Microwave Measurements of High Electron Densities*

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The conventional microwave method for measuring plasma electron densities is limited 'n its validity to relatively low concentrations ($\sim 10^9$ cm^{-3}). Following a theoretical development by Persson, a method is presented whereby much higher densities can be measured. The method is based on eliminating the effect of ac space charge on the probing microwave field. This is accomplished by ensuring that the electric field be everywhere perpendicular to electron-density gradients. A discussion is presented on the effect of neighboring modes on the measuring mode in a microwave cavity.

THE conventional microwave method for measuring plasma electron densities consists of introducing the plasma into a resonant microwave cavity and measuring the amount of detuning which the plasma causes. The equations commonly used to relate the amount of detuning to electron density and collision frequency are[1]

$$\frac{\Delta f}{f_0}=\frac{1}{2}\int\frac{\eta}{1+\gamma^2}E_0^2dv\Big/\int E_0^2dv=\frac{1}{2}\frac{\langle\eta\rangle}{1+\gamma^2},\qquad(1)$$

and

$$\frac{1}{Q}-\frac{1}{Q_0}=\int\frac{\gamma\eta}{1+\gamma^2}E_0^2dv\Big/\int E_0^2dv=\frac{\gamma\langle\eta\rangle}{1+\gamma^2}.\qquad(2)$$

Here, $\eta=ne^2/m\epsilon_0\omega^2$ is a measure of electron density; $\gamma=\nu_m/\omega$ (where ν_m is the collision frequency for momentum transfer) is a measure of the damping in the plasma; Q denotes the unloaded Q value of the cavity; f is its resonant frequency. The subscript zero refers to quantities without the plasma and the absence of the subscript indicates quantities with the plasma. Equations (1) and (2) are usually derived by considering that the plasma is a medium with complex conductivity $\sigma=ne^2/[m(\nu_m+j\omega)]$. The imaginary part of σ causes the shift in the resonant frequency $\Delta f/f_0$ of the cavity, and the real part of the conductivity causes the decrease in the Q value of the cavity.

A question arises concerning the limits of validity of Eqs. (1) and (2). (These limits are discussed extensively by Persson.[2]) In a more correct form, Eq. (1) can be rewritten as

$$\frac{\Delta f}{f_0}=\frac{1}{2}\int\frac{\eta}{1+\gamma^2}\mathbf{E}\cdot\mathbf{E}_0dv\Big/\int\mathbf{E}\cdot\mathbf{E}_0dv.\qquad(3)$$

Equation (3) is valid only when $\Delta f/f_0$ is very small. It exhibits one of the approximations made in deriving Eq. (1), namely, that $\mathbf{E}$, the field in the cavity with the plasma present, has been replaced by $\mathbf{E}_0$, the field without the plasma. When the electron density is low ($\eta\ll1$), the approximation is good, since the plasma does not appreciably disturb the electric field. This is the region in which the conventional microwave method is very successful. As the density is increased, however, $\mathbf{E}$ becomes appreciably different from $\mathbf{E}_0$—for three reasons: (a) ac space charge, commonly called "plasma resonance," makes itself felt as η approaches unity; (b) as η is increased beyond unity, the plasma begins to shield its interior from the field outside; (c) when both η and the pressure are high enough so that the Q value is lowered, the overlapping of higher modes may also cause $\mathbf{E}$ to be different from $\mathbf{E}_0$ by adding to $\mathbf{E}_0$ some of the fields of the higher modes. The major effect, because it usually enters first, is the ac space charge.

By proper methods of design, the space-charge effect may not only be reduced but eliminated. This can be shown theoretically as follows. By combining the first Maxwell equation with the continuity equation and assuming harmonic time variations as $\exp(j\omega t)$, an equation for the space charge ρ is obtained:

$$\nabla\cdot\mathbf{E}=\rho/\epsilon_0=-(\nabla K)\cdot\mathbf{E}/K,\qquad(4)$$

the coefficient K being defined in terms of electron density by $K=1+\sigma/j\omega\epsilon_0$. Equation (4) states that ρ will be zero when the applied field $\mathbf{E}$ is normal to the density gradient. If, therefore, by a proper experimental arrangement, this condition is satisfied, the space-charge effect will not limit the microwave method from measuring high electron density.

