outgoing* spherical waves in the present region, $z>0$, $r\sim\epsilon$, and small angles of order $1/\epsilon$ about $\mathbf{p}$. This may be expected by simple geometrical considerations: In this region spherical outgoing waves themselves are almost plane waves traveling in the direction of $\mathbf{p}$, away from the origin, while ingoing waves are almost plane waves moving in the opposite direction. The latter waves must therefore be absent in a description of the plane wave.

The plane wave part of the scattering state wave function $e^{i\mathbf{p}\cdot\mathbf{r}}F_+$ can likewise be represented as a superposition of only outgoing waves in this region. Thus this wave function has only outgoing waves; all ingoing waves are absent. This is indeed indicated by the result (5a.8).

The phase $i\int_r^\infty V dr'$ is of no importance to us as it drops out in the matrix elements. In fact, since this phase does not depend upon energy, and is the same for the outgoing type wave function as for the ingoing type [see (5a.7, 8) and later (5a.10, 11)], these phases from the initial and from the final state will exactly cancel in the case of bremsstrahlung. In pair production, where both particles are created, the corresponding factor in the matrix element would be, for $z>0$, $\exp[-i\int_r^\infty V_{\text{el}} dr' - i\int_r^\infty V_{\text{pos}} dr']$. Because of charge conservation, these phases again cancel. It is shown, moreover, in (6a.2), that derivatives of $\exp[i\int_r^\infty V dr']$ in the matrix element will always give contributions which are negligible.

We note [(A.21), ff.] that in the case of a pure Coulomb potential the term $i\int_r^\infty V dr'$ must be replaced by $ia\log 2r$ which is the well known anomalous term[11] in the asymptotic radial wave function, (5a.3). Again, in this case, the deformation of the plane wave is, according to (5a.7), $\exp[ia\log 2r]$. This is clearly effectively the same as the usual expression

$$\exp[ia\log(r+z)]$$

in the present region of small angles.

It is interesting to calculate the wave functions explicitly for the case of a pure Coulomb potential. Then $\delta_l = -a\log(l/p)$ [note (A.11), ff.] and

$$e^{i\mathbf{p}\cdot\mathbf{r}}F_+ = \frac{e^{ipr+ia\log 2r}}{ipr}\int l dl (l/p)^{-2ia} \times \exp(il^2/2pr) J_0(l\vartheta)$$
$$= p^{ia} e^{\frac{1}{2}\pi a}\Gamma(1-ia)\exp[ipr(1-\tfrac{1}{2}\vartheta^2)] \times {}_1F_1(ia;1;\tfrac{1}{2}ipr\vartheta^2). \quad (5a.9)$$

This is indeed the small-angle approximation to the exact Sommerfeld-Maue wave function

$$p^{ia}e^{\frac{1}{2}\pi a}\Gamma(1-ia)e^{i\mathbf{p}\cdot\mathbf{r}}\,{}_1F_1(ia;1;ipr-i\mathbf{p}\cdot\mathbf{r}).$$

The difference in normalization from customary usage has already been discussed in Sec. 5. The constant phase factor p^{ia} [note (A.11), ff.] is of course completely unimportant. It is a remarkable feature of the WKB method to give these accurate wave functions.

For $z<0$ we find corresponding to (5a.7, 8),

$$e^{i\mathbf{p}\cdot\mathbf{r}}F_- = -\frac{\exp\left(-ipr - i\int_r^\infty V dr'\right)}{ipr}\int l dl \times \exp(-2i\delta_l - il^2/2pr) J_0(l\vartheta), \quad (5a.10)$$

$$e^{i\mathbf{p}\cdot\mathbf{r}}F_+ = -\frac{\exp\left(-ipr - i\int_r^\infty V dr'\right)}{ipr}\int l dl \times \exp(-il^2/2pr) J_0(l\vartheta)$$
$$= \exp\left[-ipr\left(1-\frac{\vartheta^2}{2}\right) - i\int_r^\infty V dr'\right]$$
$$= \exp\left(i\mathbf{p}\cdot\mathbf{r} - i\int_r^\infty V dr'\right). \quad (5a.11)$$

It should be remembered that according to the definition (5a.1), ϑ is the angle between $\mathbf{r}$ and $-\mathbf{p}$ in the present region $z<0$, and that here $\mathbf{p}\cdot\mathbf{r} \approx -pr(1-\vartheta^2/2)$.

It is easy to see that the ingoing type wave function (5a.7 and 10) is indeed the space and time reversed outgoing type wave function (5a.8 and 11).

b. Wave Functions for $\rho \sim \varepsilon$, $|z| \sim \varepsilon$

This is the region of angles ϑ of order 1; hence we cannot use the method applied in Sec. 5a. In the present case, as we shall see, we may obtain a good approximation to the wave function by applying the WKB method to the wave equation (5.2) for F itself rather than to the radial wave equation, as was done in the case of small angles in Sec. 5a. We therefore write

$$F = e^{i\chi}. \quad (5b.1)$$

The equation for χ is then, from (5.2),

$$-i\nabla^2\chi + (\nabla\chi)^2 + 2\mathbf{p}\cdot\nabla\chi + 2\epsilon V = 0. \quad (5b.2)$$

Since both ρ and z are of order ϵ, both $\partial\chi/\partial\rho$ and $\partial\chi/\partial z$ are of order χ/ϵ, and the terms in (5b.2) are of orders χ/ϵ^2, χ^2/ϵ^2, χ and 1, respectively. It follows that χ is of order 1. Expanding χ in powers of $1/\epsilon$

$$\chi = \chi_0 + \chi_1 + \cdots,$$

one easily finds

$$\mathbf{p}\cdot\nabla\chi_0 + \epsilon V = 0,$$

and

$$2\mathbf{p}\cdot\nabla\chi_1 - i\nabla^2\chi_0 + (\nabla\chi_0)^2 = 0.$$

From this it then follows that

$$\chi_0 = -\frac{\epsilon}{p}\int_{-\infty}^{z} V(\rho,\zeta)d\zeta, \quad (5b.3)$$

which is of order 1, while χ_1 is of order $1/\epsilon^2$. As is shown in Sec. 7, it is really necessary that the error in χ be of order $1/\epsilon^2$ rather than $1/\epsilon$ for the approximation to be applicable. This is required since an important part of the difference $\chi_{\text{final}} - \chi_{\text{initial}}$ is of order $1/\epsilon$. In fact, this part accounts for the Born approximation result.

It should be pointed out that the procedure used here in Sec. 5b, i.e., a WKB method applied to the wave function F, is not feasible in the region considered

[11] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Clarendon Press, Oxford, 1949), second edition, p. 46.

in Sec. 5a, *viz.*, z of order ϵ but ρ of order 1. In this latter case each of the terms in (5b.2) is of the same order of magnitude and hence the first two terms cannot be neglected.

The χ_0 we have written down is zero for $z\to-\infty$. The wave function ψ in (5.1), therefore, is a plane wave in this limit, and thus describes the normal scattering state. It is shown in Appendix B that this state is really an approximation to the state satisfying the Sommerfeld radiation condition: plane wave plus outgoing spherical wave at infinity. We denote χ_0 in (5b.3) by χ_+ (dropping the subscript zero). But we shall also need ψ_-, the wave function describing a particle *leaving* the nucleus, which therefore must reduce to a plane wave for $z\to+\infty$. It is clear that the corresponding χ_- must have the form $(\epsilon/p)\int_z^\infty V(\rho,\zeta)d\zeta$. However, the sign has to be determined by space and time reversal of ψ_+. These two transformations applied to an "outgoing" state change it into an "ingoing" state.[12,13]

$$\psi_-(\mathbf{r},t)=ST\psi_+(\mathbf{r},t),$$

where $S=\beta\times$ space inversion and $T=i\sigma_y\times$ complex conjugation. It is easy to see that space and time reversal, ST, commutes with the factor

$$e^{i\mathbf{p}\cdot\mathbf{r}}\left(1-\frac{i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}}{2\epsilon}\right)$$

in the wave function ψ. The effect of the spin flip[13] caused by $\beta\sigma_y$ is of no importance to us, since we sum over spins. The only relevant change is in F: space and time reversal changes the sign of χ and of z. Since

$$\chi_+^{\mathrm{el}}(\rho,z)=-\frac{\epsilon}{p}\int_{-\infty}^{z}V(\rho,\zeta)d\zeta, \qquad (5b.4)$$

$$\chi_-^{\mathrm{el}}(\rho,z)=-\chi_+^{\mathrm{el}}(\rho,-z)=+\frac{\epsilon}{p}\int_{z}^{\infty}V(\rho,\zeta)d\zeta. \qquad (5b.5)$$

The expression for χ and the phase shift δ_l of Sec. Va look very much the same. Their appearance in the wave functions ψ are however completely different: while δ_l is the usual scattering phase shift, $\chi(\rho,z)$ is not.

The χ of (5b.4) and (5b.5) describe electrons. The corresponding χ for positrons is obtained by reversing the sign of the charge:

$$\chi_\pm^{\mathrm{pos}}(\rho,z)=-\chi_\pm^{\mathrm{el}}(\rho,z). \qquad (5b.6)$$

6. MATRIX ELEMENTS FOR "LARGE" VALUES OF THE MOMENTUM TRANSFER, ($q\sim1$)

In this section we calculate the bremsstrahlung and pair production matrix elements in the upper range of values of the momentum transfer, $q\sim1$. We then use the wave functions for $\rho\sim1$, $|z|\sim\epsilon$ according to Sec. 5a.

[12] Rose, Biedenharn, and Arfken, Phys. Rev. **85**, 5 (1952).
[13] Haakon Olsen, Kgl. Norske Videnskab. Selskab, Forh. **28**, 10 (1955).

a. Bremsstrahlung

Here the initial state is a normal scattering state, ψ_+, while the final state has the asymptotic form: plane wave plus *ingoing* spherical waves, ψ_-. The matrix element is

$$\int\psi_{\mathrm{el},-}^\dagger H_{\mathrm{rad}}'\psi_{\mathrm{el},+}d^3r.$$

In pair production, when both particles are produced, both states in the matrix element are "ingoing" states, ψ_-. As will now be seen directly, this causes a big difference between the *differential* cross sections for bremsstrahlung and for pair production.

