for three different initial mode separations. The results are shown in Fig. 3, in which the arithmetic mean of the actual shifts of the resonant frequency of the two modes is plotted along the abscissa. Qualitatively, the agreement with Eq. (12) is good, in that the additional shift is proportional to the square of the density, as shown in Fig. 4, and is approximately proportional to the pressure squared. Quantitatively, the observed shifts are smaller than those predicted by Eq. (12). The reason may be that the coupling coefficients of the two modes were not the same, and because it was necessary to introduce into the cavity a tuning stub to vary the initial separation of the modes, the fields of the two modes were not normal to each other in the immediate neighborhood of the stub. However, there can be no doubt that the higher mode affects the lower one and causes the shift in its resonant frequency to be greater under the influence of the plasma than it would be if the second mode were not present.

From the preceding discussion, it is clear that with the TE_{011} mode set up for measuring high electron densities, the TM_{111} mode, or indeed any mode close to it, is not desirable. In practice, all TM and all asymmetric TE modes can be suppressed by cutting azimuthal slots in the wall of the cavity.

To summarize, even though it is not possible to prevent the plasma from shielding the microwave field, by arranging the field at right angles to the density gradients, electron densities of the order of 10^{12} cm^{-3} can be measured by the conventional microwave method, if proper care is taken to eliminate the effect of higher modes on the measuring mode.

PHYSICAL REVIEW VOLUME 106, NUMBER 2 APRIL 15, 1957

Hydrodynamic Resistance in Liquid Helium II and Determination of the Normal Concentration

S. M. BHAGAT AND R. K. PATHRIA
Department of Physics, University of Delhi, Delhi, India
(Received August 6, 1956; revised manuscript received January 29, 1957)

The forces of hydrodynamic reaction in liquid helium II have been studied. Since the superfluid component, being inviscid, may not be expected to contribute to such forces, a measurement of these may provide direct information regarding the concentration of the normal component in the liquid. Experiments along these lines are suggested and it is hoped that they may throw light on the phenomena of "critical velocity" and on the irrotational nature of the superfluid motion.

I. INTRODUCTION

IT is well known that according to the two-fluid model,[1] liquid helium II is looked upon as an intimate mixture of two components, the normal and the superfluid. Here, the former is regarded as behaving like any other ordinary liquid whereas the latter is taken to be a perfect inviscid one, at least for low enough velocities.[2] It is evident that in order to make a direct determination of the concentration of the normal component one should look for those hydrodynamic properties in which the viscous nature of the fluid is straightaway effective. In the present paper we have attempted to suggest certain lines along which it may be interesting to perform experiments in order to obtain such information directly. In this connection, we propose to study the forces of reaction which would come into play when liquid helium II flows past a solid body or, alternatively, when the body is made to move through an otherwise stationary bath of the liquid.

In the general hydrodynamical theory of an inviscid fluid one meets with the apparent paradox that the fluid offers no resistance to the motion of a solid body through it. However, this is far from being true in the case of real fluids and in fact one does obtain a resistive force, the so-called profile drag, when account is taken of the skin-friction forces (due effectively to the finite viscosity of real fluids) and the dissipation of energy through the eddying wake. Clearly, the drag force should be absent in the case of the superfluid whereas one should obtain a finite contribution from the normal component. Hence, measurements on the drag force would be of interest, as discussed in detail in Sec. II.

Another hydrodynamic force in which the two components of liquid helium II may be expected to behave differently from each other is the well-known cross-wind force experienced in a uniform stream by a solid body with circulation around it. Here again the superfluid may, for velocities less than a certain critical one, remain free from participating in the rotatory motion and the observed force may give direct information regarding the normal concentration in the liquid. The expected results in this case are elaborated in Sec. III.

[1] L. Tisza, Nature **141**, 913 (1938); F. London, Phys. Rev. **54**, 947 (1938).
[2] K. R. Atkins, *Advances in Physics* (Taylor and Francis, Ltd., London, 1952), Vol. 1, p. 169.