A cylindrical cavity that oscillates in the TE_{011} mode, with a cylindrical plasma column placed along the axis of the cavity, satisfies the required conditions. The electric field of the TE_{011} mode, in the absence of the plasma, is given by

$$E_\theta=E_0J_1(\chi_{01}r/a)\sin(\pi z/L);\quad E_r=E_z=0,\qquad(5)$$

where $\chi_{01}=3.832$ is the first root of $J_1(X)=0$, a is the radius of the cavity, and L is its length. Since the field possesses only an azimuthal component, a plasma with density gradients in the axial and radial directions, but not in the azimuthal direction, will satisfy the condition

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[1] Rose, Kerr, Biondi, Everhart, and Brown, Technical Report 140, Research Laboratory of Electronics, Massachusetts Institute of Technology, October 17, 1949 (unpublished).

[2] K. B. Persson, Phys. Rev. **106**, 191 (1957), preceding paper.

$\rho=0$. The additional advantage of the TE_{011} mode is that its electric field goes to zero on the axis of the cavity. Hence, even a dense plasma that is placed in a small region where the field is weak will scarcely disturb the field and will cause only a small shift in the resonant frequency of the cavity. Accordingly, we can expect the linear relationship between the frequency shift and the plasma density, as given by Eq. (1), to be valid, even for high densities.

In order to check this validity, the resonant frequency of a TE_{011} cavity with a uniform plasma column whose radius was one-tenth that of the cavity was computed rigorously in the limit $\gamma=0$. The result is plotted in Fig. 1. It is seen that the frequency increases linearly with density, for values of η that are at least as large as 10. At S band this corresponds to densities of the order of 10^{12} cm^{-3}. It is only at densities higher than 10^{12} cm^{-3} that the plasma begins to shield its interior, and the frequency shift flattens off. It is interesting to note that, for η as large as 20 000 ($n\approx 10^{15}$ cm^{-3}), the resonant frequency is still different from the resonant frequency of a cavity with a perfectly conducting metal post instead of the plasma. There seems to be no way in which to prevent the plasma from shielding the field at microwave frequencies. However, as pointed out in the preceding paper, this can be accomplished by lowering the operating frequency ω by several decades. The slope of the straight line in Fig. 1 is equal to that given by Eq. (1). This equation can, therefore, be used successfully, at least for low pressures, for all values of n less than 10^{12} cm^{-3}. The disadvantage of the TE_{011} mode is that it is insensitive to low electron densities. In order to measure densities from 10^{12} cm^{-3} down to the lowest density measurable by microwave techniques, the TE_{011} mode must be used in conjunction with a conventional mode.

The behavior of the TE_{011} mode was checked experimentally; representative results are shown in Fig. 2. The plasma that was used was the positive column of an oxide-coated-cathode arc discharge in argon mixed with mercury. Since the field in the positive column is not a very strong function of the current through the tube, the current through the tube is a good measure of the average electron density in the plasma. The resultant

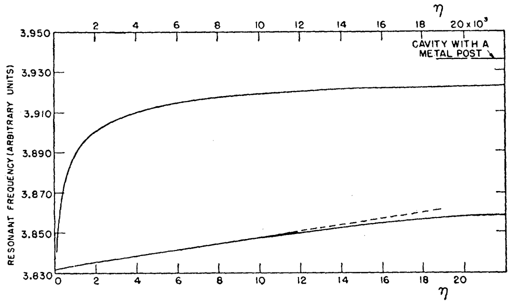

FIG. 1. Resonant frequency of a TE_{011} mode cylindrical cavity as a function of electron density in an axially located plasma column. Plasma radius$=0.1$ cavity radius; $\eta=ne^2/m\epsilon_0\omega^2$.