The I's of B-M, which we write I^{-+} to indicate the types of the initial and final states, are found by using the expressions for $e^{i\mathbf{p}\cdot\mathbf{r}}F_\pm$ from (5a.7, 8) and (5a.10, 11):

$$I_1^{-+}=\frac{1}{p_1p_2}\int ldl\int l'dl'\Bigg\{\int_{z>0}\frac{d^3r}{r^2}J_0(l\vartheta_1)J_0(l'\vartheta_2)\times\exp\left(i\delta r+\frac{ikr}{2}\vartheta^2\right)\exp\left[i\left(\frac{l^2}{2p_1r}-\frac{l'^2}{2p_2r}\right)+2i\delta_l\right]$$
$$+\int_{z<0}\frac{d^3r}{r^2}J_0(l\vartheta_1)J_0(l'\vartheta_2)\exp\left(-i\delta r-\frac{ikr}{2}\vartheta^2\right)\times\exp\left[-i\left(\frac{l^2}{2p_1r}-\frac{l'^2}{2p_2r}\right)+2i\delta_{l'}\right]\Bigg\}, \qquad (6a.1)$$

where $\delta=p_1-p_2-k$ for bremsstrahlung. ϑ_1 and ϑ_2 are the angles between $\mathbf{r}$ and $\mathbf{p}_1$, and $\mathbf{r}$ and $\mathbf{p}_2$, respectively, for $z>0$ (the first integral), and between $\mathbf{r}$ and $-\mathbf{p}_1$, and $\mathbf{r}$ and $-\mathbf{p}_2$, respectively, for $z<0$ (the second integral). They are always assumed to be small of order $1/\epsilon$.

We may write

$$I_1^{-+}=I_1(z>0)+I_1(z<0),$$

where $I_1(z>0)$ is given by the first term in (6a.1) and $I_1(z<0)$ by the second.

Before calculating $I_1(z>0)$ and $I_1(z<0)$, we consider the expressions for $\mathbf{I}_2$ and $\mathbf{I}_3$. It has been shown in B-M [Eq. (7.12)] that only the components of $\mathbf{I}_2$ and $\mathbf{I}_3$ perpendicular to $\mathbf{k}$, $\mathbf{I}_{2\perp}$ and $\mathbf{I}_{3\perp}$, are of any importance for high energies.

According to the definition of $\mathbf{I}_2$, (4.2), we find, using the wave functions (5a.7, 8) and (5a.10, 11):

$$\mathbf{I}_{2\perp}=\frac{-i}{2\epsilon_1}\left[\int_{z>0}d^3r\exp\left(-i\int_r^\infty Vdr'\right)e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\nabla}_\perp F_{1,+}+\int_{z<0}d^3rF_{2,-}^*e^{i\mathbf{q}\cdot\mathbf{r}}\boldsymbol{\nabla}_\perp\exp\left(-i\int_r^\infty Vdr'\right)\right].$$

The magnitude of the second integral relative to the

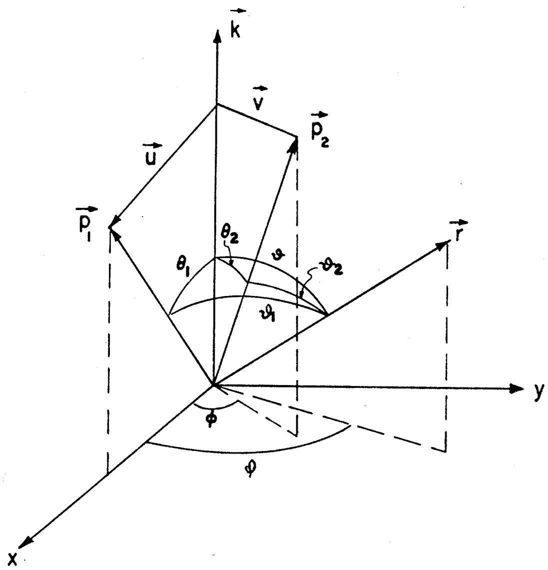

FIG. 1. Angles occuring in the matrix elements for momentum transfer $q\sim 1$.

first one is

$$\frac{\rho}{r}VF\bigg/\frac{\partial F}{\partial\rho}\sim\frac{1}{\epsilon^2},$$

and may be neglected. Performing a partial integration on the first integral we find, again neglecting a term of relative order $1/\epsilon^2$:

$$\mathbf{I}_{2\perp}=\frac{-\mathbf{q}_\perp}{2\epsilon_1}\int_{z>0}d^3re^{i\mathbf{q}\cdot\mathbf{r}}F_{1,+}=\frac{-\mathbf{q}_\perp}{2\epsilon_1}I_1(z>0).\quad(6\text{a}.2)$$

Similarly

$$\mathbf{I}_{3\perp}=\frac{\mathbf{q}_\perp}{2\epsilon_2}I_1(z<0).\quad(6\text{a}.3)$$

For the actual calculation of the I's, we recall the definition

$$I_1(z>0)=\frac{1}{p_1p_2}\int ldl\int l'dl'\int_{z>0}\frac{d^3r}{r^2}J_0(l\vartheta_1)J_0(l'\vartheta_2)$$
$$\times\exp\left[i\delta r+\frac{ikr}{2}\vartheta^2+i\left(\frac{l^2}{2p_1r}-\frac{l'^2}{2p_2r}\right)+2i\delta_l\right].\quad(6\text{a}.4)$$

Now we express $J_0(l\vartheta_1)$ and $J_0(l'\vartheta_2)$ by the addition theorem

$$J_0(l\vartheta_1)=\sum_{m=-\infty}^{+\infty}J_m(l\vartheta)J_m(l\theta_1)e^{im(\varphi-\phi_1)},$$
$$(6\text{a}.5)$$
$$J_0(l'\vartheta_2)=\sum_{m'=-\infty}^{+\infty}J_{m'}(l'\vartheta)J_{m'}(l'\theta_2)e^{im'(\varphi-\phi_2)}.$$

The angles are indicated in Fig. 1. Because of the rapid oscillations of the integrand in (6a.4) for large values of ϑ, the integration over ϑ is extended to ∞. We now perform the straightforward integrals over φ, ϑ and l' in this order, using integrals[14] of the form

$$\int_0^\infty\vartheta d\vartheta J_m(l\vartheta)J_m(l'\vartheta)\exp\left(i\frac{kr\vartheta^2}{2}\right)$$
$$=\frac{i}{kr}\exp\left(-i\frac{l^2+l'^2}{2kr}\right)J_m\left(\frac{ll'}{kr}\right)\exp\left(\frac{im\pi}{2}\right).\quad(6\text{a}.6)$$

This gives

$$I_1(z>0)=\frac{2\pi}{p_1^2}\int ldle^{2i\delta_l}\sum_m J_m(l\theta_1)J_m\left(l\theta_2\frac{p_2}{p_1}\right)e^{im\phi}$$
$$\times\int dr\exp\left[i\left(\delta+\frac{\theta_2^2p_2k}{2p_1}\right)r\right].\quad(6\text{a}.7)$$

Now the sum over m can be performed, giving $J_0(lq_\perp/p_1)$, where

$$q_\perp^2=p_1^2\theta_1^2+p_2^2\theta_2^2-2p_1p_2\theta_1\theta_2\cos\phi\equiv(\mathbf{u}-\mathbf{v})^2.$$

Thus the integral becomes

$$-\frac{4\pi\epsilon_2}{ik\epsilon_1(1+p_2^2\theta_2^2)}\int_0^\infty ldle^{2i\delta_l}J_0(lq_\perp/p_1).\quad(6\text{a}.8)$$

In the l integral only $l\sim p_1/q_\perp\sim p_1$ will contribute. Accordingly in the phase shift [see (7b.12) and (A.27)]

$$\delta_l=\lim_{\eta\to 0}+a\int_0^\infty\frac{xdx}{x^2+\eta^2}[1-F(x)]J_0(xl/p_1)\quad(6\text{a}.9)$$

only values of the argument x of order 1 in the atomic form factor $F(x)$ will be of importance. But for such values $F(x)=0$, so that

$$\delta_l=-a\log(l/p)\quad(6\text{a}.10)$$

in the present range, giving

$$\int_0^\infty ldl\left(\frac{l}{p_1}\right)^{-2ia}J_0\left(l\frac{q_\perp}{p_1}\right)=\frac{2p_1^2}{q_\perp^2}\left(\frac{q_\perp^2}{4}\right)^{ia}\frac{\Gamma(1-ia)}{\Gamma(ia)}.$$

Thus

$$I_1(z>0)=\frac{\Gamma(1-ia)}{\Gamma(1+ia)}\left(\frac{q_\perp^2}{4}\right)^{ia}$$
$$\times\left(-\frac{8\pi a}{q_\perp^2}\right)\frac{\epsilon_1}{(k/\epsilon_2)(1+p_2^2\theta_2^2)}.\quad(6\text{a}.11)$$

$I_1(z<0)$ is obtained from $I_1(z>0)$ by the transformations $p_1\rightleftarrows+p_2$, $k\to-k$, $\theta_1\rightleftarrows\theta_2$, as is clear from the

[14] W. Magnus and F. Oberhettinger, *Formeln und Sätze für die Speziellen Funktionen der mathematischen Physik* (Springer-Verlag, Berlin, 1948), second edition, p. 49 and reference 18, p. 395.

definition. Thus

$$I_1(z<0)=\frac{\Gamma(1-ia)}{\Gamma(1+ia)}\left(\frac{q_\perp^2}{4}\right)^{ia}\times\left(-\frac{8\pi a}{q_\perp^2}\right)\frac{-\epsilon_2}{(k/\epsilon_1)(1+p_1^2\theta_1^2)}. \quad (6a.12)$$

The three I's are, therefore, in the present range $q\sim 1$, equal to the Born approximation values apart from the unimportant common phase factor

$$(q_\perp^2/4)^{ia}\Gamma(1-ia)/\Gamma(1+ia).$$

This is easily seen by comparison with the Born approximation expressions (4.2) for the case of no screening, $V=-a/r$, as then

$$\int e^{i\mathbf{q}\cdot\mathbf{r}}V(r)d^3r=-4\pi a/q^2.$$

The present result, including the phase factor, coincides with the result of B-M, apart from the factor $\Gamma(1-ia)/\Gamma(1+ia)$ which appears because we have used different normalization than in B-M [see (5.3)].

