Relativistic Dispersion Relation Approach to Photomeson Production*†

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Relativistic dispersion relations for photomeson production, analogous to the pion-nucleon scattering dispersion relations, are formulated without proof. The assumption that the 33 resonance dominates the dispersion integrals then leads to detailed predictions about the photomeson amplitude. An attempt is made to keep first order (in v/c) nucleon recoil effects. Except for the latter, the predictions of the cutoff model are generally reproduced.

A. INTRODUCTION

The preceding article[1] has applied the relativistic dispersion relation method to a discussion of low-energy pion-nucleon scattering; the purpose of the present paper is to extend the dispersion approach to photomeson production. Two distinct problems will be faced: First the general dispersion relations for photomeson production must be formulated, and second a way must be found for approximately evaluating at low energies the integrals which occur.

Sections B and C of this paper will be concerned with the formulation of the dispersion relations while D and E describe an attempt to evaluate them on the basis of the assumption that the 33 resonance dominates all integrals. In Sec. F a formula for the complete low-energy photoproduction amplitude is written down and discussed.

B. KINEMATICAL CONSIDERATIONS

2. Let the four-vector momenta of the incident photon and outgoing pion be denoted by k and q, respectively, while those of the initial and final nucleons are p_1 and p_2. Momentum-energy conservation,

$$p_1+k=p_2+q, \tag{2.1}$$

means that of these four momenta only three are independent. We choose to consider the combination

$$P=\tfrac{1}{2}(p_1+p_2) \tag{2.2}$$

together with k and q as the three independent four-vectors.

The mass shell restrictions, $p_1^2=p_2^2=-M^2$, $q^2=-1$, and $k^2=0$, means that only two independent scalars can be formed from our three independent vectors. We choose

$$\nu=-P\cdot k/M=-P\cdot q/M, \tag{2.3}$$

and

$$\nu_1=-q\cdot k/2M. \tag{2.4}$$

To form further invariants, one must use the photon polarization ϵ and (or) the nucleon Dirac operator γ.

3. The most general S-matrix element must be a function of Lorentz invariants so it is useful to enumerate all the independent invariant quantities involving ϵ and γ. Remembering that

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}, \tag{3.1}$$

and that

$$(i\gamma\cdot p_1+M)u_1=0, \tag{3.2}$$

$$(i\gamma\cdot p_2+M)u_2=0, \tag{3.3}$$

where u_1 and u_2 are the Dirac spinors associated with the initial and final nucleon states, it follows that the only independent invariants involving γ are $\gamma\cdot\epsilon$ and $\gamma\cdot k$. The reason is that $\gamma\cdot p_1$ and $\gamma\cdot p_2$ can be moved via (3.1) either to the extreme right or the extreme left of the S-matrix element, where they may be eliminated by use of (3.2) or (3.3). The quantity $\gamma\cdot q$ can be replaced by $\gamma\cdot(p_1-p_2+k)$ from momentum conservation. The only independent scalar products involving ϵ, in addition to $\gamma\cdot\epsilon$, are $P\cdot\epsilon$ and $q\cdot\epsilon$, since $k\cdot\epsilon=0$.

The matrix element must, of course, be linear and homogeneous in ϵ but from (3.1) it follows also that it can at most contain $\gamma\cdot k$ linearly. Because $\gamma\cdot k$ anticommutes with $\gamma\cdot\epsilon$ all factors of $\gamma\cdot k$ may be brought together and by (3.1) reduced to the zeroth or first power. A substantial further restriction on the form of the matrix element results if we consider in addition the requirement of gauge invariance.

4. Stated concisely, gauge invariance demands that if ϵ is formally replaced by k the matrix element must vanish. Taken together with the considerations of the preceding section, this requirement allows only four

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† Dispersion relations for photomeson production have also been studied by A. A. Logunov and B. M. Stepanov, Doklady **110**, 3 (1956) and by E. Corinaldesi, Nuovo cimento **IV**, 6 (1956).
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[1] Chew, Goldberger, Low, and Nambu, preceding paper [Phys. Rev. **106**, 1337 (1957)].

TABLE I. Matrix elements of $\mathcal{g}^{(\pm,0)}$ for the four possible charge configurations.

	$\gamma+p\to\pi^0+p$	$\gamma+n\to\pi^0+n$	$\gamma+p\to\pi^++n$	$\gamma+n\to\pi^-+p$
$\mathcal{g}^{(+)}$	1	1	0	0
$\mathcal{g}^{(-)}$	0	0	$\sqrt{2}$	$-\sqrt{2}$
$\mathcal{g}^{(0)}$	1	-1	$\sqrt{2}$	$\sqrt{2}$

independent functions of γ and ϵ: $\gamma\cdot\epsilon\gamma\cdot k$, $P\cdot\epsilon\gamma\cdot k-\gamma\cdot\epsilon P\cdot k$, $q\cdot\epsilon\gamma\cdot k-\gamma\cdot\epsilon q\cdot k$, and $P\cdot\epsilon q\cdot k-q\cdot\epsilon P\cdot k$. It will turn out that a certain linear combination of the first and second of these forms is more convenient than the second alone. Also a factor γ_5 must be added to each because the meson being produced is pseudoscalar. We are thus led to define the four fundamental forms:

$$M_A=i\gamma_5\gamma\cdot\epsilon\gamma\cdot k, \tag{4.1}$$

$$M_B=2i\gamma_5(P\cdot\epsilon q\cdot k-P\cdot kq\cdot\epsilon), \tag{4.2}$$

$$M_C=\gamma_5(\gamma\cdot\epsilon q\cdot k-\gamma\cdot kq\cdot\epsilon), \tag{4.3}$$

$$M_D=2\gamma_5(\gamma\cdot\epsilon P\cdot k-\gamma\cdot kP\cdot\epsilon-iM\gamma\cdot\epsilon\gamma\cdot k). \tag{4.4}$$

The factors i and 2 are for convenience in subsequent calculations.[2]

Combining all results to this point, we see that the complete invariant photomeson transition matrix element may be written

$$H=M_AA+M_BB+M_CC+M_DD, \tag{4.5}$$

where the quantities A, B, C, D are functions of ν and ν_1 as well as the nucleon isotopic spin τ. It is understood, of course, that (4.5) is to be sandwiched between initial and final nucleon spinors in the standard manner $\bar{u}_2Hu_1$.[3]

5. The isotopic spin analysis has already been given by Watson.[4] If one denotes the isotopic spin index of the outgoing pion by β, there are three independent nucleon isotopic spin combinations possible:

$$\mathcal{g}_\beta^{(+)}=\tfrac{1}{2}(\tau_\beta\tau_3+\tau_3\tau_\beta)=\delta_{\beta 3}, \tag{5.1}$$

$$\mathcal{g}_\beta^{(-)}=\tfrac{1}{2}(\tau_\beta\tau_3-\tau_3\tau_\beta)=\tfrac{1}{2}[\tau_\beta,\tau_3], \tag{5.2}$$

$$\mathcal{g}_\beta^{(0)}=\tau_\beta. \tag{5.3}$$

These particular combinations are chosen so as to be either Hermitian or anti-Hermitian. For future reference we present in Table I values of the matrix elements of $\mathcal{g}^{(\pm,0)}$ for the four possible charge configurations. Note that the superscript $(\pm,0)$ bears no simple relation to the charge of the meson being produced.