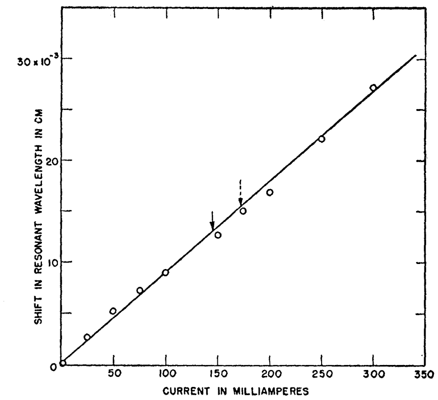

FIG. 2. Shift in the resonant frequency of a TE_{011} mode cylindrical cavity as a function of current through an axially located plasma column.

frequency shift is seen to be linear with increasing current. The highest electron density achieved was about 3×10^{11} per cm^3. One point on the electron-density scale was measured by an independent method. This method is based on the fact that when a plane electromagnetic wave polarized in a direction perpendicular to the plasma cylinder axis impinges on that cylinder, the scattered wave exhibits a resonance when the real part of the coefficient K (defined earlier) goes through minus unity.[3] The two arrows indicate the discrepancy in the value of η as measured by the two methods. The discrepancy is approximately 15%.

Since the TE_{011} mode is degenerate in its resonant frequency with the TM_{111} mode (in practice, the degeneracy is removed by introducing into the cavity the glass tube which will contain the plasma), it is necessary to determine the effect on the measuring mode of other modes that are present in the cavity. Following Slater,[4] this effect can be exhibited through the input impedance Z of the cavity as seen from some point on the line.

$$Z=\sum_{a=1}^{\infty}\frac{v_a^2}{\omega_a\epsilon_0}\bigg/\left\{j\left(\frac{\omega}{\omega_a}-\frac{\omega_a}{\omega}\right)+\left[\int\sigma\mathbf{E}\cdot\mathbf{E}_a dv\bigg/\left(\omega_a\epsilon_0\int\mathbf{E}\cdot\mathbf{E}_a dv\right)\right]\right\},\quad (6)$$

where ω_a is the characteristic frequency of the ath mode, $\mathbf{E}_a$ is its characteristic field, $\mathbf{E}$ is the field in the cavity when the plasma is present, and v_a is related to the coupling coefficient between the ath mode and the line. In the absence of the plasma, when the cavity is assumed to be lossless, the cavity presents a line spectrum with resonant frequencies at $\omega=\omega_a$, $a=1, 2, \cdots$, provided that the v_a are not zero. With the plasma present, the resonant condition is different. Since we allow the modes to interact, the electric field $\mathbf{E}$ is given

[3] T. R. Kaiser and R. L. Closs, Phil. Mag. **43**, 1 (1952).
[4] J. C. Slater, Revs. Modern Phys. **18**, 441 (1946).

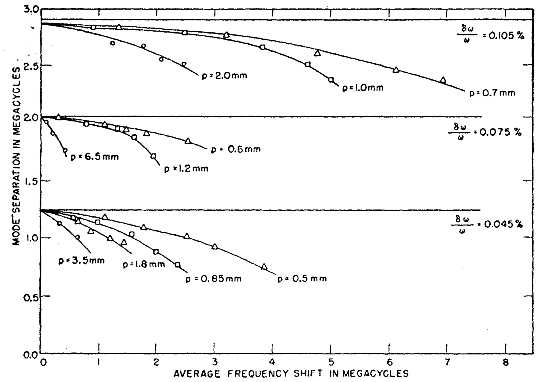

FIG. 3. Frequency separation of the two lowest modes in a rectangular cavity as a function of plasma density.

by some linear combination of the characteristic fields of all modes.