It is interesting to note that the Born approximation cross section comes out, although the wave functions are not the Born approximation wave functions. In fact, it was shown in Sec. 5a that our wave function $e^{i\mathbf{p}\cdot\mathbf{r}}F_\pm$ is, in the case of no screening, the small angle approximation of the exact wave function. This is analogous to and closely related to the fact that the Born approximation cross section is exact for the scattering of a (spinless) particle in an unscreened Coulomb potential, and in the case of small angle scattering even for a particle with spin. The case of scattering will be discussed in greater detail in Sec. 9.

The present effect is probably the reason why the correction to the Born approximation is so much smaller than expected. In the case of lead, for instance, the Coulomb correction amounts to about 10% [15] instead of $a^2=36\%$.

b. Pair Production

The matrix element

$$\int \psi_{\mathrm{el},-}^{\dagger}H_{\mathrm{el\ mag}}'\psi_{\mathrm{pos},-}^{*}d^3r$$

has ingoing type wave functions in both initial and final states. The I integrals are accordingly I^{--}. In the present case it is advantageous to consider I_0 and derive I_1, $\mathbf{I}_2$, and $\mathbf{I}_3$ by means of the rules given in B-M. This is possible since screening is unimportant in the present range. We will accordingly immediately use the phase shifts for an unscreened Coulomb potential $\mp a/r$:

$$\delta_l^{\mathrm{el}}\equiv\delta_l=-a\log(l/p)+\mathrm{const},$$
$$\delta_l^{\mathrm{pos}}=+a\log(l/p)+\mathrm{const}, \quad (6b.1)$$

as given by (5a.4).

Using the wave functions (5a.7) and (5a.10), one easily finds

$$I_0^{--}=\int_{z>0}e^{i\mathbf{q}\cdot\mathbf{r}}\frac{d^3r}{r}+\int ldl\int l'dl'\int_{z<0}\frac{d^3r}{r^3}J_0(l\vartheta_1)J_0(l'\vartheta_2)\times\exp\left[-i\delta r+\frac{ikr\vartheta^2}{2}+i\left(\frac{l^2}{2p_1r}+\frac{l'^2}{2p_2r}\right)+2i\delta_l^{\mathrm{pos}}+2i\delta_{l'}^{\mathrm{el}}\right]. \quad (6b.2)$$

Here $\delta=k-p_1-p_2$ for pair production. Since, as was shown in Sec. 5a there are no scattered waves in the forward direction, the phases drop completely out of the integral for $z>0$. In fact, the contribution to I_1 from the first term in I_0 would be

$$\int_{z>0}d^3re^{i\mathbf{q}\cdot\mathbf{r}}\sim\delta(\mathbf{q}_\perp),$$

which is zero in the present region, $q\sim 1$.

It was shown in (5a.9) that our wave functions are the small angle approximation of the exact Sommerfeld-Maue wave functions. The integral I_0^{--} may therefore be evaluated using the method first given by Sommerfeld.[3] (See also Bess[8] and Nordsieck.[16]) The result could be written down immediately and must coincide with the result of B-M. However, the integrals in their present form are much simpler, because of the small angle approximation. The calculation is somewhat analogous to Bess's computation. There is no need of introducing parabolic coordinates, however, which simplifies the calculation considerably.

Using the addition theorem (6a.5) as in the case of bremsstrahlung, the integrals over φ and ϑ are readily done by means of (6a.6).

In the case of bremsstrahlung, as can be seen from the position of the terms $\exp(2i\delta_l)$ and $\exp(2i\delta_{l'})$ in (6a.1), only the initial state wave function is scattered in the region $z>0$ and only the final state wave function in the region $z<0$. This is the situation which obtains for simple scattering and leads us in that case, as well as for bremsstrahlung, to the Born approximation matrix element. For pair production, however, both states are scattered in the region $z<0$ and this gives a result different from the Born approximation.

It is convenient to introduce as a new variable the ratio of the two angular momenta, $s=l'/l$. This removes the phase from one of the l integrals. From (6b.2) we

[15] Davies, Bethe, and Maximon, Phys. Rev. **93**, 788 (1954).

[16] A. T. Nordsieck, Phys. Rev. **93**, 785 (1954).

find, therefore,

$$I_0^{--}=\frac{2\pi i}{p_1p_2k}\left(\frac{p_2}{p_1}\right)^{2ia}\sum_m\int s^{1-2ia}ds\int l^3dl\int\frac{dr}{r^2}$$
$$\times J_m\left(\frac{l^2s}{kr}\right)J_m(l\theta_1)J_m(ls\theta_2)$$
$$\times\exp\left[\frac{il^2}{2kr}\left(\frac{p_2}{p_1}+s^2\frac{p_1}{p_2}\right)-i\delta r+im\left(\phi+\frac{\pi}{2}\right)\right].$$

Introducing now the new variable $\xi=l^2/r$ instead of r, the integral over l involves only the product of *two* Bessel functions and can thus be done, again using (6a.6). Then the sum over m can be performed as in bremsstrahlung, giving

$$I_0^{--}=\frac{\pi}{p_1p_2k\delta}\int s^{1-2ia}ds\int\xi d\xi$$
$$\times\exp\left[\frac{i\xi}{4k\delta}\left(\frac{D_1}{p_1}+s^2\frac{D_2}{p_2}\right)\right]J_0(\alpha s\xi)\left(\frac{p_2}{p_1}\right)^{2ia}.$$

Here

$$\alpha^2=\frac{1}{k^2}+\left(\frac{\theta_1\theta_2}{2\delta}\right)^2-\frac{\theta_1\theta_2}{\delta k}\cos\phi,$$

$$D_1=kp_1\theta_1^2+2p_2\delta\quad\text{and}\quad D_2=kp_2\theta_2^2+2p_1\delta.$$

The ξ integral is elementary, yielding

$$I_0^{--}=\frac{-16i\pi k\delta}{p_1p_2}\int s^{1-2ia}ds\left(\frac{D_1}{p_1}+\frac{D_2}{p_2}s^2\right)$$
$$\times\left[(4k\delta\alpha s)^2-\left(\frac{D_1}{p_1}+\frac{D_2}{p_2}s^2\right)^2\right]^{-\frac{3}{2}}\left(\frac{p_2}{p_1}\right)^{2ia}.$$

Introducing the new variable $t=s^2(D_2/D_1)(p_1/p_2)$, the integral is seen to be a function of only one variable, $x=4\alpha^2p_1p_2k^2\delta^2/D_1D_2$. With these new variables the integral simplifies to

$$I_0^{--}=\frac{8\pi k\delta}{D_1D_2}\left(\frac{p_2D_2}{p_1D_1}\right)^{ia}\int_0^\infty dt\,t^{-ia}(1+t)^{-2}\times\left[1-\frac{4xt}{(1+t)^2}\right]^{-\frac{3}{2}}.$$

By expanding $[1-4xt/(1+t)^2]^{-\frac{3}{2}}$, the pertinent integrals are very simple:

$$\int_0^\infty t^{-ia+n}(1+t)^{-2n-2}dt=\frac{\Gamma(1+ia+n)\Gamma(1-ia+n)}{\Gamma(2n+2)},$$

and accordingly

$$I_0^{--}=\frac{8\pi k\delta}{D_1D_2}\left(\frac{p_2D_2}{p_1D_1}\right)^{ia}|\Gamma(1+ia)|^2F(1+ia,1-ia;1;x),$$

F being the hypergeometric function. Using the transformation

$$F(1-ia,1+ia;1;x)=\frac{1}{1-x}F(ia,-ia;1;x),$$

and the fact that

$$1-x=\frac{1}{D_1D_2}(D_1D_2-4\alpha^2k^2p_1p_2\delta^2)=\frac{2k\delta}{D_1D_2}q_\perp^2,$$

we may write

$$I_0^{--}=\frac{4\pi}{q_\perp^2}\left(\frac{p_2D_2}{p_1D_1}\right)^{ia}|\Gamma(1+ia)|^2F(ia,-ia;1;x).\quad(6\text{b}.3)$$

$q_\perp$ is the transverse momentum transfer: $\mathbf{q}_\perp=-\mathbf{u}-\mathbf{v}$.

To derive the quantity I_1^{--} it should be noticed that $I_1=i\partial I_0/\partial\delta$; thus δ (as well as $\mathbf{p}_1$ and $\mathbf{p}_2$) has been kept as an independent variable. In contrast to the Sommerfeld rules[1,3] no extra parameter λ is therefore necessary. This simplifies the derivation of I_1.

From the definitions it follows that

$$\frac{\partial D_1}{\partial\delta}=2p_2,\quad\frac{\partial D_2}{\partial\delta}=2p_1,$$
$$\frac{\partial x}{\partial\delta}=-\frac{2kq_\perp^2}{D_1D_2}\left(1-\frac{2p_2\delta}{D_1}-\frac{2p_1\delta}{D_2}\right),\quad(6\text{b}.4)$$

and I_1^{--} is easily written down. ($V=F$, $W=a^{-2}dF/dx$ as in B-M.)

$$I_1^{--}=\frac{8\pi a}{q_\perp^2}\left(\frac{p_2D_2}{p_1D_1}\right)^{ia}|\Gamma(1+ia)|^2\left\{V\left(\frac{p_2}{D_1}-\frac{p_1}{D_2}\right)-ia\frac{q_\perp^2kW}{D_1D_2}\left(1-\frac{2p_1\delta}{D_2}-\frac{2p_2\delta}{D_1}\right)\right\}.\quad(6\text{b}.5)$$

Before $\mathbf{I}_2$ and $\mathbf{I}_3$ can be found by the differentiations

$$\left(\frac{p_1}{p_2}\right)^{ia}\mathbf{I}_2=\frac{i}{2}\nabla_{p1}\left[\left(\frac{p_1}{p_2}\right)^{ia}I_0\right],$$
$$\left(\frac{p_1}{p_2}\right)^{ia}\mathbf{I}_3=\frac{i}{2}\nabla_{p2}\left[\left(\frac{p_1}{p_2}\right)^{ia}I_0\right]$$

one should substitute $\delta=k-p_1-p_2$ and $\mathbf{k}=\mathbf{q}+\mathbf{p}_1+\mathbf{p}_2$ in accordance with the fact that I_0 should be a function of $\mathbf{q}$, $\mathbf{p}_1$ and $\mathbf{p}_2$ in order that the Sommerfeld rules may be applied. The factor $(p_1/p_2)^{ia}$ should also be noticed. It comes from the different normalization condition for the wave functions.