It is now possible to make the isotopic spin dependence of the amplitude explicit by writing

$$A(\nu,\nu_1,\tau)=A^{(+)}(\nu,\nu_1)\mathcal{g}_\beta^{(+)}+A^{(-)}(\nu,\nu_1)\mathcal{g}_\beta^{(-)}+A^{(0)}(\nu,\nu_1)\mathcal{g}_\beta^{(0)}, \tag{5.4}$$

with a similar decomposition for B, C, and D. The problem has thus been reduced to 12 invariant functions of the two variables ν and ν_1. These will be referred to in general as $H_j(\nu,\nu_1)$ where j runs from 1 to 12.

6. Now let us investigate the consequences of crossing symmetry. The most convenient expression of this symmetry arises from an exchange of incoming and outgoing *nucleon* lines in the Feynman diagrams, which one can show is achieved by setting $p_1=-p_2$, $p_2=-p_1$ and taking the transpose conjugate of the nucleon spin and isotopic spin operators. The photon and meson variables are untouched. Since $P\to-P$ and since each of the γ's which occurs anticommutes with each of the others, we see that under crossing

$$M_A\to M_A,\quad M_B\to M_B,\quad M_C\to -M_C,\quad M_D\to M_D. \tag{6.1}$$

The chosen isotopic spin operators also have a pure crossing symmetry. Evidently

$$\mathcal{g}^{(+)}\to\mathcal{g}^{(+)},\quad \mathcal{g}^{(-)}\to-\mathcal{g}^{(-)},\quad \mathcal{g}^{(0)}\to\mathcal{g}^{(0)}. \tag{6.2}$$

Finally observe that under crossing

$$\nu\to-\nu,\quad \nu_1\to\nu_1,$$

so that in order to satisfy the crossing requirement $A^{(+,0)}$, $B^{(+,0)}$, $C^{(-)}$, and $D^{(+,0)}$ must all be even functions of ν while $A^{(-)}$, $B^{(-)}$, $C^{(+,0)}$, and $D^{(-)}$ are odd functions.

The standard symmetry considerations have now been exhausted but one general principle still remains unexploited—the unitarity of the S matrix. It is well known[4] that, for photoproduction, unitarity relates the phase of an outgoing state of well-defined angular momentum, isotopic spin, and parity to the phase of the corresponding scattering amplitude. The above decomposition of the photomeson amplitude into twelve parts H_j, however, does not correspond to an eigenstate expansion. In order to apply unitarity, it is necessary to find the relation getween the amplitudes H_j and eigenamplitudes.

7. Let the complete photoproduction amplitude be denoted by f, such that the differential cross section for meson production in the barycentric system is

$$\frac{d\sigma}{d\Omega}=\frac{q}{k}|\langle 2|\mathfrak{F}|1\rangle|^2, \tag{7.1}$$

where the matrix element indicated is taken between initial and final Pauli (not Dirac) spinors. For a given isotopic spin configuration, it is then possible to write $\mathfrak{F}$ as follows

$$\mathfrak{F}=i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}f_1+\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})}{qk}\mathfrak{F}_2+\frac{i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}}{qk}\mathfrak{F}_3+\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}}{q^2}\mathfrak{F}_4, \tag{7.2}$$

[2] For purposes of orientation, one may remark that the nonrelativistic limits of these forms are as follows: $M_A\to i\omega\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}$, $M_B\to i\omega\mathbf{q}\cdot\boldsymbol{\varepsilon}\boldsymbol{\sigma}\cdot(\mathbf{k}-\mathbf{q})$, $M_C\to i\boldsymbol{\sigma}\cdot\mathbf{q}\times(\mathbf{k}\times\boldsymbol{\varepsilon})+i\omega^2\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}$, and $M_D\to\mathbf{q}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})$, where $\boldsymbol{\sigma}$, $\mathbf{q}$, $\mathbf{k}$, and ω are as defined in Sec. (7).

[3] A discussion of the limitations imposed by gauge invariance essentially equivalent to that given here has been published by Z. Koba *et al.*, Progr. Theoret. Phys. (Japan) **VI**, 849 (1951).

[4] K. M. Watson, Phys. Rev. **95**, 228 (1954).

where $f_1 \cdots f_4$ are functions of energy and angle in the barycentric system and $\mathbf{q}$ and $\mathbf{k}$ are the meson and photon three-momenta. The angular dependence may be made explicit through an expansion involving derivatives of Legendre polynomials:

$$\mathfrak{F}_1=\sum_{l=0}^{\infty}[lM_{l+}+E_{l+}]P_{l+1}'(x)+[(l+1)M_{l-}+E_{l-}]P_{l-1}'(x), \quad (7.3)$$

$$\mathfrak{F}_2=\sum_{l=1}^{\infty}[(l+1)M_{l+}+lM_{l-}]P_l'(x), \quad (7.4)$$

$$\mathfrak{F}_3=\sum_{l=1}^{\infty}[E_{l+}-M_{l+}]P_{l+1}''(x)+[E_{l-}+M_{l-}]P_{l-1}''(x), \quad (7.5)$$

$$\mathfrak{F}_4=\sum_{l=1}^{\infty}[M_{l+}-E_{l+}-M_{l-}-E_{l-}]P_l''(x). \quad (7.6)$$

Here x is the cosine of the angle of emission in the barycentric system and is related to ν_1 by

$$x=(k\omega_q-2M\nu_1)/kq, \quad (7.7)$$

if $\omega_q=(q^2+1)^{\frac{1}{2}}$. The derivation of formulas (7.3) to (7.6) is lengthy but can be carried out by straightforward methods which have nothing to do with meson theory.

The energy-dependent amplitudes $M_{l\pm}$ and $E_{l\pm}$ refer to transitions initiated by magnetic and electric radiation, respectively, leading to final states of orbital angular momentum l and total angular momentum $l\pm\frac{1}{2}$. Superscripts $(\pm,0)$ may be added to each quantity in formulas (7.2) to (7.6) in order to designate the isotopic spin character of the transition.

