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Similarity Relations and the Cathode Dark Space

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In a study of the cathode dark space of the glow discharge von Engel, Seeliger, and Steenbeck found deviations from the similarity laws at high pressures. By using a correct solution to their (slightly modified) energy-balance equation and more recent values of the constants, agreement is obtained between the calculated and measured values of current density over the pressure range 1 to 760 mm Hg. This, together with other experimental evidence, shows that the similarity laws are valid over this pressure range, thus enabling various properties of the dark space to be determined and indicating that the fundamental mechanisms are the same as at low pressures.

INTRODUCTION

As part of a wider investigation into the properties and mechanisms of the high-pressure glow discharge it was obviously of interest to determine whether, as at low pressures, the cathode dark space conforms to the requirements of the similarity principle. This problem has been investigated by von Engel, Seeliger, and Steenbeck[1] who found deviations from similarity at high pressures. However these authors obtain an incorrect solution to their energy-balance equation and make no attempt to justify some of their assumptions. The present work re-examines the problem using a correct solution to the (slightly modified) differential equation and more recent values of the constants.

ANALYSIS

The temperature in the cathode dark space is first calculated using an energy-balance equation. This assumes (see I) that the electric field, E, and the positive-ion current density, j^+, fall linearly with distance, x, from the cathode. The first assumption is true,[2,3] but the second does not, as stated in I, follow immediately from this. Because the positive-ion density is constant throughout most of the dark space, j^+ will vary in the same way as the drift velocity. Thus if the ions were everywhere in equilibrium with the field, the curve of j^+ would be asymptotic to $j^+ \propto E$ near the negative glow and to $j^+ \propto E^{\frac{1}{2}}$ near the cathode. However because of the rapid variation in the field, the ions have less than the equilibrium energy, so that the curve obtained in this way may be taken as an upper limit. On the other hand it may be shown from a consideration of the ionization produced by the electrons, that since most of the ions reaching the cathode are formed in the cathode dark space[3,4] then

$$j^+ = J\left[1-\exp\left(-\int_x^d \alpha dx\right)\right], \tag{1}$$

where J is the total current density, d is the length of the dark space, and α is the first Townsend coefficient. Equation (1) is quite general and does not depend on any specified field distribution. However, the electrons will not be in equilibrium with the field so that the curve obtained from Eq. (1) forms a lower limit for j^+. When the curves showing the upper and lower limits of j^+ are plotted it is found that they both lie close to the curve showing a linear fall. Thus the assumption of a linear fall in j^+ is in fact justified but only as a fair approximation.

A more accurate expression than used in I for the thermal conductivity, k, is

$$k = aT^n, \tag{2}$$

where a and n are constants. The modified energy-balance equation may now be written:

$$E_0 J\left(1-\frac{x}{d}\right)^2 + \frac{d}{dx}\left(aT^n\frac{dT}{dx}\right) = 0 \tag{3}$$

and the solution with boundary conditions $T=T_0$, when $x=0$, and $dT/dx=0$, when $x=d$ is:

$$T^{n+1} = T_0^{n+1} + \frac{(n+1)}{12a} E_0 J d^2\left[1-\left(1-\frac{x}{d}\right)^4\right], \tag{4}$$

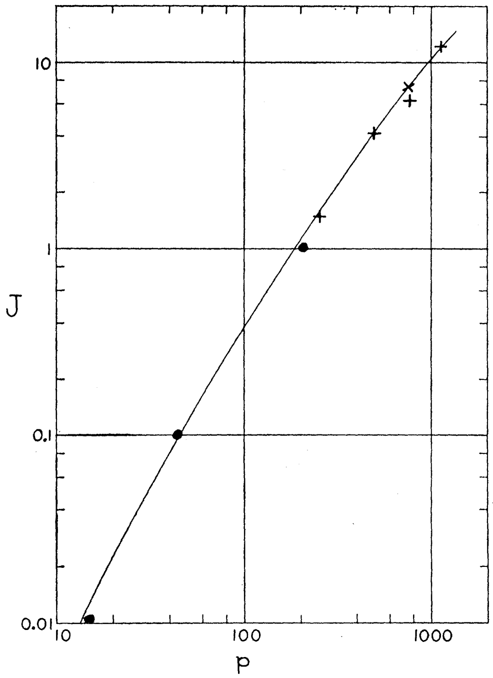
where E_0 and T_0 are the electric field and temperature at the cathode surface. The corresponding solution (with $n=1$) in I is not correct.[5] Instead of deducing an

TABLE I. Constants used in the evaluation of Eq. (6).