As in bremsstrahlung, Sec. 6a, we shall only be interested in the components of $\mathbf{I}_2$ and $\mathbf{I}_3$ perpendicular to $\mathbf{k}$, as only these are of importance for high energies [see B-M, Eq. (7.12)]. It should also be noticed that, the z components I_{2z} and I_{3z} being of relative order $1/\epsilon$

compared to $\mathbf{I}_{2\perp}$ and $\mathbf{I}_{3\perp}$, they do not follow correctly from our approximation.

One easily finds

$$\nabla_{p_1}k=\mathbf{k}/k,\ \nabla_{p_1}\delta=\mathbf{k}/k-\mathbf{p}_1/p_1,\ (\nabla_{p_1}D_2)_\perp=2\mathbf{q}_\perp,$$

$$(\nabla_{p_1}x)_\perp=\frac{2kq_\perp^2}{D_1D_2}\left[\frac{\mathbf{u}}{p_1}+\frac{2\delta\mathbf{q}_\perp}{D_2}\right],\quad (\nabla_{p_2}x)_\perp=\frac{2kq_\perp^2}{D_1D_2}\left[\frac{\mathbf{v}}{p_2}+\frac{2\delta\mathbf{q}_\perp}{D_1}\right].$$

Thus

$$\mathbf{I}_{2\perp}=\frac{4\pi a}{q_\perp^2}\left(\frac{p_2D_2}{p_1D_1}\right)^{ia}|\Gamma(1+ia)|^2 \times\left\{-V\frac{\mathbf{q}_\perp}{D_2}+ia\frac{q_\perp^2k}{D_1D_2}W\left(\frac{\mathbf{u}}{p_1}+\frac{2\delta\mathbf{q}_\perp}{D_2}\right)\right\},$$

$$\mathbf{I}_{3\perp}=\frac{4\pi a}{q_\perp^2}\left(\frac{p_2D_2}{p_1D_1}\right)^{ia}|\Gamma(1+ia)|^2 \times\left\{V\frac{\mathbf{q}_\perp}{D_1}+ia\frac{q_\perp^2k}{D_1D_2}W\left(\frac{\mathbf{v}}{p_2}+\frac{2\delta\mathbf{q}_\perp}{D_1}\right)\right\}.$$

It should be noticed that the sign of $\mathbf{I}_2$ in B-M Eqs. (6.3a) and (6.10) is incorrect[17]; the final expression for $\mathbf{I}_2$ in (6.23) is however, correct. Our result is consistent with the B-M result [B-M Eq. (6.23)]. It is comforting that our result agrees, to first order in a, with the Born approximation for pair production, as obtained from the bremsstrahlung result (4.2) by changing the sign of ϵ_1.

7. MATRIX ELEMENTS FOR SMALL VALUES OF THE MOMENTUM TRANSFER ($q\sim 1/\epsilon$)

We now calculate the matrix elements for pair production and bremsstrahlung for the range of momentum transfers $q\sim 1/\epsilon$. (See Fig. 2.) The appropriate wave functions are given by (5b.1) and (5b.4–6). In the present region screening will be of importance, and the matrix elements will be given for arbitrary screening. We again start with the simplest case, which here is pair production.

a. Pair Production

The matrix element I_1 is

$$I_1^{--}=\int \exp(-i\chi_2^{\rm el})\exp(i\mathbf{q}\cdot\mathbf{r})\exp(-i\chi_1^{\rm pos})d^3r,$$

where $\chi_2^{\rm el}+\chi_1^{\rm pos}\equiv\chi_{2,-}^{\rm el}+\chi_{1,-}^{\rm pos}$ as ingoing waves have to be taken for both states.

We now separate the difference in energy from the difference in directions of $\mathbf{p}_1$ and $\mathbf{p}_2$ in the χ's:

$$\chi_2^{\rm el}(\rho_2,z_2)+\chi_1^{\rm pos}(\rho_1,z_1)=\chi_1^{\rm pos}(\rho_1,z_1)+\chi_2^{\rm el}(\rho_1,z_1) -\chi_2^{\rm el}(\rho_1,z_1)+\chi_2^{\rm el}(\rho_2,z_2). \quad (7a.1)$$

It should be remembered that $\chi^{\rm pos}=-\chi^{\rm el}$. Since the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$ is small, of order $1/\epsilon$, and both ρ and z are of order ϵ in the present region we may expand, taking for the moment the direction of $\mathbf{p}_2$ as

[17] This is also clear from B-M Eqs. (6.10)–(6.12).

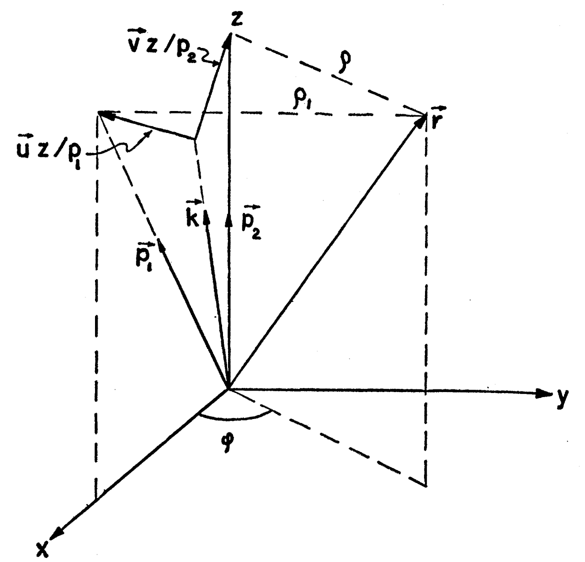

FIG. 2. Cylindrical coordinates used in the matrix elements for momentum transfer $q\sim 1/\epsilon$.

z axis, and omitting the subscript 2:

$$\chi_2(\rho_2,z_2)-\chi_2(\rho_1,z_1)=\frac{\partial\chi_2}{\partial\rho}(\rho-\rho_1)+\frac{\partial\chi_2}{\partial z}(z-z_1)+O(1/\epsilon^2).$$

Using, from geometry,

$$\rho(\rho-\rho_1)=-z(z-z_1),\quad \rho-\rho_1=\frac{z}{\rho}\boldsymbol{\rho}\cdot\left(\frac{\mathbf{u}}{p_1}-\frac{\mathbf{v}}{p_2}\right),\quad (7a.2)$$

we find

$$-\chi_{2,-}^{\rm el}-\chi_{1,-}^{\rm pos}=-\left(\frac{\epsilon_2}{p_2}-\frac{\epsilon_1}{p_1}\right)\int_z^\infty V(\rho,\zeta)d\zeta -\left(\frac{\boldsymbol{\rho}\cdot\mathbf{u}}{p_1}-\frac{\boldsymbol{\rho}\cdot\mathbf{v}}{p_2}\right)\left[\frac{z}{\rho}\frac{\partial}{\partial\rho}-\frac{\partial}{\partial z}\right]\int_z^\infty V(\rho,\zeta)d\zeta. \quad (7a.3)$$

The first term here is of order $1/\epsilon^2$ and should accordingly be dropped, while the second term is of order $1/\epsilon$. Thus we may expand the exponent, giving

$$I_1^{--}=-i\int d^3re^{i\mathbf{q}\cdot\mathbf{r}}\left(\frac{\boldsymbol{\rho}\cdot\mathbf{u}}{p_1}-\frac{\boldsymbol{\rho}\cdot\mathbf{v}}{p_2}\right) \times\left[\frac{z}{\rho}\frac{\partial}{\partial\rho}-\frac{\partial}{\partial z}\right]\int_z^\infty V(\rho,\zeta)d\zeta. \quad (7a.4)$$

We have here dropped a term proportional to $\delta(\mathbf{q})$, which is always zero. Since I_1 as given in (7a.4) is proportional to the charge number, it must be the Born approximation result. In fact, this may be written

$$I_1^{--}=-i\int d^3re^{i\mathbf{q}\cdot\mathbf{r}}\left(\frac{\boldsymbol{\rho}\cdot\mathbf{u}}{p_1}-\frac{\boldsymbol{\rho}\cdot\mathbf{v}}{p_2}\right)\frac{1}{\rho}\frac{\partial}{\partial\rho} \times\int_z^\infty (z-\zeta)V(\rho,\zeta)d\zeta,$$

using the fact that for a spherically symmetric potential

$$\frac{1}{\zeta}\frac{\partial}{\partial\zeta}V[(\rho^2+\zeta^2)^{\frac{1}{2}}]=\frac{1}{\rho}\frac{\partial}{\partial\rho}V[(\rho^2+\zeta^2)^{\frac{1}{2}}].$$

Now the φ integration gives

$$I_1^{--}=\frac{1}{q_\perp}\left(\frac{\mathbf{q}_\perp\cdot\mathbf{u}}{p_1}-\frac{\mathbf{q}_\perp\cdot\mathbf{v}}{p_2}\right)2\pi\int\rho d\rho J_1(q_\perp\rho)\frac{\partial}{\partial\rho}\times\int dz e^{iq_z z}\int_z^\infty(z-\zeta)V(\rho,\zeta)d\zeta.$$

Integrating by parts once with respect to ρ and twice with respect to z, we are left with

$$I_1^{--}=-\frac{1}{q_z^2}\left(\frac{\mathbf{q}_\perp\cdot\mathbf{u}}{p_1}-\frac{\mathbf{q}_\perp\cdot\mathbf{v}}{p_2}\right)2\pi\int\rho d\rho J_0(q_\perp\rho)\times\int dz V(r)e^{iq_z z}\quad(7a.5)$$

$$=-\frac{1}{q_z^2}\left(\frac{\mathbf{q}_\perp\cdot\mathbf{u}}{p_1}-\frac{\mathbf{q}_\perp\cdot\mathbf{v}}{p_2}\right)\int d^3r e^{i\mathbf{q}\cdot\mathbf{r}}V(r).$$

In order to compare with the Born approximation we substitute $\mathbf{q}_\perp=-\mathbf{u}-\mathbf{v}$, taking into account the fact that for $q\sim1/\epsilon$:

$$u-v=O(1/\epsilon)\quad(7a.6)$$

angle between $\mathbf{u}$ and $-\mathbf{v}=O(1/\epsilon)$.