It is possible to express the operators $M_A \cdots M_D$, defined by (4.1) $\cdots$ (4.4), in terms of the spin combinations introduced in (7.2) so long as the initial and final states do not contain antinucleons. By a straightforward comparison, one then arrives at a set of linear equations connecting the four amplitudes A, B, C, D to the four amplitudes $F_1 \cdots F_4$:

$$F_1=4\pi\frac{2W}{W-M}\frac{\mathfrak{F}_1}{[(M+E_2)(M+E_1)]^{\frac{1}{2}}}=A+(W-M)D+\frac{2M\nu_1}{W-M}(C-D), \quad (7.8)$$

$$F_2=4\pi\frac{2W}{W-M}\left(\frac{M+E_2}{M+E_1}\right)^{\frac{1}{2}}\frac{\mathfrak{F}_2}{q}=-A+(W+M)D+\frac{2M\nu_1}{W+M}(C-D), \quad (7.9)$$

$$F_3=4\pi\frac{2W}{W-M}\frac{1}{[(M+E_2)(M+E_1)]^{\frac{1}{2}}}\frac{\mathfrak{F}_3}{q}=(W-M)B+(C-D), \quad (7.10)$$

$$F_4=4\pi\frac{2W}{W-M}\left(\frac{M+E_2}{M+E_1}\right)^{\frac{1}{2}}\frac{\mathfrak{F}_4}{q^2}=-(W+M)B+(C-D). \quad (7.11)$$

As in the scattering problem, W is the total energy in the barycentric system. However, here we must distinguish between the initial nucleon energy $E_1=(k^2+M^2)^{\frac{1}{2}}$ and the final nucleon energy $E_2=(q^2+M^2)^{\frac{1}{2}}$. Some identities helpful in deriving the above relations and in further calculation are as follows:

$$\nu=[(W^2-M^2)/2M]-\nu_1, \quad (7.12)$$

$$\nu_1=(1/2M)(k\omega_q-\mathbf{k}\cdot\mathbf{q}), \quad (7.13)$$

$$\omega_q=k+(1/2W), \quad (7.14)$$

$$k/(M+E_1)=(W-M)/(W+M), \quad (7.15)$$

$$W^2-M^2=2kW. \quad (7.16)$$

The preliminary machinery is now complete. To proceed further we need a new physical principle—in this case to be provided by dispersion relations.

C. THE DISPERSION RELATIONS

8. The assumptions which led in the case of scattering to the forms given by Eqs. (3.1) and (3.2) of the preceding paper lead to a similar result here. That is,

$$\mathrm{Re}H_j(\nu,\nu_1)=R_j\left(\frac{1}{\nu_B-\nu}\pm\frac{1}{\nu_B+\nu}\right)+\frac{1}{\pi}\int_{\nu_0}^{\infty}d\nu'\,\mathrm{Im}H_j(\nu',\nu_1)\left[\frac{1}{\nu'-\nu}\pm\frac{1}{\nu'+\nu}\right], \quad (8.1)$$

where

$$\nu_B=-\nu_1=k\cdot q/2M, \quad (8.2)$$

and

$$\nu_0=1+\frac{1}{2M}(1+k\cdot q). \quad (8.3)$$

These last two forms are the same as for scattering if $k\cdot q$ is replaced by $q_1\cdot q_2$. The principal value of the integral in (8.1) is as usual to be understood; obviously the plus signs are to be used with the even H_j's and the minus signs with the odd ones.

The poles at $\pm\nu_B$ in (8.1) correspond once again to the renormalized Born approximation, with the Pauli magnetic moment term included explicitly. The residues turn out to be:

$$R[A^{(\pm,0)}]=-\tfrac{1}{2}e_rf_r, \tag{8.4}$$

$$R[B^{(\pm,0)}]=-\left(\frac{1}{2M\nu_1}\right)\tfrac{1}{2}e_rf_r, \tag{8.5}$$

$$R[C^{(\pm)}]=R[D^{(\pm)}]=\tfrac{1}{2}f_r(\mu_{Pr}'-\mu_{Nr}), \tag{8.6}$$

$$R[C^{(0)}]=R[D^{(0)}]=\tfrac{1}{2}f_r(\mu_{Pr}'+\mu_{Nr}), \tag{8.7}$$

where μ_{Pr}' and μ_{Nr} are the rationalized anomalous static nucleon moments (no form factor) and e_r and f_r are the rationalized and renormalized electronic charge and pion-nucleon coupling constant,[1] respectively. That is,

$$\begin{aligned}\mu_{Pr}'&=1.78e_r/2M, & \mu_{Nr}&=-1.91e_r/2M,\\ e_r^2/4\pi&=1/137, & f_r^2/4\pi&\approx 0.08.\end{aligned} \tag{8.8}$$

An important special characteristic of the photoproduction problem, when electromagnetic radiative corrections are ignored, is the linear dependence of the amplitude on e and μ. In other words it is possible to consider those parts of the amplitude generated by e separately from those generated by the magnetic moment.[5] We make this separation explicit by writing

$$A^{(\pm,0)}=A_e^{(\pm,0)}+A_\mu^{(\pm,0)},\quad \text{etc.}, \tag{8.9}$$

doubling the number of independent amplitudes but simplifying the Born terms. In other words,

$$R[A_\mu^{(\pm,0)}]=R[B_\mu^{(\pm,0)}]=R[C_e^{(\pm,0)}]=R[D_e^{(\pm,0)}]=0, \tag{8.10}$$

while the residues for $A_e^{(\pm,0)}$, $B_e^{(\pm,0)}$, $C_\mu^{(\pm,0)}$, and $D_\mu^{(\pm,0)}$ are as given in formulas (8.4) to (8.7).

Practically all the general remarks made in the preceding article about the scattering dispersion relations apply also to (8.1). One must assume that each of the amplitudes $H_j(\nu,\nu_1)$ vanishes for infinite ν and that a continuation to the nonphysical range of the variable ν_1 (which corresponds to κ^2 in the scattering problem) is possible. Fourth-order perturbation calculations give support to these assumptions.[3] In what follows, the ν_1 continuation will be made via Legendre polynomials even though considerations pointed out by Symanzik[6] indicate this method to be questionable.

9. The first step in the implementation of (8.1) is the change of variable from ν to W and the replacement of A, B, C, D by $F_1\cdots F_4$, using (7.8)–(7.11). These changes produce relations much more complicated than (8.1) but if unitary is to be employed so as to express the imaginary parts of the amplitudes in terms of scattering phase shifts, the complication seems unavoidable. By a lengthy but straightforward calculation we find

$$\begin{aligned}\mathrm{Re}F_1(W,\nu_1)=F_1{}^B+\frac{1}{\pi}\int_{M+1}^{\infty}dW'\Big\{&\mathrm{Im}F_1(W',\nu_1)\\ &\times\left[\frac{1}{W'-W}\pm\frac{W'+W-4M\nu_1/(W-M)}{W'^2+W^2-2M^2-4M\nu_1}\right]\\ &-\mathrm{Im}F_2(W',\nu_1)\left[\frac{1}{W'+W}\pm\frac{W'-W+4M\nu_1/(W-M)}{W'^2+W^2-2M^2-4M\nu_1}\right]\\ &+\binom{0}{1}\mathrm{Im}\left[\frac{F_3(W',\nu_1)}{W'-M}+\frac{F_4(W',\nu_1)}{W'+M}\right]\frac{4M\nu_1}{W-M}\Big\},\end{aligned} \tag{9.1}$$