II. DRAG FORCE

(a) Theoretical Observations

It is well known that the drag force may be expressed by the equation

$$D=C_D'(\tfrac{1}{2}\rho V^2)A(\rho Vl/\eta)^n, \quad (1)$$

where the symbols have the following meanings: A is the area of the body projected on a plane normal to the direction of motion; l is a characteristic linear dimension of the body; ρ and η are the density and coefficient of viscosity, respectively, of the fluid; V is the uniform velocity of the body relative to the regions of the fluid far removed from it; and finally, C_D' is a dimensionless constant depending upon the form of the body. The quantity $C_D'(\rho Vl/\eta)^n$ is defined as the drag coefficient C_D, a function of the dimensionless combination $\rho Vl/\eta$—the Reynolds number N_R. Evidently, the Reynolds number for any flow may be considered as a criterion for deciding the relative importance of the inertia forces on the one hand and the viscous forces on the other. In the limit of small N_R, the index n in Eq. (1) tends to -1, whence we get

$$D\propto \eta Vl \quad (\text{since } A\propto l^2), \quad (2)$$

indicating that the drag force is independent of the density, that is, of the inertia of the fluid and is determined directly by its viscosity. On the other hand, when N_R is large, n approaches zero, and one obtains

$$D\propto \rho V^2 A, \quad (3)$$

from which it is obvious that in this case the density of the fluid, and not its viscosity, is the dominating factor. For intermediate values of N_R, the dependence of D on various factors is also intermediate between the two extremes expressed by Eqs. (2) and (3).

Next, it is generally held that the two components of liquid helium II may be assumed to be capable of maintaining independent velocity fields[3] and therefore would require separate hydrodynamical equations to represent their flow. Also, these equations contain certain extra terms representing the thermomechanical flow under a temperature gradient besides the ordinary flow under the pressure gradient. Consequently, in addition to the velocity and pressure fields, one has to investigate simultaneously the temperature field as well. This has been done by Woldringh[4] for the case of flow around a spherical obstacle under the usual assumption of Stokes, that is, that the terms of second order in velocity (which represent the effect of inertia) may be neglected in the solution of the equations. His result for the drag is

$$D=6\pi\eta_n aVq, \quad (4)$$

which is similar to that of Stokes derived in the single-fluid hydrodynamics, except for the dimensionless factor q which is ordinarily not very much different from unity. Here, V is the relative velocity of the body and the normal component (at large distances, of course) and η_n is its viscosity. As expected, the result is independent of the density of the fluid. Evidently, the conclusions arrived at by Woldringh are true only for small values of the Reynolds number for the flow of the normal component, that is, of $\rho_n Vl/\eta_n$.

It is, however, important to remember that the viscosity of the fluid under investigation is so low that for ordinary experimental conditions, the magnitude of the Reynolds number would probably be fairly high. In that case, the inertia terms in the equations of the two-fluid hydrodynamics would no longer be negligible and the treatment given by Woldringh would not hold. One should, however, expect that the normal component of liquid helium would behave like any ordinary liquid even at high values of N_R, an expectation supported by the similarity of the results obtained by Stokes and by Woldringh at small values of N_R. Consequently, in the limit of large N_R, the contribution of the normal component to the drag force experienced by the solid body would be of the form

$$D=C_D(\tfrac{1}{2}\rho_n V^2)A, \quad (5)$$

where ρ_n is the density of the normal component; C_D is a constant independent of the Reynolds number in this domain and may be expected to be of the same order of magnitude as in the case of experiments with ordinary fluids.[5]

On the other hand, the superfluid component, envisaged as a perfectly inviscid fluid, may not be expected, for velocities less than a certain critical value, to contribute to the drag force. This is the well-known result that follows from the theory of inviscid flow and becomes a paradox in the case of a real fluid. The net drag therefore consists merely of the contribution from the normal component as given by Eq. (5) and is directly proportional to its density. If measurements of this force are made at various temperatures, its variation would give directly the relative fraction of the normal component in liquid helium II,

$$D_T/D_\lambda=\rho_n/\rho, \quad (6)$$

for velocities less than the critical velocity of flow V_c. For velocities greater than V_c, the superfluid is no longer inviscid and such a variation of D as given by Eq. (6) may not be expected.

At the absolute zero of temperature, all of the liquid helium becomes superfluid so that it offers to the solid body no resistance at all; evidently, the d'Alembert paradox would be valid in this case.

[3] J. G. Daunt and R. S. Smith, Revs. Modern Phys. 26, 172 (1954).

[4] H. H. Woldringh, Physica 18, 277 (1952).

[5] In case there is circulation around the body and the latter is of a finite span, there would be a contribution to the profile drag due to the induced drag arising from the cross-wind force. The dependence of D on ρ_n, however, would remain unchanged because the superfluid, being incapable of bearing vortices, may not be expected to convey the induced drag as well.