$$\mathbf{E}=\sum_{a=1}^{\infty} e_a\mathbf{E}_a. \tag{7}$$

The characteristic fields are orthonormal, that is,

$$\int \mathbf{E}_a\cdot\mathbf{E}_b dv=\delta_{ab}. \tag{8}$$

To simplify the discussion, we assume that there is no cross coupling arising from the nonuniformity of the plasma; that is, we assume that

$$\int \sigma\mathbf{E}\cdot\mathbf{E}_a dv=\int \sigma E_a^2 dv. \tag{9}$$

Equation (9) will hold if the plasma is uniform or, if the mode configuration is such that the characteristic fields that make up the measuring field **E** are normal to each other. The second condition is actually well approximated in a rectangular cavity in which the three fundamental modes have their fields at right angles. By making the end walls of the cavity almost square, the resonant frequencies of the two lowest modes are almost equal, while the frequency of the third fundamental mode used to produce the plasma can be made higher. In this manner, the overlapping of modes is experimentally confined to the two lowest modes. This degenerates the infinite sum in Eq. (6) to the first two terms and satisfies Eq. (9). With the additional definition:

$$j\left(\frac{\omega}{\omega_a}-\frac{\omega_a}{\omega}\right)+\left[\int \sigma E_a^2 dv\Big/\left(\omega_a\epsilon_0\int E_a^2 dv\right)\right]=j\left(\frac{\omega}{\omega_a'}-\frac{\omega_a'}{\omega}\right)+\frac{1}{Q_a}, \tag{10}$$

the resonance condition, given by the imaginary part of Z equal to zero, can be written as

$$\frac{(\omega/\omega_1'-\omega_1'/\omega)}{[(\omega/\omega_1'-\omega_1'/\omega)^2+1/Q_1^2]}=-\left(\frac{v_2^2}{v_1^2}\right)\frac{(\omega/\omega_2'-\omega_2'/\omega)}{[(\omega/\omega_2'-\omega_2'/\omega)^2+1/Q_2^2]}. \tag{11}$$

When ω_2 is much larger than ω, or if v_2 is zero (so that the higher mode is not coupled), the right-hand side of Eq. (11) is small or zero, the resonant frequency is $\omega=\omega_1'$, and the simple formula given by Eq. (1) holds. To obtain an idea of how much the resonant frequency departs from ω' when the modes do interact, let us assume that the Q values and the coupling coefficients are the same for both modes. Then, if we set $\omega=\omega_1'+\Delta\omega$ and $\omega_2=\omega_1+\delta\omega$, the frequency shift $\Delta\omega$ of the lowest mode resulting from the presence of the higher mode is

$$(\Delta\omega/\omega)(1-\Delta\omega/\delta\omega)=(1/4Q^2)/(\delta\omega/\omega)=\gamma^2\frac{\langle\eta/[2(1+\gamma^2)]\rangle^2}{(\delta\omega/\omega)}. \tag{12}$$

The extra shift is approximately proportional to density squared, to pressure squared (if $\gamma^2\ll 1$), and inversely proportional to the original mode separation. If Eq. (1) were used to calculate the electron density from the measured shift, the relative error made would be

$$\frac{(\Delta\omega/\omega)}{\langle\eta/[2(1+\gamma^2)]\rangle}\approx\gamma^2\frac{\langle\eta/[2(1+\gamma^2)]\rangle}{(\delta\omega/\omega)}, \tag{13}$$

and it may be appreciable at high pressures and electron densities.

These ideas were checked experimentally in the rectangular cavity (described earlier) by measuring the separation in frequency of the two lowest modes as a function of electron density for various pressures and

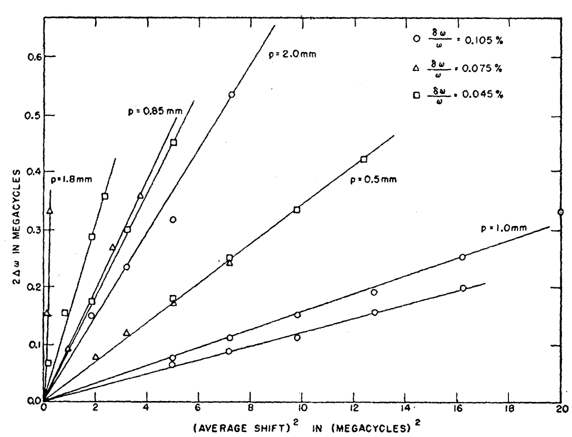

FIG. 4. Frequency shift of the lowest mode caused by the higher mode as a function of plasma density squared.