$$\begin{aligned}\mathrm{Re}F_2(W,\nu_1)=F_2{}^B+\frac{1}{\pi}\int_{M+1}^{\infty}dW'\Big\{&\mathrm{Im}F_2(W',\nu_1)\\ &\times\left[\frac{1}{W'-W}\pm\frac{W'+W-4M\nu_1/(W+M)}{W'^2+W^2-2M^2-4M\nu_1}\right]\\ &-\mathrm{Im}F_1(W',\nu_1)\left[\frac{1}{W'+W}\pm\frac{W'-W+4M\nu_1/(W+M)}{W'^2+W^2-2M^2-4M\nu_1}\right]\\ &+\binom{0}{1}\mathrm{Im}\left[\frac{F_3(W',\nu_1)}{W'-M}+\frac{F_4(W',\nu_1)}{W'+M}\right]\frac{4M\nu_1}{W+M}\Big\},\end{aligned} \tag{9.2}$$

$$\begin{aligned}\mathrm{Re}F_3(W,\nu_1)=F_3{}^B+\frac{1}{\pi}\int_{M+1}^{\infty}dW'\Big\{&\mathrm{Im}F_3(W',\nu_1)\\ &\times\left[\frac{1}{W'-W}\pm\frac{-W'+W-2M+4M\nu_1/(W'-M)}{W'^2+W^2-2M^2-4M\nu_1}\right]\\ &+\mathrm{Im}F_4(W',\nu_1)\left[\frac{1}{W'+W}\pm\frac{2M-W'-W+4M\nu_1/(W'+M)}{W'^2+W^2-2M^2-4M\nu_1}\right]\\ &\mp 2\,\mathrm{Im}[F_1(W',\nu_1)+F_2(W',\nu_1)]\times\frac{1}{W'^2+W^2-2M^2-4M\nu_1}\Big\},\end{aligned} \tag{9.3}$$

$$\begin{aligned}\mathrm{Re}F_4(W,\nu_1)=F_4{}^B+\frac{1}{\pi}\int_{M+1}^{\infty}dW'\Big\{&\mathrm{Im}F_4(W',\nu_1)\\ &\times\left[\frac{1}{W'-W}\pm\frac{-W'+W+2M+4M\nu_1/(W'+M)}{W'^2+W^2-2M^2-4M\nu_1}\right]\\ &+\mathrm{Im}F_3(W',\nu_1)\left[\frac{1}{W'+W}\pm\frac{-2M-W'-W+4M\nu_1/(W'-M)}{W'^2+W^2-2M^2-4M\nu_1}\right]\\ &\mp 2\,\mathrm{Im}[F_1(W',\nu_1)+F_2(W',\nu_1)]\times\frac{1}{W'^2+W^2-2M^2-4M\nu_1}\Big\},\end{aligned} \tag{9.4}$$

[5] A physical basis for this separation may be seen if one reflects that the nucleon anomalous magnetic moment is independent of e in the sense that it can be changed by the addition of new (heavy) meson fields. Formally the separability of e and μ parts is made possible by the linear character of the photomeson unitarity condition. In contrast to the case of scattering, where the imaginary part of the amplitude is given by a quadratic form, the imaginary part of the photoproduction amplitude is a bilinear function of scattering and production amplitudes—so long as electromagnetic radiative reaction is ignored.

[6] K. Symanzik (private communication).

where in each case the upper signs go with the isotopic superscripts (+) and (0) and the lower signs go with the superscript (−).

The Born terms induced by e are

$$F_{1e}{}^{B}=-F_{2e}{}^{B}=\tfrac{1}{2}e_rf_r\left[\frac{2M}{W^2-M^2}\mp\frac{2M}{W^2-M^2-4M\nu_1}\right], \quad (9.5)$$

$$F_{3e}{}^{B}=\tfrac{1}{2}e_rf_r\frac{1}{2M\nu_1}\left[\frac{2M}{W+M}\mp\frac{2M(W-M)}{W^2-M^2-4M\nu_1}\right], \quad (9.6)$$

$$F_{4e}{}^{B}=-\tfrac{1}{2}e_rf_r\frac{1}{2M\nu_1}\left[\frac{2M}{W-M}\mp\frac{2M(W+M)}{W^2-M^2-4M\nu_1}\right], \quad (9.7)$$

while those induced by μ are

$$F_{1\mu}{}^{B}=-\tfrac{1}{2}f_r\mu_r \times\left[\frac{2M}{W+M}\pm\frac{2M(W-M)-8M^2\nu_1/(W-M)}{-W^2+M^2+4M\nu_1}\right], \quad (9.8)$$

$$F_{2\mu}{}^{B}=-\tfrac{1}{2}f_r\mu_r \times\left[\frac{2M}{W-M}\pm\frac{2M(W+M)-8M^2\nu_1/(W+M)}{-W^2+M^2+4M\nu_1}\right], \quad (9.9)$$

$$F_{3\mu}{}^{B}=F_{4\mu}{}^{B}=\pm\tfrac{1}{2}f_r\mu_r\frac{4M}{4M\nu_1-W^2+M^2}, \quad (9.10)$$

where the abbreviation μ_r means $\mu_{pr}'-\mu_{nr}$ in the case of isotopic superscripts (+) and (−) and means $\mu_{pr}'+\mu_{nr}$ in the case of superscript (0).

10. From this point, the most sensible path to follow is not clear. We shall assume that the polynomial expansions (7.3)–(7.6) are legitimate (even though under the dispersion integrals values of x outside the range −1 to +1 are involved) and by projection we shall form dispersion relations in which individual multipole amplitudes occur on the left-hand side. In each case on the right-hand side, under the integral, an infinite sum of multipole amplitudes will occur and a method must be found for evaluating this sum.

An exact evaluation is, of course, out of the question until an understanding of very high-energy phenomena, including K-particles and hyperons, is achieved. The success of the cutoff model,[7] however, suggests that if only the sub-Bev range of the integration is considered, sensible results may be obtained. The dominance of the low-energy 33 resonance is presumed to be responsible for this circumstance.

We shall consequently keep only $l=0$ and $l=1$ (S and P) amplitudes under the integrals and in addition neglect multiple-meson production. Furthermore we shall everywhere expand in powers of $1/M$ and keep only terms of zeroth and first order in $1/M$. It is conceivable that the main results to be obtained are more generally valid than the method of derivation indicates. The approach described here should be regarded only as a first attempt.

[7] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1579 (1956).