(b) Suggestions for Experimental Studies

Experiments which may be performed in order to investigate the resistive force that comes into play whenever there is a uniform relative motion between a solid body and a fluid may be classified into two groups, depending upon whether it is the solid body that moves through the fluid, which is otherwise stationary, or whether it is the fluid that flows past the solid body which stays as an obstacle in the stream. The important experiments in the first group are concerned with the free fall of the bodies under the action of gravity and involve measurements of the terminal velocity acquired by the body when the gravitational pull, the buoyancy of the fluid, and the hydrodynamic resistance are in equilibrium. Those in the second group are concerned mainly with the measurement of the reaction as it is communicated from the obstacle to the balance beams through wire suspensions.

(*1*) *Freely Falling Bodies*

At first sight it might appear worthwhile to try to use fine particles, such as lycopodium powder, but the possibility of experimenting with such minute bodies is easily ruled out when we recall that our primary aim is to determine the normal fraction of liquid helium II and hence to investigate only that region wherein the drag force depends directly upon the density, and not upon the viscosity, of the fluid. The Reynolds number N_R $(=Vl\rho_n/\eta_n)$ must, therefore, be of the order of 10^3 and may be as high as 10^5. Since the magnitude of η_n/ρ_n is of the order of 10^{-3} to 10^{-4} cgs units, Vl should preferably lie in the range 1 to 10 cgs units. Further, V must not exceed the critical value (which may be of the order of 1 cm/sec); otherwise the superfluidity exhibited by a part of the liquid may disappear and our previous considerations may accordingly cease to hold. Consequently, fine particles with diameters of the order of 10^{-3} cm would not work for our purpose; they would throw us into the Stokes region or, more probably, into the intermediate region.

We, therefore, resort to spheres of ordinary sizes (diameter $\sim$0.5 cm). Let ρ', ρ, and ρ_n be, respectively, the effective density of the sphere, the total density of the liquid and the density of the normal component. If a be the radius of the sphere, the equation of motion becomes

$$(\tfrac{4}{3}\pi a^3\rho')vdv/dx=\tfrac{4}{3}\pi a^3(\rho'-\rho)g-C_D\cdot\tfrac{1}{2}\rho_n v^2\cdot\pi a^2, \quad (7)$$

with the solution

$$v=v_0[1-\exp(-Bx)]^{\frac{1}{2}}, \quad (8)$$

where

$$B=\tfrac{3}{4}(C_D/a)(\rho_n/\rho'),$$

and the terminal velocity is

$$v_0=\left[\frac{8}{3}\left(\frac{ga}{C_D}\right)\frac{(\rho'-\rho)}{\rho_n}\right]^{\frac{1}{2}};$$

this terminal velocity is almost fully attained when Bx becomes of the order of 5, that is, for $x_0\sim(20/3)\times(a/C_D)(\rho'/\rho_n)$.

Now $C_D\sim 0.5$ in the region where it is independent of N_R. Let $2a=0.5$ cm. We then get

$$v_0=36[(\rho'-\rho)/\rho_n]^{\frac{1}{2}}, \quad x_0\sim(10/3)(\rho'/\rho_n).$$

Since v_0 has to be of a magnitude low enough not to exceed the critical value, the experiment can be successfully performed only if $(\rho'-\rho)$ is made as small as possible in comparison with ρ_n. This means, of course, that ρ' should be as close as possible to ρ, which may be achieved by choosing a material of small density or better by making the sphere hollow. This will not affect the drag or the buoyancy forces but will effectively reduce the density of the sphere.

At this stage it seems worthwhile to point out that better velocities can be obtained by using a disk, of radius a and thickness t, instead of a sphere. The terminal velocity in this case would be given by

$$v_0=\left[\frac{2gt}{C_D}\left(\frac{(\rho'-\rho)}{\rho_n}\right)\right]^{\frac{1}{2}}.$$

Here $C_D\sim 1.1$ and, with $t=1$ mm, one obtains

$$v_0=13[(\rho'-\rho)/\rho_n]^{\frac{1}{2}}.$$

In the present case, however, another difficulty arises which is the following: A disk as thin as the one taken here and falling at such a high Reynolds number as 10^4 is very likely to execute oscillations and the motion therefore may not be steady. This would affect not only the reckoning of the time of fall but also the value of C_D. However, the latter effect is not important in the present context as long as C_D does not vary appreciably with N_R.

