tions are obeyed by the cathode dark space in hydrogen and air over the full pressure range up to and above one atmosphere.

The similarity relations serve two purposes. Firstly, they enable properties of the dark space such as field strength, ion density, length, and temperature to be obtained at high pressures from the simple measurement of current density.[11] Because the dark space is very short ($\sim 10^{-3}$ cm at 760 mm Hg) these quantities cannot be measured by any other means. Secondly they often shed light on the discharge mechanism since the fundamental processes must also conform to the similarity requirements. It appears likely therefore that these processes are the same in the cathode dark space at high, as at low, pressures, namely ionization by electron impact, and electron emission from the cathode by positive-ion bombardment or photoemission.

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Adiabatic Invariant of the Harmonic Oscillator

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The problem of a vibrating harmonic oscillator whose frequency is changing in time is considered in the case where the frequency ω is initially constant, varies in an arbitrary fashion and becomes constant again. It is found that the relative change of the quantity, the energy divided by the frequency, in the final region from its value in the initial region is zero to as many orders in the rate of change of ω as ω has continuous derivatives. For the case where there is a break in the Nth derivative of ω the relative change is given to this order.

INTRODUCTION

THERE are many problems in physics in which there exist quantities which change so slowly that they may be taken as constants of the motion to a high degree of accuracy. Any such quantity whose change approaches zero asymptotically as some physical parameter approaches zero or infinity is an adiabatic invariant. For instance, in Fermi's theory[1] for the acceleration of cosmic rays, it is assumed that the magnetic moment of a spiraling particle in a varying magnetic field remains constant. Combined with the conservation of energy this enables one to show that a magnetic field can reflect such a spiraling particle. The magnetic moment of this particle is not really a rigorous constant but is nearly so if the relative space change in the magnetic field over the Larmor radius of the particle is small, or when the field is changing in time if its relative change during a Larmor period is also small. These conditions are satisfied to a high degree in many astrophysical situations.[2]

The constancy of the magnetic moment to first order in these parameters of smallness was first derived by Alfvén[3] and was later shown to be true in the next order by Helwig[4] for a general field. Later, Kruskal[5] showed that it was valid to all orders for the special case of a particle moving in a magnetic field in the z direction which varies only in the y direction and a constant electric field in the x direction. From these results it seemed possible that the adiabatic constancy of the magnetic moment to all orders was a result of general validity.

That the magnetic moment of the particle is a constant in all orders would imply that any change in it must vanish more rapidly than any power of the parameter of smallness, i.e., the relative change of the field over the Larmor radius. This does not imply that it must be a rigorous constant. For instance, $\Delta c = \exp(-1/\lambda)$ has this behavior since at $\lambda=0$ all derivatives of Δc vanish.

An example of an adiabatic invariant in quantum mechanics would be the distribution over energy states of a system as the Hamiltonian is changed by external means, such as changing the volume of the boundaries of the system without adding heat to the system.

In order to approach the problem of the constancy of adiabatic invariants to all orders, this paper treats another simpler problem in which an adiabatic invariant exists. Consider the classic one-dimensional problem of

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[1] E. Fermi, Phys. Rev. **75**, 1169 (1949).
[2] L. Spitzer, Astrophys. J. **116**, 299 (1952).
[3] H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, 1950), p. 19.
[4] G. Helwig, Z. Naturforsch. **10a**, 508 (1955).
[5] M. Kruskal (private communication).

an oscillator whose spring constant is slowly varied by some external means, such as a varying temperature, which only affects the motion through its spring constant. The counterpart of this problem was first considered by Einstein[6] at the Solvay Congress of 1911 on the old quantum theory. Lorentz asked how the amplitude of a simple pendulum would vary if its period were slowly changed by shortening its string. Would the number of quanta of its motion change? Einstein immediately gave the answer that the action, E/ω, where E is its energy and ω its frequency, would remain constant and thus the number of quanta would remain unchanged, if $(1/\omega)(d\omega/dt)$ were small enough. Birkhoff[7] showed for problems such as these, that one can write the displacement

$$x=W(t)\sin\left(\int_0^t\omega(t)dt+\delta\right),$$

where $\omega(t)$ is the frequency and $W(t)$ can be developed in a series which converges asymptotically. However, he did not evaluate the higher order terms and find the variation of E/ω. This is easier to do if one writes

$$x=W(t)\sin\left(\int_0^t S(t)dt+\delta\right),$$

and develops both W and S in asymptotic series. The relation between W and S may be chosen to simplify the problem and it can be seen that E/ω is indeed invariant to all orders in $(1/\omega)(d\omega/dt)$.