Therefore

$$-\frac{1}{q_z^2}\left(\frac{\mathbf{q}_\perp\cdot\mathbf{u}}{p_1}-\frac{\mathbf{q}_\perp\cdot\mathbf{v}}{p_2}\right)=\frac{k(u-v)u}{q_z^2p_1p_2}.\quad(7a.7)$$

On the other hand, since

$$q_z=\frac{1+u^2}{2\epsilon_1}+\frac{1+v^2}{2\epsilon_2}=\frac{k}{2\epsilon_1\epsilon_2}(1+u^2),\quad D_1=2\epsilon_2q_z,\quad D_2=2\epsilon_1q_z,$$

the factor in the Born approximation for pair production corresponding to (4.2) is, in the present region,

$$2\left(\frac{\epsilon_1}{D_2}-\frac{\epsilon_2}{D_1}\right)=\frac{ku(u-v)}{q_z^2p_1p_2},\quad(7a.8)$$

which indeed coincides with (7a.7).

The calculation of $\mathbf{I}_2$ and $\mathbf{I}_3$ is even simpler. Dropping terms of relative order $1/\epsilon$, we find

$$\mathbf{I}_2^{--}=\frac{i}{2\epsilon_1}\int\exp(-i\chi_2^{\mathrm{el}}+i\mathbf{q}\cdot\mathbf{r})\nabla\exp(-i\chi_1^{\mathrm{pos}})$$
$$=\frac{1}{2\epsilon_1}\int e^{i\mathbf{q}\cdot\mathbf{r}}\nabla\chi_1^{\mathrm{pos}}d^3r=\frac{-i\mathbf{q}}{2\epsilon_1}\int e^{i\mathbf{q}\cdot\mathbf{r}}\chi_1^{\mathrm{pos}}d^3r,$$

the last step by partial integration. Substituting for χ the expression

$$\chi_{1,-}^{\mathrm{pos}}=-\int_z^\infty V(\rho,\zeta)d\zeta,$$

and performing a partial integration with respect to z, we obtain

$$\mathbf{I}_2=\frac{\mathbf{q}}{2p_2q_z}\int e^{i\mathbf{q}\cdot\mathbf{r}}Vd^3r,\quad(7a.9)$$

which is the Born approximation result ($D_2=2p_1q_z$). In the same way

$$\mathbf{I}_3=\frac{-\mathbf{q}}{2p_1q_z}\int e^{i\mathbf{q}\cdot\mathbf{r}}Vd^3r.\quad(7a.10)$$

We have thus proved explicitly that the pair production cross section is given by the Born approximation in the region $q\sim1/\epsilon$, for any amount of screening, a result which was anticipated in B-M.

It is interesting to note that the terms of order 1 in the phases $\int_z^\infty V(\rho,\zeta)d\zeta$ from the initial and final states just cancel [see (7a.3)], leaving exactly the *Born approximation matrix element*, (7a.5). This is true for any potential $V(r)$. This is then in a certain sense to be contrasted to the case of bremsstrahlung for $q\sim1$, in which case we also obtained the Born approximation result. In that case it was only because the potential was the pure Coulomb potential, $V=-a/r$.

b. Bremsstrahlung

Here the final state describes an electron leaving the nucleus so that the matrix element is

$$I_1^{-+}=\int\exp(-i\chi_{2,-}^{\mathrm{el}})e^{i\mathbf{q}\cdot\mathbf{r}}\exp(i\chi_{1,+}^{\mathrm{el}})d^3r.$$

Handling the phases in a manner analogous to (7a.1) we find

$$\chi_{1,+}-\chi_{2,-}=\chi_{1,+}(\rho_1,z_1)-\chi_{2,-}(\rho_1,z_1)+\chi_{2,-}(\rho_1,z_1)-\chi_{2,-}(\rho_2,z_2)$$
$$=-\frac{\epsilon_1}{p_1}\int_{-\infty}^z Vd\zeta-\frac{\epsilon_2}{p_2}\int_z^\infty Vd\zeta-(\rho-\rho_1)\left[\frac{\partial}{\partial\rho}-\frac{\rho}{z}\frac{\partial}{\partial z}\right]\int_z^\infty V(\rho,\zeta)d\zeta.\quad(7b.1)$$

The first two terms add up to $\int_{-\infty}^\infty V(\rho,\zeta)d\zeta$. This is the important difference when compared with pair production, (7a.3), where this term was absent. Using the relation (7a.2) we may write

$$\chi_{1,+}-\chi_{2,-}=-\int_{-\infty}^\infty Vd\zeta-\left(\frac{\boldsymbol{\rho}\cdot\mathbf{u}}{p_1}-\frac{\boldsymbol{\rho}\cdot\mathbf{v}}{p_2}\right)\times\left[\frac{z}{\rho}\frac{\partial}{\partial\rho}-\frac{\partial}{\partial z}\right]\int_z^\infty V(\rho,\zeta)d\zeta.\quad(7b.2)$$

The first term here is of order 1 and must be kept in the exponent, while the second is of order $1/\epsilon$, and may be expanded as in (7a.4):

$$I_1{}^{-+}=\int d^3r \exp\left[i\mathbf{q}\cdot\mathbf{r}-i\int_{-\infty}^{\infty} V(\rho,\zeta)d\zeta\right]$$
$$\times\left[1-i\left(\frac{\boldsymbol{\varrho}\cdot\mathbf{u}}{p_1}-\frac{\boldsymbol{\varrho}\cdot\mathbf{v}}{p_2}\right)\left[\frac{z}{\rho}\frac{\partial}{\partial\rho}-\frac{\partial}{\partial z}\right]\times\int_z^{\infty} V(\rho,\zeta)d\zeta\right]. \quad (7b.3)$$

The first term is proportional to a δ function:

$$2\pi\delta(q_z)\int d^2\rho \exp\left[i\mathbf{q}_\perp\cdot\boldsymbol{\varrho}-i\int_{-\infty}^{\infty} V(\rho,\zeta)d\zeta\right] \quad (7b.4)$$

and is to be dropped. This term will be discussed further in Sec. 9 for the case of scattering.

In the present case we have, rewriting as in Sec. 7a,

$$I_1{}^{-+}=-i\int d^3r \exp\left(i\mathbf{q}\cdot\mathbf{r}-i\int_{-\infty}^{\infty} Vd\zeta\right)\times\left(\frac{\boldsymbol{\varrho}\cdot\mathbf{u}}{p_1}-\frac{\boldsymbol{\varrho}\cdot\mathbf{v}}{p_2}\right)\frac{1}{\rho}\frac{\partial}{\partial\rho}\int_z^{\infty}(z-\zeta)V(\rho,\zeta)d\zeta.$$

The angular integration gives, exactly as in (7a.5),

$$I_1{}^{-+}=\frac{2\pi}{q_\perp}\left(\frac{\mathbf{q}_\perp\cdot\mathbf{u}}{p_1}-\frac{\mathbf{q}_\perp\cdot\mathbf{r}}{p_2}\right)\int \rho d\rho J_1(q_\perp\rho)\times\exp\left[-i\int_{-\infty}^{\infty} V(\rho,\zeta)d\zeta\right]\int_{-\infty}^{\infty} e^{iq_z z}\frac{\partial}{\partial\rho}\times\int_z^{\infty}(z-\zeta)V(\rho,\zeta)d\zeta dz.$$

Furthermore, integrating by parts twice with respect to z we have, finally,

$$I_1{}^{-+}=\frac{2\pi}{q_\perp q_z{}^2}\left(\frac{\mathbf{q}_\perp\cdot\mathbf{u}}{p_1}-\frac{\mathbf{q}_\perp\cdot\mathbf{v}}{p_2}\right)\int \rho d\rho J_1(q_\perp\rho)\times\exp\left[-i\int_{-\infty}^{\infty} V(\rho,\zeta)d\zeta\right]\frac{\partial}{\partial\rho}\int_{-\infty}^{\infty} e^{iq_z z}V(\rho,z)dz \quad (7b.5)$$
$$=-\frac{ku(u-v)}{q_z{}^2p_1p_2}\frac{2\pi}{q_\perp}\int \rho d\rho J_1(q_\perp\rho)\times\exp\left[-i\int_{-\infty}^{\infty} V(\rho,\zeta)d\zeta\right]\frac{\partial}{\partial\rho}\int_{-\infty}^{\infty} e^{iq_z z}V(\rho,z)dz,$$

where we have used the fact that $\mathbf{q}_\perp=\mathbf{u}-\mathbf{v}$, and that in the present range ($q\sim1/\epsilon$) [see (7a.6)]:

$$u-v=O(1/\epsilon), \quad (7b.6)$$

angle between $\mathbf{u}$ and $\mathbf{v}=O(1/\epsilon)$.

We now turn to $\mathbf{I}_2$ and $\mathbf{I}_3$ and, as before, we shall be interested only in the components of these quantities perpendicular to $\mathbf{k}$:

$$\mathbf{I}_{2\perp}{}^{-+}=\frac{-i}{2\epsilon_1}\int d^3r e^{-i\chi_{2,-}+i\mathbf{q}\cdot\mathbf{r}}\nabla_\perp e^{i\chi_{1,+}}$$
$$=\frac{1}{2\epsilon_1}\int d^3r \exp\left[i\mathbf{q}\cdot\mathbf{r}-i\int_{-\infty}^{\infty} V(\rho,\zeta)d\zeta\right]\nabla_\perp\chi_{1,+}$$

up to relative order $1/\epsilon$.