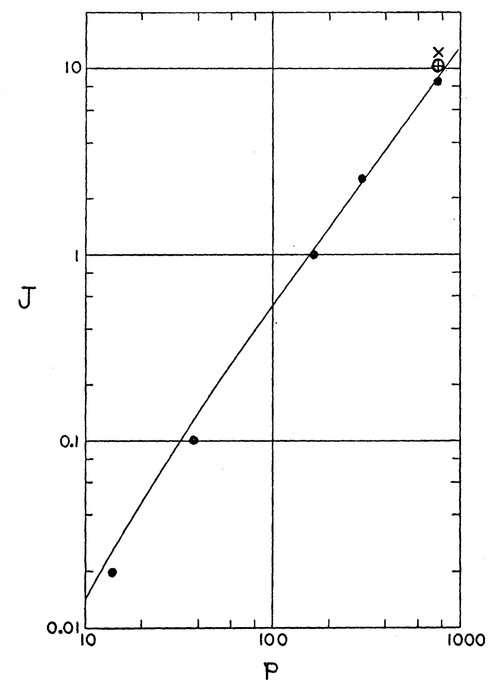
Quantity	Hydrogen	Air	Units	Reference
V_c	285	285	volts	a
a	3.3×10^{-5}	$\cdots$	watt cm^{-1} $(°K)^{-1.7}$	b
a	$\cdots$	8.0×10^{-7}	watt cm^{-1} $(°K)^{-2}$	b
n	0.7	1	$\cdots$	b
d_1	0.80	0.23	cm	c
J_1	6.4×10^{-5}	2.4×10^{-4}	amp cm^{-2}	c
T_1	285	285	°K	$\cdots$

a See reference 8.
b See reference 6.
c See reference 7.

[1] von Engel, Seeliger, and Steenbeck, Z. Physik **85**, 144 (1933). This paper will be referred to as I.
[2] F. W. Aston, Proc. Roy. Soc. (London) **A84**, 526 (1911); R. P. Stein, Phys. Rev. **89**, 134 (1953).
[3] P. F. Little and A. von Engel, Proc. Roy. Soc. (London) **A224**, 209 (1954).
[4] R. Warren, Phys. Rev. **98**, 1658 (1955).
[5] Dr. von Engel has pointed out (private communication) that an approximation was used.



(a)



(b)

FIG. 1. Variation of total current density (amp cm^{-2}) with pressure (mm Hg), (a) in hydrogen and (b) in air. The full lines show the calculated curves and the experimental points were obtained as follows: ● von Engel *et al.*,[1] × Fan,[10] + Gambling,[11] ⊕ Gambling and Edels.[8]

expression for the mean temperature, T_m, as in I, it is now assumed that $T=T_m$ at $x=\frac{1}{2}d$. Subsequent analysis shows that even at high pressures the error involved is only $\sim5\%$. Thus,

$$T_m^{n+1}=T_0^{n+1}+\frac{5(n+1)}{64a}E_0Jd^2. \tag{5}$$

The cathode-fall voltage, V_c, is given by $V_c=\frac{1}{2}E_0d$ and substitution of the similarity relations in (5) gives:

$$p^{n+1}=\left(\frac{T_0}{T_1}\right)^{n+1}\left(\frac{J}{J_1}\right)^{\frac{1}{2}(n+1)} +\left(\frac{5(n+1)}{32a}\right)\frac{V_cd_1J_1}{T_1^{n+1}}\left(\frac{J}{J_1}\right)^{\frac{1}{2}(n+2)}, \tag{6}$$

where the subscript 1 indicates the value of a quantity at $p=1$ mm Hg. Equation (6) shows that at low pressure $J\propto p^2$ in accordance with experiment, and at high pressure $J\propto p^\beta$, where $\beta=(2n+2)/(n+2)$ and not $\frac{4}{3}$ as in I. A comparison of the current densities predicted by (6) with those obtained experimentally, for a water-cooled copper cathode 1 mm thick in hydrogen and air, is shown in Fig. 1. The constants used,[6–8] which are more recent than those given in I, are shown in Table I. It is seen that there is good agreement between the calculated and experimental values over the wide pressure range from 1 to 760 mm Hg.

DISCUSSION

It is assumed in I that all the energy acquired by the positive ions is transferred to the gas by collisions. This is not true since the ions gain energy in a high field over the last free path before striking the cathode, and Campan[9] has shown that in a normal glow the energy retained might amount to $\sim$60 volts. This is about $0.25V_c$ so that V_c in Eq. (6) must be replaced by $V_c'=0.75V_c$. Taking $J=10$ amp/cm^2 in air, as an example, p is now 745 mm Hg compared with the previously calculated value of 840 mm Hg. It is seen that the ions may reach the cathode with an appreciable mean energy (at least up to $0.25V_c$) without significantly affecting the agreement between theory and experiment.