D. EVALUATION OF THE DISPERSION INTEGRALS IN THE STATIC LIMIT

11. To gain an orientation in the photoproduction problem, where the details can be very complicated, we begin by writing down the static ($M=\infty$) limit of the dispersion relations for the electric dipole amplitude leading to a final S state and the electric quadrupole and magnetic dipole amplitudes leading to final P states. These relations are obtained by projection from (9.1)$\cdots$(9.4), using (7.3)$\cdots$(7.6) in order to identify the individual multipole amplitudes. Introducing $\omega=W-M$ and then setting $M=\infty$, one finds

$$\mathrm{Re}E_{0+}(\omega)=E_{0+}{}^{B}+\frac{\omega}{\pi}\int_1^\infty d\omega' \times\left\{\frac{1}{\omega'}\,\mathrm{Im}E_{0+}(\omega')\left[\frac{1}{\omega'-\omega}\pm\frac{1}{\omega'+\omega}\right] -2\binom{1}{0}\mathrm{Im}\frac{M_{1-}(\omega')-M_{1+}(\omega')}{k'q'} +\frac{6}{\omega'}\binom{\omega'}{2\omega}\mathrm{Im}\frac{E_{1+}(\omega')}{k'q'}\right\}, \quad (11.1)$$

$$\mathrm{Re}\frac{E_{1+}(\omega)}{kq}=\frac{E_{1+}{}^{B}}{kq}+\frac{\omega}{\pi}\int_1^\infty\frac{d\omega'}{\omega'} \times\mathrm{Im}\frac{E_{1+}(\omega')}{k'q'}\left[\frac{1}{\omega'-\omega}\pm\frac{1}{\omega'+\omega}\right], \quad (11.2)$$

$$\mathrm{Re}\frac{M_{1-}(\omega)-M_{1+}(\omega)}{kq}=\frac{M_{1-}{}^{B}-M_{1+}{}^{B}}{kq} +\frac{1}{\pi}\int_1^\infty d\omega'\left\{\mathrm{Im}\frac{M_{1-}(\omega')-M_{1+}(\omega')}{k'q'}\left[\frac{1}{\omega'-\omega}\mp\frac{1}{\omega'+\omega}\right] +\frac{6}{\omega'}\binom{0}{1}\mathrm{Im}\frac{E_{1+}(\omega')}{k'q'}\right\}, \quad (11.3)$$

$$\mathrm{Re}\frac{M_{1-}(\omega)+2M_{1+}(\omega)}{kq}=\frac{M_{1-}{}^{B}+2M_{1+}{}^{B}}{kq} +\frac{1}{\pi}\int_1^\infty d\omega'\,\mathrm{Im}\frac{M_{1-}(\omega')+2M_{1+}(\omega')}{k'q'} \times\left[\frac{1}{\omega'-\omega}\pm\frac{1}{\omega'+\omega}\right], \quad (11.4)$$

where the isotopic spin convention is the same as in Eqs. (9.1)–(9.4).

12. The Born terms associated with μ are very simple in the static limit:

$$E_{0+,\mu}{}^{B(\pm,0)}=-f\omega\begin{bmatrix}\mu_p'-\mu_n\\0\\\mu_p'+\mu_n\end{bmatrix}, \quad (12.1)$$

$$E_{1+,\mu}{}^{B(\pm,0)}=0, \quad (12.2)$$

$$\frac{M_{1-,\mu}{}^{B(\pm,0)}-M_{1+,\mu}{}^{B(\pm,0)}}{kq}=-\frac{f}{\omega}\begin{bmatrix}\mu_p'-\mu_n\\0\\\mu_p'+\mu_n\end{bmatrix}, \quad (12.3)$$

$$\frac{2M_{1+,\mu}{}^{B(\pm,0)}+M_{1-,\mu}{}^{B(\pm,0)}}{kq}=-\frac{f}{\omega}\begin{bmatrix}0\\\mu_p'-\mu_n\\0\end{bmatrix}, \quad (12.4)$$

where f, μ_p', and μ_n are now nonrationalized constants. The Born terms associated with e are more complicated so we postpone their consideration. It is worth mentioning here, however, that a well-defined group of the $1/M$ corrections to the static limit of the e terms turns out to have precisely the same form as (12.1) $\cdots$ (12.4) with μ_p' replaced by $e/2M$ and μ_n equal to zero. In other words, if the *total* nucleon moments are used in (12.1) $\cdots$ (12.4), rather than the *anomalous* moments, a well-defined group of $1/M$ effects is correctly included.

It is evidently consistent with Eq. (11.2) to set $E_{1+\mu}$ equal to zero, a result which agrees with the cutoff model[7] and with one's intuition. Further, with no electric quadrupole, Eqs. (11.3) and (11.4) become equivalent to those of the cutoff model for the amplitudes referred to in reference 7 as $\mathcal{H}^V$ and $\mathcal{H}^S$, except that the cutoff factor here is missing. If, however, all important contributions to the dispersion integrals occur for ω less than the cutoff energy ($\sim$1 Bev), the solutions of the cutoff-model equations must approximately satisfy Eqs. (11.3) and (11.4). These solutions of the dispersion equations are probably not unique and we do not understand at present how to justify their selection without using the cutoff model as a guide. It would be surprising, however, if any other solutions should be physically interesting so long as the neglect of high-energy effects is correct.

13. The solutions of the four equations for $M_{1\pm\mu}{}^{(\pm)}$ indicated by the cutoff model are simple multiples of the corresponding P-wave scattering amplitudes:

$$\frac{1}{kq}M_{1\pm,\mu}{}^{(\pm)}=\frac{\mu_p-\mu_n}{2f}h_{1\pm}{}^{(\pm)}=\frac{\mu_p-\mu_n}{2f}f_{1\pm}{}^{(\pm)}/q^2. \quad (13.1)$$

The P amplitudes $f_{1\pm}{}^{(\pm)}$ are defined by formula (2.20) of the preceding paper and comparison of the static dispersion relations (4.2) of the preceding paper satisfied by $f_{1\pm}{}^{(\pm)}$ shows immediately that (13.1) is consistent with (11.4). The group of four amplitudes $M_{1\pm,\mu}{}^{(\pm)}$ is precisely equivalent to $\mathcal{H}^V$ in reference 7 if the full nucleon magnetic moments are used.[8]

Reference 6 gives no closed form solution for $\mathcal{H}^S$, the amplitude equivalent to $M_{1\pm,\mu}{}^{(0)}$ here, but from a practical standpoint the Born approximation should be adequate. The Born term to begin with is small (since $\mu_p+\mu_n=0.88e/2M$, in contrast to $\mu_p-\mu_n=4.69e/2M$), but also the dispersion integrals, in equations for isotopic type (0) amplitudes, contain only $I=\frac{1}{2}$ contributions, which are uniformly small at low energies. We shall therefore make no effort to improve the Born approximation for $M_{1\pm,\mu}{}^{(0)}$.