Because of the small angles between $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{k}$, and since ρ and z are of order ϵ, it is of no importance whether we take the direction of $\mathbf{p}_1$, $\mathbf{p}_2$, or $\mathbf{k}$ for the z axis. The symbol $\perp$ then denotes the component of any vector perpendicular to any of these vectors. For the moment we take the z axis along $\mathbf{p}_2$ and $\mathbf{q}$ in the x-z plane. Then

$$\int d\varphi \exp(i\mathbf{q}_\perp\cdot\boldsymbol{\varrho})\nabla_\perp\chi$$
$$=\int d\varphi \exp(iq_\perp\rho\cos\varphi)\{\cos\varphi, \sin\varphi\}\frac{\partial\chi}{\partial\rho}$$
$$=2\pi iJ_1(q_\perp\rho)\{1,0\}\frac{\partial\chi}{\partial\rho}=2\pi i\frac{\mathbf{q}_\perp}{q_\perp}J_1(q_\perp\rho)\frac{\partial\chi}{\partial\rho}.$$

Performing a partial integration with respect to z, we find

$$\mathbf{I}_{2\perp}{}^{-+}=\frac{\mathbf{q}_\perp}{2\epsilon_1q_z}\frac{2\pi}{q_\perp}\int \rho d\rho J_1(q_\perp\rho)\times\exp\left(-i\int_{-\infty}^{\infty} V(\rho,\zeta)d\zeta\right)\frac{\partial}{\partial\rho}\int_{-\infty}^{\infty} e^{iq_z z}V(\rho,z)dz. \quad (7b.7)$$

$\mathbf{I}_{3\perp}$ is obtained from $\mathbf{I}_{2\perp}$ by substituting the index 2 for 1, as is evident from the definition:

$$\mathbf{I}_{3\perp}{}^{-+}=\frac{i}{2\epsilon_2}\int(\nabla_\perp e^{-i\chi_2})e^{i\mathbf{q}\cdot\mathbf{r}+i\chi_1}d^3r$$
$$=\frac{\mathbf{q}_\perp}{2\epsilon_2q_z}\frac{2\pi}{q_\perp}\int \rho d\rho J_1(q_\perp\rho)\times\exp\left(-i\int_{-\infty}^{\infty} V(\rho,\zeta)d\zeta\right)\frac{\partial}{\partial\rho}\times\int_{-\infty}^{\infty} e^{iq_z z}V(\rho,z)dz. \quad (7b.8)$$

Now it is also clear that for $q\sim 1/\epsilon$:

$$D_1=2\epsilon_2 q_z,\quad D_2=2\epsilon_1 q_z$$

and

$$2\left(\frac{\epsilon_1}{D_2}-\frac{\epsilon_2}{D_1}\right)=\frac{ku(u-v)}{q_z^2 p_1 p_2}. \tag{7b.9}$$

Our results (7b.6–8) are therefore equal to the Born approximation times the common factor

$$A=-\frac{2\pi}{q_\perp}\int_0^\infty \rho d\rho J_1(q_\perp\rho)\exp\left[-i\int_{-\infty}^{\infty}V(\rho,\zeta)d\zeta\right]\frac{\partial}{\partial\rho}\times\int_{-\infty}^{\infty}V(\rho,z)e^{iq_z z}dz\Big/\int d^3r e^{i\mathbf{q}\cdot\mathbf{r}}V(r). \tag{7b.10}$$

Since the cross section is a homogeneous, quadratic function in the I's, the result is that *the Born approximation gets multiplied by* $|A|^2$.

Again we obtain the B-M result in the case of no screening[18]:

$$\begin{aligned}A&=\frac{q^2q_z}{q_\perp}\int_0^\infty \rho d\rho \rho^{-2ia}J_1(q_\perp\rho)K_1(q_z\rho)\\&=(q^2/4)^{ia}\Gamma(1-ia)\Gamma(2-ia)F(ia,1-ia;2;q_\perp^2/q^2)\\&=\left(\frac{q^2}{4}\right)^{ia}\frac{\Gamma(1-ia)}{\Gamma(1+ia)}\frac{\pi a}{\sinh\pi a}\{F(ia,-ia;1;q_\perp^2/q^2)\\&\qquad-ia(q_z^2/q^2)F(1+ia,1-ia;2;q_\perp^2/q^2)\},\end{aligned} \tag{7b.11}$$

which is exactly their result [B-M Eq. (8.20)], as $1-x\equiv D_1D_2/4\epsilon_1\epsilon_2q^2=q_z^2/q^2$ in the present range [see (7b.9)]. The phase factor is of course the same as in (6a.11).

We may express A in terms of the atomic form factor $F(x)$: Let $aQ(r)$ be the charge. Then

$$1-F(p)=\int e^{-i\mathbf{p}\cdot\mathbf{r}}Q(r)d^3r.$$

The potential is

$$V(r)=\lim_{\eta\to 0}-\frac{2a}{(2\pi)^2}\int\frac{d^3p e^{i\mathbf{p}\cdot\mathbf{r}}[1-F(p)]}{p^2+\eta^2},$$

and the integrals

$$\begin{aligned}&\int_{-\infty}^{\infty}e^{iq_z z}V(\rho,z)dz\\&=\lim_{\eta\to 0}\left[-2a\int\frac{p_\perp dp_\perp\{1-F[(p_\perp^2+q_z^2)^{1/2}]\}}{p_\perp^2+q_z^2+\eta^2}J_0(p_\perp\rho)\right],\\&\int_{-\infty}^{\infty}V(\rho,z)dz=\lim_{\eta\to 0}-2a\int\frac{p_\perp dp_\perp(1-F(p_\perp))}{p_\perp^2+\eta^2}J_0(p_\perp\rho).\end{aligned} \tag{7b.12}$$

[18] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1952), second edition, p. 410.

This expression was used in (6a.9). η is necessary in order to obtain the correct expressions for the unscreened case $F(x)\equiv 0$.

The expression for A is then

$$\begin{aligned}A=\lim_{\substack{\eta_1\to 0\\ \eta_2\to 0}}&\frac{q^2}{1-F(q)}\frac{1}{q_\perp}\int\frac{p_\perp^2dp_\perp\{1-F[(p_\perp^2+q_z^2)^{1/2}]\}}{p_\perp^2+q_z^2+\eta_1^2}\\&\times\int\rho d\rho J_1(q_\perp\rho)J_1(p_\perp\rho)\\&\times\exp\left[2ia\int\frac{xdx}{x^2+\eta_2^2}(1-F(x))J_0(x\rho)\right].\end{aligned} \tag{7b.13}$$

In cases where screening is important $[F(x)\neq 0]$ this expression must be handled by numerical calculation. This will be left for later work.

8. INTEGRATION OVER ELECTRON ANGLES IN THE BREMSSTRAHLUNG CROSS SECTION

Now we may show that our result for the bremsstrahlung cross section agrees with the result previously shown more generally[2]. The cross section integrated over the motion of the electron is additive just as in the case of pair production: one part is the Born approximation cross section *including* screening; to this is then added the effect of the Coulomb correction, which is *independent* of screening.

The expression for the square of the matrix element [B-M Eq. (7.11)] is

$$\begin{aligned}\sum|H_{12}'|^2=\sum_\nu|C'|^2\frac{1}{\epsilon_1\epsilon_2}\Big\{&\frac{1}{2\epsilon_1\epsilon_2}[k^2+\epsilon_2^2u^2+\epsilon_1^2v^2]I_{1\nu}^2\\&+2I_{1\nu}(\epsilon_2\mathbf{u}\cdot\mathbf{I}_{2\nu\perp}+\epsilon_1\mathbf{v}\cdot\mathbf{I}_{3\nu\perp})\\&+2\epsilon_1\epsilon_2(I_{2\nu\perp}^2+I_{3\nu\perp}^2)\Big\}.\end{aligned} \tag{8.1}$$

This expression is the same for pair production as for bremsstrahlung.

The cross section differs from the Born approximation only in the region $q\sim 1/\epsilon$. Here we find

$$\sum|H_{12}'|^2=|C'|^2|\mathcal{T}|^2\times\frac{1}{q_z^2\epsilon_1\epsilon_2}\frac{1}{q_\perp^2}\left\{-\frac{k^2u^2\chi^2}{q_z^2\epsilon_1^2\epsilon_2^2}+\frac{q_\perp^2}{2}\left(\frac{\epsilon_1}{\epsilon_2}+\frac{\epsilon_2}{\epsilon_1}\right)\right\}, \tag{8.2}$$

where $\chi=u-v$, and

$$\mathcal{T}=-2\pi\int_0^\infty\rho d\rho J_1(q_\perp\rho)\exp\left[-i\int_{-\infty}^{\infty}V(\rho,\zeta)d\zeta\right]\frac{\partial}{\partial\rho}\times\int_{-\infty}^{\infty}e^{iq_z z}V(\rho,z)dz. \tag{8.3}$$

We now choose the variables q_z, $\alpha=\chi/q_\perp$ and $q_\perp$. The angular part of the phase space volume element is then

$$u du\, v dv\, d\varphi = dq_z d\alpha q_\perp dq_\perp / [2\delta(1-\alpha^2)^{\frac{1}{2}}].$$

With these variables the explicit dependence on $q_\perp$ in (8.2) vanishes and the differential cross section depends on $q_\perp$ only through $|\mathcal{T}|^2 q_\perp dq_\perp$. The integration over φ, which is equivalent to integration over $q_\perp$, therefore gives

$$\int \sigma d\varphi \sim \int_0^{q_{0\perp}} \sigma q_\perp dq_\perp \sim \int_0^{q_{0\perp}} |\mathcal{T}|^2 q_\perp dq_\perp,$$

where $q_{0\perp}$ is a number small compared to 1, but large enough so that all screening effects are contained below $q_{0\perp}$:

$$q_{0\perp} \gg 1/r_{\text{screening}}.$$

Now the integral is

$$\int_0^{q_{0\perp}} |\mathcal{T}|^2 q_\perp dq_\perp = (2\pi)^2 \int_0^\infty \rho d\rho \int_0^\infty \rho' d\rho' \int_0^{q_{0\perp}} q_\perp dq_\perp J_1(q_\perp\rho) J_1(q_\perp\rho') \times \exp\left[-i\int_{-\infty}^\infty V(\rho,\zeta)d\zeta + i\int_{-\infty}^\infty V(\rho',\zeta)d\zeta\right] \frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho'} \times \int_{-\infty}^\infty \int_{-\infty}^\infty e^{iq_z z - iq_z z'} V(\rho,z) V(\rho',z') dz dz'.$$

Let us now subtract from this the Born approximation contribution for the screened potential:

$$\int_0^{q_{0\perp}} q_\perp dq_\perp \int_0^\infty \rho d\rho \int_0^\infty \rho' d\rho' J_1(q_\perp\rho) J_1(q_\perp\rho') \times \left\{\exp\left[-i\int_{-\infty}^\infty V(\rho,\zeta)d\zeta + i\int_{-\infty}^\infty V(\rho',\zeta)d\zeta\right] - 1\right\} \times \frac{\partial}{\partial\rho}\frac{\partial}{\partial\rho'} \int_{-\infty}^\infty \int_{-\infty}^\infty e^{iq_z z - iq_z z'} V(\rho,z) \times V(\rho',z') dz dz'. \quad (8.4)$$