There is further independent experimental evidence which indicates that the similarity relations are obeyed in the cathode dark space in hydrogen and air. For example the cathode current density, except at low currents when edge effects are no longer negligible, is independent of current,[1,8,10,11] and the cathode-fall voltage is independent of current and pressure.[8] All the evidence indicates therefore that the similarity rela-

[6] F. G. Keyes, Trans. Am. Soc. Mech. Engr. **73**, 589 (1951); *Reactor Handbook: Engineering* (U. S. Atomic Energy Commission, Washington, D. C., 1955), Vol. II.

[7] A. von Engel and M. Steenbeck, *Elektrische Gasentladungen* (Verlag Julius Springer, Berlin, 1934).

[8] W. A. Gambling and H. Edels, Brit. J. Appl. Phys. **5**, 36 (1954); **7**, 376 (1956).

[9] T. I. Campan, Z. Physik **91**, 111 (1934).

[10] H. Y. Fan, Phys. Rev. **55**, 769 (1939).

[11] W. A. Gambling, Can. J. Phys. **34**, 1466 (1956).

tions are obeyed by the cathode dark space in hydrogen and air over the full pressure range up to and above one atmosphere.

The similarity relations serve two purposes. Firstly, they enable properties of the dark space such as field strength, ion density, length, and temperature to be obtained at high pressures from the simple measurement of current density.[11] Because the dark space is very short ($\sim 10^{-3}$ cm at 760 mm Hg) these quantities cannot be measured by any other means. Secondly they often shed light on the discharge mechanism since the fundamental processes must also conform to the similarity requirements. It appears likely therefore that these processes are the same in the cathode dark space at high, as at low, pressures, namely ionization by electron impact, and electron emission from the cathode by positive-ion bombardment or photoemission.

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Adiabatic Invariant of the Harmonic Oscillator

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The problem of a vibrating harmonic oscillator whose frequency is changing in time is considered in the case where the frequency ω is initially constant, varies in an arbitrary fashion and becomes constant again. It is found that the relative change of the quantity, the energy divided by the frequency, in the final region from its value in the initial region is zero to as many orders in the rate of change of ω as ω has continuous derivatives. For the case where there is a break in the Nth derivative of ω the relative change is given to this order.

INTRODUCTION

THERE are many problems in physics in which there exist quantities which change so slowly that they may be taken as constants of the motion to a high degree of accuracy. Any such quantity whose change approaches zero asymptotically as some physical parameter approaches zero or infinity is an adiabatic invariant. For instance, in Fermi's theory[1] for the acceleration of cosmic rays, it is assumed that the magnetic moment of a spiraling particle in a varying magnetic field remains constant. Combined with the conservation of energy this enables one to show that a magnetic field can reflect such a spiraling particle. The magnetic moment of this particle is not really a rigorous constant but is nearly so if the relative space change in the magnetic field over the Larmor radius of the particle is small, or when the field is changing in time if its relative change during a Larmor period is also small. These conditions are satisfied to a high degree in many astrophysical situations.[2]

The constancy of the magnetic moment to first order in these parameters of smallness was first derived by Alfvén[3] and was later shown to be true in the next order by Helwig[4] for a general field. Later, Kruskal[5] showed that it was valid to all orders for the special case of a particle moving in a magnetic field in the z direction which varies only in the y direction and a constant electric field in the x direction. From these results it seemed possible that the adiabatic constancy of the magnetic moment to all orders was a result of general validity.

That the magnetic moment of the particle is a constant in all orders would imply that any change in it must vanish more rapidly than any power of the parameter of smallness, i.e., the relative change of the field over the Larmor radius. This does not imply that it must be a rigorous constant. For instance, $\Delta c = \exp(-1/\lambda)$ has this behavior since at $\lambda = 0$ all derivatives of Δc vanish.

An example of an adiabatic invariant in quantum mechanics would be the distribution over energy states of a system as the Hamiltonian is changed by external means, such as changing the volume of the boundaries of the system without adding heat to the system.

In order to approach the problem of the constancy of adiabatic invariants to all orders, this paper treats another simpler problem in which an adiabatic invariant exists. Consider the classic one-dimensional problem of

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[2] L. Spitzer, Astrophys. J. **116**, 299 (1952).
[3] H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, 1950), p. 19.
[4] G. Helwig, Z. Naturforsch. **10a**, 508 (1955).
[5] M. Kruskal (private communication).