By generalizing this device Kruskal[8] is able to express the motion of a spiraling particle in a general electromagnetic field in terms of two such independent variables. Specifying a relation between them to simplify the problem, he is able to demonstrate that the magnetic moment of the particle is also invariant to all orders as has been suspected. It might be that there are many such invariants which are known to be constant to lowest order but which are actually invariant to all orders.

In this paper the details of the solution of the simple harmonic oscillator are given. The solution of the particle in the electromagnetic field will be published in a separate paper by Kruskal.[8]

CALCULATION

Since, when ω is changing E is changing also, we restrict ourselves to the case where the frequency ω as a function of the time t is at first infinitely flat, then changes in some arbitrary way, and finally becomes infinitely flat again. Thus the energy is well defined and constant in the initial and final regions and we compare E/ω in these two regions. In this case it is found that if ω has N continuous derivatives throughout, than the final value of E/ω is the same as its initial value to $N+1$ orders in $(1/\omega)(d\omega/dt)$. The change in the $(N+2)$nd order is calculated. If $N=\infty$, that is if, ω has all derivatives continuous, then E/ω is the same to all orders, although as already remarked this does not imply it is rigorously constant, since the result is only an asymptotic one.

If the mass is one, the equation of motion is

$$d^2x/dt^2+\omega^2(t/T)x=0, \tag{1}$$

where x is the displacement, $\omega(t/T)$ is a function with the properties listed above, and large T corresponds to small $(1/\omega)(d\omega/dt)$. Let $\tau=t/T$, so that Eq. (1) becomes

$$T^{-2}d^2x/d\tau^2+\omega^2(\tau)x=0, \tag{2}$$

which is to be solved for large T. Since ω is changing slowly in time, we anticipate that x may be written in the form

$$x(\tau)=W\sin\left(T\int^\tau Sd\tau\right), \tag{3}$$

where W and S are slowly varying functions of τ, between which we may specify an arbitrary relation. Then

$$x'(\tau)=W'\sin\left(T\int^\tau Sd\tau\right)+TSW\cos\left(T\int^\tau Sd\tau\right), \tag{4}$$

$$x''(\tau)=(W''-T^2S^2W)\sin\left(T\int^\tau Sd\tau\right)+(2TSW'+TS'W)\cos\left(T\int^\tau Sd\tau\right), \tag{5}$$

where primes represent derivatives with respect to τ. Thus Eq. (2) becomes

$$(W''/T^2-WS^2+\omega^2(\tau)W)\sin\left(T\int^\tau Sd\tau\right)+T^{-1}(2W'S+WS')\cos\left(T\int^\tau Sd\tau\right). \tag{6}$$

We choose the relationship between S and W to make the two coefficients in Eq. (6) vanish separately, so that W and S are given by

$$W''/T^2-WS^2+\omega^2(\tau)W=0, \tag{7}$$

$$2W'S+WS'=0. \tag{8}$$

Equation (8) may be integrated to give

$$W^2S=a, \tag{9}$$

where a is a constant.

Let us assume that the oscillator is started in the initial region with a displacement x_0 and zero velocity.

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[6] P. Langevin and M. DeBroglie, *La Theorie du Rayonnement et les Quanta* (*Repoıt on meeting at Institute Solvay, Brussels, 1911*) (Gauthier-Villars, Paris, 1912), p. 450.

[7] G. D. Birkhoff, Trans. Am. Math. Soc. **9**, 219 (1908).

[8] M. Kruskal (to be published).

The initial conditions on W and S are chosen such that W is constant throughout the initial region. From Eqs. (3), (4), (7), and (9).

$$S=\pm\omega \quad \text{(in the initial region)}, \qquad (10)$$

$$W=x_0 \quad \text{(in the initial region)}, \qquad (11)$$

$$\pm x_0^2\omega=a \quad \text{(in the initial region)}. \qquad (12)$$

We may take the positive sign in Eqs. (10) and (12). Note that if we choose x_0 independent of T, a is also independent of T.