But now since[19]

$$\int_0^\infty q_\perp dq_\perp J_1(q_\perp\rho) J_1(q_\perp\rho') = \delta_2(\rho-\rho'),$$

and

$$\left\{\exp\left[-i\int_{-\infty}^\infty V(\rho,\zeta)d\zeta + i\int_{-\infty}^\infty V(\rho',\zeta)d\zeta\right] - 1\right\}_{\rho=\rho'} = 0,$$

it is clear that with the integrand as in (8.4)

$$\int_0^{q_{0\perp}} dq_\perp = -\int_0^\infty dq_\perp + \int_0^{q_{0\perp}} dq_\perp = -\int_{q_{0\perp}}^\infty dq_\perp,$$

[19] The function $\delta_2(\rho-\rho')$ is defined by $\int \rho' d\rho' f(\rho') \delta_2(\rho-\rho') = f(\rho)$.

so that the integral over $q_\perp$ may be taken in the *upper* range of $q_\perp$. But in this range screening is of no importance:

$$\rho q_\perp \sim 1 \quad \text{and hence} \quad \rho \sim \frac{1}{q_\perp} \ll r_{\text{screening}}.$$

In (8.4), therefore, the potential may be taken to be the unscreened Coulomb potential, although $q_\perp < q_{0\perp}$. Thus we simply find

$$\int \sigma_{\text{exact}}^{\text{screened}} d\Omega_{p2} = \int \sigma_{\text{Born}}^{\text{screened}} d\Omega_{p2} + \int (\sigma_{\text{exact}}^{\text{no screening}} - \sigma_{\text{Born}}^{\text{no screening}}) d\Omega_{p2},$$

as mentioned at the beginning of Sec. 8, which will lead to Eq. (4) in reference 2, if the integrations are carried out

9. SCATTERING

The scattering amplitude for high-energy small angle scattering was derived by Molière,[4] from the phase shifts obtained by the WKB method. In this section we rederive his results using the wave functions of Sec. 5. Scattering is closely related to bremsstrahlung: The S matrix for scattering is

$$S = \int \psi_{\text{el},-}^\dagger \psi_{\text{el},+} d^3r = \int F_{2,-}^* e^{i\mathbf{q}\cdot\mathbf{r}} F_{1,+} d^3r = I_1^{-+},$$

in which $\mathbf{q} = \mathbf{p}_1 - \mathbf{p}_2$. It is therefore obtained from the I_1^{-+} of bremsstrahlung by putting $\mathbf{k}=0$, and leaving $\mathbf{p}_1$ and $\mathbf{p}_2$ arbitrary. It should be noticed that, although the z axis is now arbitrary, it has to be chosen so that $q_z \sim 1/\epsilon$, as before.

From (6a.7), we find, in the case $q \sim 1$,

$$I_1(z>0) = \frac{2\pi}{p_1^2} \int l dl e^{2i\delta_l} J_0\left(\frac{l}{p_1} q_\perp\right) \int_0^\infty e^{i\delta r} dr,$$

where $\delta = p_1 - p_2$. Now

$$\int_0^\infty e^{i\delta r} dr = \delta_+(p_1-p_2) = \pi\delta(p_1-p_2) - \frac{i}{p_1-p_2}.$$

Thus, using (5a.4)

$$I_1(z>0) = \left\{\pi\delta(p_1-p_2) - \frac{i}{p_1-p_2}\right\} 2\pi \int x dx \times \exp\left[-2i\int_0^\infty V(x,\zeta)d\zeta\right] J_1(xq_\perp),$$

with the substitution $x = l/p_1$. Only the term corresponding to $i/(p_1-p_2)$ was kept in bremsstrahlung, (6a.7). As in (6a.12) we find $I_1(z<0)$ by the substitution $p_1\rightleftarrows p_2$ in $I_1(z>0)$:

$$I_1(z<0)=\left\{\pi\delta(p_1-p_2)+\frac{i}{p_1-p_2}\right\}2\pi\int xdx \times\exp\left[-2i\int_0^\infty V(x,\zeta)d\zeta\right]J_0(xq_\perp).$$

Thus

$$I_1=I_1(z>0)+I_1(z<0) =2\pi\delta(p_1-p_2)2\pi\int xdx \times\exp\left[-2i\int_0^\infty V(x,\zeta)d\zeta\right]J_0(xq). \quad (9.1)$$

since $q_z\ll q_\perp$ for $q\sim 1$. The T matrix is defined by

$$S_{21}=\delta_{21}-2\pi i\delta(\epsilon_1-\epsilon_2)T_{21}.$$

Therefore,

$$T=2\pi i\int_0^\infty xdxJ_0(xq)\left\{\exp\left[-2i\int_0^\infty V(x,\zeta)d\zeta\right]-1\right\}. \quad (9.2)$$

The wave functions used in calculating (9.2) are, however, only valid in the case $q\sim 1$, in which case the screening is of no importance, thus reducing T as given in (9.2) effectively to the Born approximation value as was true in the case of bremsstrahlung [see (6a.11, 12)]. For $q\sim 1$ (9.2) thus reduces to

$$T=2\pi i\int_0^\infty xdxJ_0(xq)x^{-2ia}=-\frac{4\pi a}{q^2}\left(\frac{q^2}{4}\right)^{ia}\frac{\Gamma(1-ia)}{\Gamma(1+ia)},$$

for scattering angle $\vartheta\neq 0$, which gives the Rutherford formula.

But now the expression for T in the region $q\sim 1/\epsilon$ happens to be exactly the same as (9.2): Using our result (7b.4) we find, for $q\sim 1/\epsilon$,

$$S=I_1^{-+}=2\pi\delta(q_z)2\pi\int\rho d\rho J_0(\rho q) \times\exp\left[-2i\int_0^\infty V(\rho,\zeta)d\zeta\right],$$

which is the same as (9.1), since

$$q_z=p_{1z}-p_{2z}=(p_1-p_2)[1+u^2/(2p_1p_2)]+O(1/\epsilon^2) =(p_1-p_2)[1+O(1/\epsilon^2)].$$

We thus find the same expression for the scattering amplitude (9.2) in both regions, $q\sim 1$ and $q\sim 1/\epsilon$. Therefore the distinction which must be made in the case of bremsstrahlung and pair production between the wave function for the region $q\sim 1$ and the wave function for the region $q\sim 1/\epsilon$ need not be made in the case of elastic scattering.

ACKNOWLEDGMENT

We are indebted to Professor H. A. Bethe for several stimulating discussions and for the hospitality extended during our various residences at Cornell University.

APPENDIX A

The radial wave equation is

$$\frac{d^2u_l}{dr^2}+\left(p^2-2\epsilon V-\frac{l(l+1)}{r^2}\right)u_l=0. \quad (A.1)$$

Since we want u_l for l's of order ϵ, there is effectively only one nondimensional parameter in (A.1), *viz.*, ϵ. We then use a WKB technique to obtain an approximate solution of (A.1) for large ϵ, expanding the solution in powers of $1/\epsilon$.

We require a wave function which agrees to within terms of relative order $1/\epsilon$ with the exact solution of (A.1) having the proper behavior at $r=0$. Using the method of Langer,[20–22] we have found that for $r\sim\epsilon$ this approximate wave function should be taken as

$$u_l=\cos\left\{\int_{r_0}^r\eta(r')dr'-\tfrac{1}{4}\pi\right\}, \quad (A.2)$$

where

$$\eta^2=p^2-2\epsilon V-(l+\tfrac{1}{2})^2/r^2, \quad (A.3)$$

and r_0 is defined by $\eta(r_0)=0$. Since for $r\sim 1$, $2\epsilon V\ll(l+\frac{1}{2})^2/r^2$ for the Coulomb potential (with arbitrary screening),

$$r_0=(l+\tfrac{1}{2})/p+O(1/\epsilon)=O(1). \quad (A.4)$$

Since we only need the wave function for both l and r of order ϵ one might be inclined to presume that it is immaterial whether we choose, in the last term in η^2, $(l+\frac{1}{2})^2$ or $l(l+1)$. Inspection of the error by means of an iteration procedure shows that we must indeed choose η^2 as given above, even in this case. Now if V is a screened potential then we may write

$$\int_{r_0}^r\eta dr'=\lim_{N\to\infty}\left\{\int_{r_0}^N\eta dr'-\int_{r_1}^N\eta_0 dr'\right\} +\lim_{N\to\infty}\left\{\int_{r_1}^N\eta_0 dr'-\int_r^N\eta dr'\right\}, \quad (A.5)$$

where

$$(\eta_0)^2=p^2-(l+\tfrac{1}{2})^2/r^2,$$

and

$$r_1=(l+\tfrac{1}{2})/p$$

[20] R. E. Langer, Phys. Rev. **51**, 669 (1937); Bull. Am. Math. Soc. **40**, 545 (1934).

[21] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Part II, pp. 1092–1106.