for three different initial mode separations. The results are shown in Fig. 3, in which the arithmetic mean of the actual shifts of the resonant frequency of the two modes is plotted along the abscissa. Qualitatively, the agreement with Eq. (12) is good, in that the additional shift is proportional to the square of the density, as shown in Fig. 4, and is approximately proportional to the pressure squared. Quantitatively, the observed shifts are smaller than those predicted by Eq. (12). The reason may be that the coupling coefficients of the two modes were not the same, and because it was necessary to introduce into the cavity a tuning stub to vary the initial separation of the modes, the fields of the two modes were not normal to each other in the immediate neighborhood of the stub. However, there can be no doubt that the higher mode affects the lower one and causes the shift in its resonant frequency to be greater under the influence of the plasma than it would be if the second mode were not present.

From the preceding discussion, it is clear that with the TE_{011} mode set up for measuring high electron densities, the TM_{111} mode, or indeed any mode close to it, is not desirable. In practice, all TM and all asymmetric TE modes can be suppressed by cutting azimuthal slots in the wall of the cavity.

To summarize, even though it is not possible to prevent the plasma from shielding the microwave field, by arranging the field at right angles to the density gradients, electron densities of the order of 10^{12} cm^{-3} can be measured by the conventional microwave method, if proper care is taken to eliminate the effect of higher modes on the measuring mode.

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Hydrodynamic Resistance in Liquid Helium II and Determination of the Normal Concentration

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The forces of hydrodynamic reaction in liquid helium II have been studied. Since the superfluid component, being inviscid, may not be expected to contribute to such forces, a measurement of these may provide direct information regarding the concentration of the normal component in the liquid. Experiments along these lines are suggested and it is hoped that they may throw light on the phenomena of "critical velocity" and on the irrotational nature of the superfluid motion.

I. INTRODUCTION

It is well known that according to the two-fluid model,[1] liquid helium II is looked upon as an intimate mixture of two components, the normal and the superfluid. Here, the former is regarded as behaving like any other ordinary liquid whereas the latter is taken to be a perfect inviscid one, at least for low enough velocities.[2] It is evident that in order to make a direct determination of the concentration of the normal component one should look for those hydrodynamic properties in which the viscous nature of the fluid is straightaway effective. In the present paper we have attempted to suggest certain lines along which it may be interesting to perform experiments in order to obtain such information directly. In this connection, we propose to study the forces of reaction which would come into play when liquid helium II flows past a solid body or, alternatively, when the body is made to move through an otherwise stationary bath of the liquid.

In the general hydrodynamical theory of an inviscid fluid one meets with the apparent paradox that the fluid offers no resistance to the motion of a solid body through it. However, this is far from being true in the case of real fluids and in fact one does obtain a resistive force, the so-called profile drag, when account is taken of the skin-friction forces (due effectively to the finite viscosity of real fluids) and the dissipation of energy through the eddying wake. Clearly, the drag force should be absent in the case of the superfluid whereas one should obtain a finite contribution from the normal component. Hence, measurements on the drag force would be of interest, as discussed in detail in Sec. II.

Another hydrodynamic force in which the two components of liquid helium II may be expected to behave differently from each other is the well-known cross-wind force experienced in a uniform stream by a solid body with circulation around it. Here again the superfluid may, for velocities less than a certain critical one, remain free from participating in the rotatory motion and the observed force may give direct information regarding the normal concentration in the liquid. The expected results in this case are elaborated in Sec. III.

[1] L. Tisza, Nature **141**, 913 (1938); F. London, Phys. Rev. **54**, 947 (1938).
[2] K. R. Atkins, *Advances in Physics* (Taylor and Francis, Ltd., London, 1952), Vol. 1, p. 169.