14. The remaining equation (11.1) for the electric dipole amplitude generated by μ, has no counterpart in the cutoff model even after the electric quadrupole term is dropped. We are correspondingly uncertain as to the correct method of treatment, but the argument can be made that under the dispersion integral only the large $J=\frac{3}{2}, I=\frac{3}{2}$ magnetic dipole amplitude need be considered; in particular the electric dipole terms under the integral, which are proportional to S phase shifts, may be dropped. We are then led to the results

$$\frac{E_{0+,\mu}{}^{(+)}(\omega)}{\omega}\approx-f(\mu_p-\mu_n)-\frac{2}{\pi}\int_1^\infty d\omega'\,\mathrm{Im}\frac{M_{1-,\mu}{}^{(+)}(\omega')-M_{1+,\mu}{}^{(+)}(\omega')}{k'q'}, \quad (14.1)$$

$$\frac{E_{0+,\mu}{}^{(-)}(\omega)}{\omega}\approx 0, \quad (14.2)$$

$$\frac{E_{0+,\mu}{}^{(0)}(\omega)}{\omega}\approx-f(\mu_p+\mu_n). \quad (14.3)$$

The contribution of the 33 resonance to the integral in (14.1) is of such a sign and order of magnitude as to make a substantial cancellation of the large Born term. It is impossible at present to make a reliable calculation of the remainder but it may be quite small.

15. Let us turn our attention now to the e amplitudes in the fixed nucleon limit. The Born terms here are well known,[7] at least before decomposition into multipoles:

$$\mathfrak{F}_e{}^{B(\pm,0)}=ef\begin{bmatrix}0\\1\\0\end{bmatrix}\left[i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}+2i\frac{\boldsymbol{\sigma}\cdot(\mathbf{k}-\mathbf{q})\mathbf{q}\cdot\boldsymbol{\varepsilon}}{(\mathbf{k}-\mathbf{q})^2+1}\right]. \quad (15.1)$$

The multipole analysis of (15.1) is perhaps not so well

[8] It should be observed that in the cutoff model the nucleon moments occurring in the formula corresponding to (13.1) carry a form factor, whereas here we have only the static moments. This puzzling circumstance, shown below to persist even when $1/M$ effects are considered, is presumably due to the basic inadequacy of the cutoff model with respect to Lorentz and gauge invariance.

known but is straightforward, the first four terms being

$$E_{0+,e}{}^{B(\pm,0)}=ef\begin{bmatrix}0\\1\\0\end{bmatrix}F_S, \tag{15.2}$$

$$\frac{1}{kq}E_{1+,e}{}^{B(\pm,0)}=\tfrac{1}{3}ef\begin{bmatrix}0\\1\\0\end{bmatrix}F_Q, \tag{15.3}$$

$$\frac{M_{1-,e}{}^{B(\pm,0)}-M_{1+,e}{}^{B(\pm,0)}}{kq}=ef\begin{bmatrix}0\\1\\0\end{bmatrix}F_M, \tag{15.4}$$

$$\frac{M_{1-,e}{}^{B(\pm,0)}+2M_{1+,e}{}^{B(\pm,0)}}{kq}=0, \tag{15.5}$$

where

$$F_S=1-\frac{1}{2}\left(1+\frac{1-v^2}{2v}\ln\frac{1-v}{1+v}\right), \tag{15.6}$$

$$F_Q=\frac{1}{\omega^2}\left[1-\frac{3}{4v^2}\left(1+\frac{1-v^2}{2v}\ln\frac{1-v}{1+v}\right)\right], \tag{15.7}$$

$$F_M=\frac{3}{4q^2}\left(1+\frac{1-v^2}{2v}\ln\frac{1-v}{1+v}\right), \tag{15.8}$$

if $v=q/\omega$ is the outgoing meson velocity.

Higher e multipoles are not negligible but are presumably well approximated by the Born terms alone. That is to say we need to add to (15.1) only the dispersion integrals associated with $E_{0+,e}$, $E_{1+,e}$, and $M_{1\pm,e}$, as given in Eqs. (11.1)–(11.4). The estimation of these integrals is the difficult part of our problem.

16. Equation (11.2) for the electric quadrupole amplitude is relatively simple, since it contains no coupling to the electric dipole and magnetic dipole amplitudes. To analyze this equation, we first introduce eigenamplitudes of total isotopic spin through the relations

$$E_{1+,e}{}^{(+)}=\tfrac{1}{3}E_{1+,e}{}^{\frac{1}{2}}+\tfrac{2}{3}E_{1+,e}{}^{\frac{3}{2}},$$
$$E_{1+,e}{}^{(-)}=\tfrac{1}{3}E_{1+,e}{}^{\frac{1}{2}}-\tfrac{1}{3}E_{1+,e}{}^{\frac{3}{2}}. \tag{16.1}$$

One of course sets $E_{1+,e}{}^{(0)}$ equal to zero. Then defining

$$Q_I=3E_{1+,e}{}^I/kq, \tag{16.2}$$

Eqs. (11.2) lead to

$$\mathrm{Re}Q_\pm(\omega)=\eta_I F_Q+\frac{\omega}{\pi}\int_1^\infty\frac{d\omega'}{\omega'}\times\left[\frac{\mathrm{Im}Q_I(\omega')}{\omega'-\omega}+\sum_{I'=\frac{1}{2}}^{\frac{3}{2}}C_{II'}\frac{\mathrm{Im}Q_{I'}(\omega')}{\omega'+\omega}\right], \tag{16.3}$$

where

$$\eta_I=ef\begin{pmatrix}2\\-1\end{pmatrix}_{I=\frac{3}{2}}^{I=\frac{1}{2}}, \tag{16.4}$$

and

$$C_{II'}=\begin{pmatrix}-\frac{1}{3}&\frac{4}{3}\\\frac{2}{3}&\frac{1}{3}\end{pmatrix}. \tag{16.5}$$

Equation (16.3) is substantially simpler than the corresponding Eq. (39) obtained in reference 7 on the basis of the cutoff model. The earlier equation contained two variables rather than one and was consequently less tractable. Presumably the physical content of the two equations is the same although this fact has not been proved.

The unitary condition tells us that $Q_{\frac{1}{2}}$ has the phase $e^{i\delta_{13}}$ while $Q_{\frac{3}{2}}$ has the phase $e^{i\delta_{33}}$. The type of argument used in reference 7 then says that if there were a cutoff a solution to (16.3) could be found of the form

$$Q_I(\omega)=\eta_I F_Q(\omega)+\frac{\omega^2}{\pi}h_{I3}(\omega)\int_1^\infty d\omega'\times\frac{q'^3v^2(q'^2)}{\omega'^2}\left[\frac{\eta_I F_Q(\omega')}{\omega'-\omega-i\epsilon}+\frac{G_{QI}(\omega')}{\omega'+\omega}\right], \tag{16.6}$$

where $v^2(q'^2)$ is the cutoff factor,

$$h_{I3}=\frac{e^{i\delta_{I3}}\sin\delta_{I3}}{q^3v^2(q^2)}, \tag{16.7}$$

and $G_{QI}(\omega')$ is an unknown function to be determined by the crossing requirement,

$$Q_I(-\omega)=-\textstyle\sum_{I'}C_{II'}Q_{I'}(\omega). \tag{16.8}$$

The introduction of a cutoff at this point is, of course, not really legitimate since we are supposed to be considering a local theory. At the present time, however, we do not know how to discuss solutions of equations of the type (16.3) in the absence of a cutoff. Therefore, to make any progress at all, we are forced to assume that those features of the cutoff solution which persist in the limit as the cutoff is removed are features of the local theory. It will be seen below that there is no cutoff parameter in our final formulas.