The energy is computed in the initial and final regions by calculating the kinetic energy when $x=0$. Thus, by Eqs. (3) and (4),

$$E=\tfrac{1}{2}x'^2/T^2=\tfrac{1}{2}W^2S^2=\tfrac{1}{2}aS \quad \text{(when } x=0). \qquad (13)$$

In the initial region, by Eq. (10),

$$E/\omega=\tfrac{1}{2}aS/\omega=\tfrac{1}{2}a, \qquad (14)$$

and if $S=\omega$ in the final region, than the value of E/ω would be the same in the two regions for all T. However, the representation (3) does not guarantee $S=\omega$, since many other functions S and W may represent a sinusoidal oscillation. For instance, if W changes in time, then Eq. (7) shows that $S\neq\omega$ even though ω is constant. In general, after ω has varied in an arbitrary way, the simple relations (10) and (11) will not hold in the final region.

To investigate the value of S in the final region, we therefore solve for S as a function of τ by developing S and W as asymptotic series in $1/T$.

$$S=S_0+S_1/T+S_2/T^2+\cdots, \qquad (15)$$

$$W=W_0+W_1/T+W_2/T^2+\cdots. \qquad (16)$$

Substituting these series in Eqs. (7) and (9), we have, to lowest order,

$$S_0=\omega, \qquad (17)$$

$$W_0=(a/\omega)^{\frac{1}{2}}. \qquad (18)$$

The nth order then reads, for an even n,

$$W_{n-2}''-W_{n-2}(S^2)_2-W_{n-4}(S^2)_4-\cdots -W_0(S^2)_n=0, \qquad (19)$$

$$(W^2)_nS_0+(W^2)_{n-2}S_2+\cdots+W_0^2S_n=0, \qquad (20)$$

where we have introduced the abbreviations

$$(S^2)_m=S_mS_0+S_{m-2}S_2+\cdots+S_0S_m, \qquad (21)$$

and

$$(W^2)_m=W_mW_0+W_{m-2}W_2+\cdots+W_0W_m. \qquad (22)$$

In deriving Eq. (19) we have used Eq. (17) to cancel the ω term and in Eq. (20) we have used the fact that a is independent of T. Note from the equations corresponding to (19) and (20) that all odd orders vanish since S_1 and W_1 vanish.

Equation (19) with (21) expresses S_n in terms of lower orders S_m and W_m and their derivatives. Similarly Eq. (20) with (22) expresses W_n in terms of S_n and lower orders. Since S_0 and W_0 are given in terms of ω, we have found asymptotic series in $1/T$ for S and W which formally satisfy Eqs. (7) and (9). Notice that W_n and S_n depend on ω and its first n derivatives. Thus if the $(N+1)$st derivative of ω has a discontinuity at some time then S_{N+2} and W_{N+2} do also. Since x and x' must be continuous in all orders in $1/T$, the series given by (21) and (22) cannot represent a solution to $(N+2)$nd order. This and higher orders in S and W must be found by solving Eqs. (7) and (9) in regions where $d^{N+1}\omega/d^{N+1}$ is continuous and matching x and x' in these orders.

Suppose that ω has N continuous derivatives and that these N derivatives are zero in the final region. Then by Eqs. (17) and (18) the first N derivatives of S_0 and W_0 are continuous and vanish in the final region. By Eqs. (19) and (20) with Eqs. (21) and (22), S_2 and W_2 have $N-2$ continuous derivatives, and S_2 and W_2 vanish together with these derivatives in the final region. Proceeding in this manner we find that S_N and W_N are continuous and vanish in the final region. In particular,

$$S_n=0, \quad 0<n\leqslant N. \qquad (23)$$

Comparing Eq. (23) with Eqs. (13) and (17), we see that in the final region

$$E/\omega=\tfrac{1}{2}a+S_{N+2}/\omega T^{N+2}, \qquad (24)$$

so that E/ω is constant to $N+1$ orders in $1/T$. If ω has a jump z_{N+1} in its $(N+1)$st derivative as a function of t, the relative change in E/ω is given by

$$\frac{\Delta(E/\omega)}{(E/\omega)}=-\frac{2z_{N+1}}{(2\omega)^{N+2}}\sin(2\varphi-\tfrac{1}{2}N\pi). \qquad (25)$$

Here φ is the phase of the oscillator when it reaches the discontinuity. N may be either even or odd.