(*2*) *Microbalance Measurements*

Here, the solid body which is to serve as an obstacle in the path of the uniform stream of liquid is secured by suspensions in such a way that it is held fast in position (see Prandtl[6]). The forces acting upon the body are made to act on the balance beams and are thus measured. Observations may be made by noting the change in the apparent weight of the body when the liquid is in motion in comparison with its weight when the liquid is at rest.

The drag force in the case of a disk or a sphere of radius a, under the condition of high Reynolds number, is given by

$$D=C_D\cdot\pi a^2\cdot\tfrac{1}{2}\rho_n V^2, \quad (9)$$

which is equivalent to the weight of D/g grams. Therefore, the effective mass change recorded would be

$$\Delta m=(1/g)C_D\cdot\pi a^2\cdot\tfrac{1}{2}\rho_n V^2. \quad (10)$$

---

[6] L. Prandtl, *Essentials of Fluid Dynamics* (Hafner Publishing Company, New York, 1952), p. 254.

In a typical case, with $2a=0.5$ cm, and $V=1$ cm/sec, the mass-equivalent is of the order of 10 micrograms, which is well within the range of measurement. Its variation with temperature would give the variation of ρ_n with T.

III. CROSS-WIND FORCE

Next, we consider those cases of flow in which the hydrodynamic reaction has a nonzero component in a direction perpendicular to the relative velocity—the cross-wind force. In this connection it is well known (the Kutta-Joukowski theorem) that for an infinitely long cylinder of any cross section having a circulation Γ around it, the cross-wind force per unit length, measured perpendicular to the undisturbed stream, is given by

$$L=\rho\Gamma V, \tag{11}$$

where the symbols have their usual meanings. A familiar example of this case is provided by the Magnus effect in which the circulation is created by setting the cylinder in uniform rotation about its axis, which is perpendicular to the direction of V. If ω is the angular velocity of rotation and a the radius of the cylinder, the circulation Γ would be given by

$$\Gamma=2\pi\omega a^2. \tag{12}$$

Again, in the case of liquid helium II the contribution to L would come from the normal component alone because the superfluid, being inviscid, may not be set into rotational motion at all, or, at any rate, may show considerable time lag in picking up the circulation from the rotating cylinder.[7] L would then be proportional to ρ_n, the density of the normal component and not to ρ, the total density. Consequently,

$$L_T/L_\lambda=\rho_n/\rho. \tag{13}$$

[7] S. T. Butler and J. M. Blatt, Phys. Rev. **100**, 495 (1955).

The experimental determination of the cross-wind force may also be affected by the method of balance measurements [see Sec. II, (ii)]. In this case, however, a cylinder of length l and of radius a is more suitable. Let it rotate about its axis with angular velocity ω and let the stream flow past it with an undisturbed velocity V. The (lift) force is then given, when one uses Eqs. (11) and (12), by

$$L'=l\cdot 2\pi\omega a^2\cdot\rho_n V. \tag{14}$$

In a typical case, with $a=0.5$ cm and $l=1$ cm while $\omega a\sim V\sim 1$ cm/sec, the mass equivalent of L' is of the order of 100 micrograms. Again, the measurements are to be made at various temperatures and at various speeds of rotation and/or of the stream flow.

This set of experiments would be important in that they may be expected to throw light upon the irrotational character of the superflow. It would be interesting to find out whether the superfluid picks up circulation from the rotor or not.[8] It may be that it does not do so at velocities (of rotation) less than a certain minimum one, and that beyond this minimum it starts contributing to the cross-wind force thus losing its "quiet" character. However, if it shows superfluidity below ω_c, may be that in this very domain (i.e., $\omega<\omega_c$) it may lose superfluidity if given considerable time to acquire the "equilibrium" state (see also reference 7). One may thus obtain reliable information regarding the nature, equilibrium or metastable, of the superfluid state.

IV. ACKNOWLEDGMENTS

We are indebted to Professor D. S. Kothari for suggesting this problem and for several helpful discussions. Our thanks are also due Professor K. Mendelssohn for his kind interest and valuable suggestions (by correspondence) and to Professor F. C. Auluck for constant encouragement during the course of this work.

[8] See discussion on the behavior of superfluid in a rotating viscometer, S. M. Bhagat and R. K. Pathria, Phys. Rev. **106**, 3 (1957).