Satisfaction of the unitarity requirement by (16.6) is easily seen if the delta-function part of the integrand is separated out. This leads to a term

$$ie^{i\delta_{I3}}\sin\delta_{I3}\eta_I F_Q, \tag{16.9}$$

which when added to the Born term gives back the latter multiplied by a factor $e^{i\delta_{I3}}\cos\delta_{I3}$. The principal value part of the integral is of course real and by itself leads to a contribution which satisfies unitarity.

The part of the principal value integral proportional to $\eta_I F_Q$ converges without the cutoff factor and is roughly independent thereof. Calculation shows this part to be small; the reason is the rapid decrease

($\sim 1/\omega^2$) of F_Q at high energies plus the extra factor of ω/ω' which somehow crept into the quadrupole integral. In configuration space one would say that electric quadrupole mesons arise from the outer surface of the nucleon and do not suffer a strong interaction after production.

Because the known part of the principal value integral in (16.6) is so small as to be marginally detectable by present experiments we have not made an effort to solve for G_{QI}. We believe that G_{QI} can be determined, however, by numerical methods if not by analytic, and improvement in the experimental determination of the quadrupole amplitude will make such a calculation worthwhile. For the moment we shall be satisfied with adding the delta-function term (16.9) alone as the quadrupole correction to the Born approximation.

17. The same type of analysis as the above can be carried out for the magnetic dipole equations (11.3) and (11.4). In terms of eigenamplitudes of isotopic spin and angular momentum, defined by

$$\mathfrak{M}_\alpha = M_{1e}{}^\alpha / kq, \tag{17.1}$$

where $\alpha=1$ for $I=\frac{1}{2}, J=\frac{1}{2}$; $\alpha=2$ for either $I=\frac{1}{2}, J=\frac{3}{2}$ or $I=\frac{3}{2}, J=\frac{1}{2}$; and $\alpha=3$ for $I=\frac{3}{2}, J=\frac{3}{2}$, we find

$$\begin{aligned}\mathrm{Re}\mathfrak{M}_\alpha(\omega) = {}& \xi_\alpha F_M(\omega) \\ &+\frac{1}{\pi}\int d\omega' \left\{ \frac{\mathrm{Im}\mathfrak{M}_\alpha(\omega')}{\omega'-\omega} + \sum_\beta A_{\alpha\beta}\frac{\mathrm{Im}\mathfrak{M}_\beta(\omega')}{\omega'+\omega} \right. \\ &\left. + \frac{2\xi_\alpha}{\omega'}\mathrm{Im}\frac{Q_{\frac{1}{2}}(\omega')-Q_{\frac{3}{2}}(\omega')}{3} \right\},\end{aligned} \tag{17.2}$$

where

$$\xi_\alpha = \tfrac{1}{3}ef\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \tag{17.3}$$

and

$$A_{\alpha\beta} = (1/9)\begin{bmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{bmatrix}. \tag{17.4}$$

The $I=\frac{1}{2}, J=\frac{3}{2}$ equivalence to $J=\frac{1}{2}, I=\frac{3}{2}$ is a characteristic of the static limit and will not persist in a fully relativistic treatment.

It is interesting to note the appearance of the electric quadrupole term under the integral in (17.2). The magnetic dipole equations derived in reference 7 from the cutoff model were completely decoupled from the quadrupole, but at the same time, of course, they contained two variables rather than one. The source of these differences deserves further study but we have nothing to report now.

If the approach used to discuss the quadrupole equation is applied to (17.2), one again arrives at the conclusion that the principal value part of the integral analogous to that occurring in (16.6) is probably so small as to be barely detectable. This time there is a logarithmic dependence on the cutoff because the extra factor of ω/ω', present in the quadrupole integral, is missing. However, the Born term in the important 33 state is smaller here by a factor 3. We therefore propose to add as the magnetic dipole correction to the Born approximation just the delta-function term

$$ie^{i\delta_\alpha}\sin\delta_\alpha \xi_\alpha F_M. \tag{17.5}$$

Further study of Eq. (17.2) should certainly be made to see if a better determination of the dispersion integral is possible.

18. The final and most difficult of the static limit dispersion relations is (11.1) for the electric dipole amplitude. The cutoff model is almost useless here as a guide and, as in the case of S-wave scattering, it seems necessary to concede ignorance and include some arbitrary quantities in the amplitude.

First of all, any attempt to evaluate the parts of the dispersion integral in (11.1) which involve M_1 and E_1 will give results proportional to the cutoff, if the cutoff model is used to estimate E_1 and M_1. The part of the integral involving E_0 can be estimated by an iteration of the Born term but current knowledge of the energy dependence of S phase shifts is so inadequate that nothing more than an order of magnitude estimate is believable. Of course, unitarity requires part of the correction to the Born term to be of the form (16.9) and (17.5).

Because the S phase shifts are small in the low-energy region under consideration, it is legitimate to replace $\sin\delta_s$ by δ_s and $\cos\delta_s$ by 1. We then write the electric dipole amplitude generated by e as

$$\frac{1}{ef}E_{0+,e}{}^{(\pm)}(\omega) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} F_S + iF_S \begin{pmatrix} \frac{2}{3}(\delta_1-\delta_3) \\ \frac{1}{3}(2\delta_1+\delta_3) \end{pmatrix} + \begin{pmatrix} \omega N^{(+)} \\ \omega^2 N^{(-)} \end{pmatrix}, \tag{18.1}$$

where $N^{(+)}$ and $N^{(-)}$ are unknown real numbers which we hope are roughly energy independent. [The other unknown electric dipole amplitude (14.1) may be included in the definition of $N^{(+)}$.] Order of magnitude theoretical estimates suggest that the quantities $N^{(\pm)}$ are probably no larger than ~ 0.2 in absolute value and they may be negligible.[9] As discussed below, experi-

[9] Assuming small S phase shifts everywhere and the approximation to $E_{1+,e}$ and $M_{1+,e}$ described above, Eq. (11.1) leads to

$$\begin{aligned}N^{(+)} = \frac{2}{\pi}\int_1^\infty d\omega' \Big\{ & \tfrac{2}{3}[\delta_1(\omega')-\delta_3(\omega')]\frac{F_S(\omega')}{\omega'^2-\omega^2} \\ & + \tfrac{2}{3}[\sin\delta_{13}\cos\delta_{13}-\sin\delta_{33}\cos\delta_{33}]F_Q(\omega') - (4/9)[2\sin\delta_{11}\cos\delta_{11} \\ & -2\sin\delta_{31}\cos\delta_{31}+\sin\delta_{13}\cos\delta_{13}-\sin\delta_{33}\cos\delta_{33}]F_M(\omega') \Big\} \\ & + \text{terms given by (14.1)},\end{aligned}$$

$$\begin{aligned}N^{(-)} = \frac{2}{\pi}\int_1^\infty \frac{d\omega'}{\omega'} \Big\{ & \tfrac{1}{3}[2\delta_1(\omega')+\delta_3(\omega')]\frac{F_S(\omega')}{\omega'^2-\omega^2} \\ & + \tfrac{2}{3}[2\sin\delta_{13}\cos\delta_{13}+\sin\delta_{33}\cos\delta_{33}]F_Q(\omega') \Big\}.\end{aligned}$$

ments indicate that both $N^{(+)}$ and $N^{(-)}$ are smaller than 0.1.