[22] R. E. Langer, Trans. Am. Math. Soc. **37**, 397 (1935).

is the zero of η_0. The first term in (A.5) is the phase shift, δ_l:

$$\delta_l=\lim_{N\to\infty}\left\{\int_{r_0}^{N}\eta dr'-\int_{r_1}^{N}\eta_0 dr'\right\}. \tag{A.6}$$

The second term may be written in the form

$$\int_{r_1}^{r}\eta_0 dr'+\int_{r}^{\infty}(\eta_0-\eta)dr'. \tag{A.7}$$

The first of these two integrals may be evaluated exactly. Making the transformation of variables

$$s^2=r^2-(l+\tfrac{1}{2})^2/p^2,$$

we find

$$\int_{r_1}^{r}\eta_0 dr'=ps-(l+\tfrac{1}{2})\tan^{-1}[ps/(l+\tfrac{1}{2})],$$

which, for $r\sim\epsilon$, $l\sim\epsilon$, simplifies to

$$\int_{r_1}^{r}\eta_0 dr'=pr-\tfrac{1}{2}l\pi-\tfrac{1}{4}\pi+l^2/(2pr)+O(1/\epsilon).$$

In the second integral in (A.7), since the *lower* limit is of order ϵ, we note that, for $r\gtrsim\epsilon$,

$$\eta_0-\eta=(\eta_0^2-\eta^2)/(\eta_0+\eta)=2\epsilon V/(\eta_0+\eta),$$
$$\eta_0+\eta=2p[1+O(1/\epsilon^2)],$$

and hence

$$\int_{r}^{\infty}(\eta_0-\eta)dr'=\int_{r}^{\infty}Vdr' + \text{terms of relative order } 1/\epsilon^2. \tag{A.8}$$

Thus, substituting (A.5–8) in (A.2) we have, for $r\sim\epsilon$,

$$u_l(r)=\sin\left\{pr-\tfrac{1}{2}l\pi+\frac{l^2}{2pr}+\delta_l+\int_{r}^{\infty}Vdr'+O(1/\epsilon)\right\}. \tag{A.9}$$

For the case of an unscreened Coulomb potential $V=-a/r$ the integral in (A.2) can be evaluated exactly:

$$\int_{r_0}^{r}\left\{p^2+\frac{2\epsilon a}{r'}-\frac{(l+\frac{1}{2})^2}{r'^2}\right\}^{\frac{1}{2}}dr'$$
$$=[p^2r^2+2\epsilon ar-(l+\tfrac{1}{2})^2]^{\frac{1}{2}}$$
$$+(l+\tfrac{1}{2})\sin^{-1}\left\{\frac{(l+\frac{1}{2})^2-\epsilon ar}{r[(l+\frac{1}{2})^2p^2+\epsilon^2a^2]^{\frac{1}{2}}}\right\}$$
$$+\frac{\epsilon a}{p}\log\{2p[p^2r^2+2\epsilon ar-(l+\tfrac{1}{2})^2]^{\frac{1}{2}}+2p^2r+2\epsilon a\}\Big|_{r_0}^{r}. \tag{A.10}$$

Neglecting terms in (A.10) which are $O(1/\epsilon)$ for $r\sim\epsilon$ and substituting in (A.2), we have

$$u_l(r)=\sin\left\{pr-\tfrac{1}{2}l\pi+\frac{l^2}{2pr}+a\log\left(\frac{2pr}{l}\right)\right\} \tag{A.11}$$

for an unscreened Coulomb potential. It may be noted that the term $a\log(2pr/l)$ in (A.11) is exactly that given in Mott and Massey,[23] where in our notation $\delta_l=\arg\Gamma(l+1-ia)=-a\log l+O(1/\epsilon)$ for $l\sim\epsilon$. However, as will be shown in the following paragraphs, for a screened potential δ_l can be written as a function of the single variable l/p. Hence also for an unscreened potential we choose to write δ_l as a function of l/p, *viz.*, $\delta_l=-a\log(l/p)$, rather than $\delta_l=-a\log l$. Thus the term replacing $\int_r^\infty Vdr'$ in (A.9) is $a\log 2r$. Defining $\delta_l=-a\log(l/p)$ is convenient for the evaluation of the matrix elements for $q\sim1$ (see Sec. 6) and changes the wave function only by an inconsequential factor p^{ia} [note (5a.9) and comments immediately following that equation in the text] from customary use.[24]

Finally, we simplify the phase shift defined in (A.6), neglecting terms of $O(1/\epsilon)$. Since the term $2\epsilon V$ is small compared to the remaining terms in η except near $r=r_0$, let us write (A.6) in the form

$$\delta_l=\int_{R}^{\infty}(\eta-\eta_0)dr'+\int_{r_0}^{R}\eta dr'-\int_{r_1}^{R}\eta_0 dr' \tag{A.12}$$

where R is chosen so that $|2\epsilon V|$ is of order $1/\epsilon$ relative to $p^2-(l+\frac{1}{2})^2/r^2$ for $r\geqq R$ and hence R is greater than either r_0 or r_1, but nevertheless of order 1. In the first integral in (A.12) we may expand the integrand in a series in V/ϵ:

$$\eta=\eta_0-\epsilon V/\eta_0+O(V^2/\epsilon),$$

and hence the first integral is

$$-\frac{\epsilon}{p}\int_{R}^{\infty}\frac{V(r')r'dr'}{[r'^2-(l+\frac{1}{2})^2/p^2]^{\frac{1}{2}}}+O(1/\epsilon). \tag{A.13}$$

The second integral in (A.12) may be evaluated using (A.10), since $R\sim1$ and hence the potential is unscreened over the range of values of r' in the integrand. We thus substitute $V=-a/r$ in the second integral and find for the second and third integrals in (A.12)

$$\int_{r_0}^{R}\eta dr'-\int_{r_1}^{R}\eta_0 dr'$$
$$=(\epsilon a/p)\{\log[R+(R^2+(l+\tfrac{1}{2})^2/p^2)^{\frac{1}{2}}]-\log[(l+\tfrac{1}{2})/p]\}+O(1/\epsilon) \tag{A.14}$$

noting that $r_0=(l+\frac{1}{2})/p+O(1/\epsilon)$ and that $2\epsilon a/R$ is $O(1/\epsilon)$ relative to $p^2-(l+\frac{1}{2})^2/R^2$. However, the integrand in (A.13) may be integrated exactly over the

[23] Reference 11, p. 53, Eq. (28).
[24] Reference 11, p. 48, Eq. (15).

range $r_1 \leqq r' \leqq R$ since in this range, as just noted, $V=-a/r$:

$$
\begin{aligned}
-\frac{\epsilon}{p}\int_{r_1}^{R} \frac{V(r')r'dr'}{[r'^2-(l+\frac{1}{2})^2/p^2]^{\frac{1}{2}}}
&=\frac{\epsilon a}{p}\left\{\log\left[R+\left(R^2+\frac{(l+\frac{1}{2})^2}{p^2}\right)^{\frac{1}{2}}\right]-\log\left[\frac{l+\frac{1}{2}}{p}\right]\right\} \\
&=\int_{r_0}^{R}\eta dr'-\int_{r_1}^{R}\eta_0 dr'+O(1/\epsilon). \qquad \text{(A.15)}
\end{aligned}
$$

Hence, substituting (A.13, 15) in (A.12) we have,[25] neglecting terms of $O(1/\epsilon)$,

$$
\begin{aligned}
\delta_l=-\frac{\epsilon}{p}\int_{r_1}^{\infty} \frac{V(r')r'dr'}{[r'^2-(l+\frac{1}{2})^2/p^2]^{\frac{1}{2}}}
&=-\frac{\epsilon}{p}\int_0^{\infty} V\{[\zeta^2+(l+\tfrac{1}{2})^2/p^2]^{\frac{1}{2}}\}d\zeta. \qquad \text{(A.16)}
\end{aligned}
$$

Since $l\sim\epsilon$, this may be written in the form

$$
\delta_l=-\int_0^{\infty} V\{[\zeta^2+(l/p)^2]^{\frac{1}{2}}\}d\zeta. \qquad \text{(A.17)}
$$

Thus δ_l is a function of the single variable l/p.

APPENDIX B

We want to show here that the wave functions $\psi_\pm=e^{i\mathbf{p}\cdot\mathbf{r}+i\chi_\pm}$ with $\chi_\pm$ given by (5b.3) and (5b.5) are the approximations in the region $r\sim\epsilon$ of wave functions which satisfy the Sommerfeld radiation condition: plane wave plus $\left\{\begin{matrix}\text{outgoing}\\ \text{ingoing}\end{matrix}\right\}$ spherical waves, corresponding to ψ_+ and ψ_-. These boundary conditions can, however, only be satisfied by solutions of the second-order equation. We accordingly include the term $\nabla^2\chi$, neglecting only the nonlinear term $(\nabla\chi)^2$ in (5b.2) in the text, since this term is small in the region $\rho\sim\epsilon$, $|z|\sim\epsilon$ and does not change the *character* of the solution, $\{\pm\}$. Two solutions are then clearly

$$
\begin{aligned}
\chi_\pm&=\lim_{\eta\to 0}\frac{2i\epsilon}{(2\pi)^3}\int d^3r'\int d^3k\frac{e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}}{k^2+2\mathbf{p}\cdot\mathbf{k}\mp i\eta}V(r') \\
&=\frac{i\epsilon}{2\pi}e^{-i\mathbf{p}\cdot\mathbf{r}}\int d^3r'\frac{e^{\pm ip|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}V(r')e^{i\mathbf{p}\cdot\mathbf{r}'}.
\end{aligned} \qquad \text{(B.1)}
$$

The wave functions $\psi_\pm=e^{i\mathbf{p}\cdot\mathbf{r}+i\chi_\pm}$ then satisfy the Sommerfeld radiation condition with $\left\{\begin{matrix}\text{outgoing}\\ \text{ingoing}\end{matrix}\right\}$ waves:

$$
\psi_\pm(r\to\infty)=e^{i\mathbf{p}\cdot\mathbf{r}}(1+i\chi_\pm)=e^{i\mathbf{p}\cdot\mathbf{r}}-\frac{\epsilon}{2\pi}\frac{e^{\pm ipr}}{r}\int d^3r' \times\exp\left(-ip\frac{\mathbf{r}}{r}\cdot\mathbf{r}'\right)V(r')e^{i\mathbf{p}\cdot\mathbf{r}'}. \qquad \text{(B.2)}
$$

It should be noted that neglecting the term $(\nabla\chi)^2$ in the equation for χ_1 introduces errors of relative order $Z/137$ in the wave function (B.2). On the other hand, when $r\sim\epsilon$ the major contribution to the integral in (B.1) comes from $k\sim1/\epsilon$. Thus, in (B.1), significant k's are $\ll p$ and the term k^2 in the denominator of the integrand may be neglected:

$$
\chi_\pm=\lim_{\eta\to 0}\frac{i\epsilon}{\pi}\int d^3r'\delta(\boldsymbol{\varrho}-\boldsymbol{\varrho}')V(r')\int_{-\infty}^{\infty}dk_z\frac{e^{ik_z(z-z')}}{2pk_z\mp i\eta}.
$$

The k_z integral is a step function, so that

$$
\chi_+=-\frac{\epsilon}{p}\int_{-\infty}^{z}V(\rho,z')dz', \quad \chi_-=\frac{\epsilon}{p}\int_{z}^{\infty}V(\rho,z')dz',
$$

which are the results given in (5b.3) and (5b.5), respectively.

[25] See reference 4, Eqs. (2.2) and (4.4).