In contrast to the situation regarding dispersion corrections to the electric quadrupole and magnetic dipole amplitudes, it seems likely that an evaluation of $N^{(\pm)}$ will require a major advance in our understanding of pion physics. In particular, the theory of S-wave scattering must be placed on at least as firm a footing as that for the P wave before progress can be expected with the electric dipole calculation.

E. 1/M CORRECTIONS TO THE STATIC LIMIT

19. Because the accuracy of many experiments on photopion production is now $\sim 10\%$ or better it is worthwhile to attempt to improve the static limit by considering first-order nucleon recoil effects, at least for the large parts of the amplitude. Let us survey the static limit results in order to identify the large terms.

First of all there is $M_{1\pm,\mu}^{(\pm)}$, which in the static limit is given by (13.1). This amplitude is large, partly because $\mu_p-\mu_n$ is large and partly because the large 33 scattering amplitude is contained in the h factors. In contrast $M_{1\pm,\mu}^{(0)}$ is small because $\mu_p+\mu_n$ is small and the 33 amplitude is absent. All the electric dipole amplitudes generated by μ are small and the electric quadrupole vanishes completely in the static limit.[10] Of the μ-generated amplitudes, then, we need to correct only $M_{1\pm,\mu}^{(\pm)}$.

Among the e amplitudes the Born term (15.1) must be considered large and requires correction. All dispersion integral additions to (15.1) are small, however, so that $1/M$ modifications are unnecessary. The recoil problem for the e part of the problem is then very simple: One simply calculates the $1/M$ parts of the standard Born approximation.

The results of this calculation are surprisingly trivial. As mentioned in Sec. (12) a group of the $1/M$ e terms simply augment the anomalous proton moment by an amount $e/2M$ so that if the full proton moment is used in all formulas involving μ_p' these corrections are accounted for. A second $1/M$ correction is easily identified with the current due to motion of the final state nucleon. This contribution turns out to be

$$ef\frac{1+\tau_3}{2}\tau_\beta\frac{1}{M\omega}i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}, \qquad (19.1)$$

which is a mixture of S and D waves vanishing at threshold and acting only when the final particle is a proton. The remaining correction is simply to multiply the entire Born term (15.1) by a factor $(1+\omega/M)^{-1}$.

The latter statement requires the qualification that in (15.1) as well as formula (7.2) the quantities q and k are the exact (or at least accurate to order $1/M$) values of the meson and photon momenta in the barycentric system. Confusion is possible here because the three quantities $\omega=W-M$, $\omega_q=(q^2+1)^{\frac{1}{2}}$, and k are all equal in the static limit but differ in order $1/M$. Also, of course, the laboratory and barycentric systems are indistinguishable in the static limit.

The factor $(1+\omega/M)^{-1}$ is often considered to be associated with phase space but because of our starting definition (7.1) of $\mathfrak{F}$ we here associate the factor with what we call the amplitude. There is, of course, no physical content in this departure from conventional procedure.

20. The essential part of the recoil problem, then, lies with the magnetic dipole amplitude induced by the nucleon magnetic moment. By a straightforward calculation, keeping $1/M$ terms, the relations which replace (11.3) and (11.4) for the μ case turn out to be

$$\mathrm{Re}\frac{M_{1-,\mu}^{(\pm)}(\omega)-M_{1+,\mu}^{(\pm)}(\omega)}{kq} = -\frac{f(\mu_p-\mu_n)}{\omega}\begin{pmatrix}1-\omega/2M\\0\end{pmatrix} +\frac{1}{\pi}\int_1^\infty d\omega'\,\mathrm{Im}\frac{M_{1-,\mu}^{(\pm)}(\omega')-M_{1+,\mu}^{(\pm)}(\omega')}{k'q'} \times\left[\frac{1}{\omega'-\omega}\mp\frac{1}{\omega'+\omega}+\begin{pmatrix}1/M\\0\end{pmatrix}\right], \qquad (20.1)$$

$$\mathrm{Re}\frac{M_{1-,\mu}^{(\pm)}(\omega)+2M_{1+,\mu}^{(\pm)}(\omega)}{kq} = -\frac{f(\mu_p-\mu_n)}{\omega}\begin{pmatrix}-\omega/2M\\1\end{pmatrix} +\frac{1}{\pi}\int_1^\infty d\omega'\,\mathrm{Im}\frac{M_{1-,\mu}^{(\pm)}(\omega')+2M_{1+,\mu}^{(\pm)}(\omega')}{k'q'} \times\left[\frac{1}{\omega'-\omega}\pm\frac{1}{\omega'+\omega}+\begin{pmatrix}1/M\\0\end{pmatrix}\right]. \qquad (20.2)$$

These equations should be compared to the P-wave scattering equations (4.1) of the preceding paper, which we reproduce here for ease of reference:

$$\mathrm{Re}[h_{\frac{1}{2}}^{(\pm)}(\omega)-h_{\frac{3}{2}}^{(\pm)}(\omega)] = -\frac{2f^2}{\omega}\begin{pmatrix}1-\omega/2M\\-\omega/2M\end{pmatrix} +\frac{1}{\pi}\int_1^\infty d\omega'\,\mathrm{Im}[h_{\frac{1}{2}}^{(\pm)}(\omega')-h_{\frac{3}{2}}^{(\pm)}(\omega')] \times\left[\frac{1}{\omega'-\omega}\mp\frac{1}{\omega'+\omega}+\frac{1}{M}\right], \qquad (20.3)$$

[10] The vanishing of $E_{1+,\mu}^{B}$ is maintained in order $1/M$.

$$\mathrm{Re}[h_{\frac{1}{2}}^{(\pm)}(\omega)+2h_{\frac{3}{2}}^{(\pm)}(\omega)]=-\frac{2f^2}{\omega}\binom{-\omega/2M}{1-\omega/2M}+\frac{1}{\pi}\int_1^\infty d\omega'\,\mathrm{Im}[h_{\frac{1}{2}}^{(\pm)}(\omega')+2h_{\frac{3}{2}}^{(\pm)}(\omega')]\times\left[\frac{1}{\omega'-\omega}\pm\frac{1}{\omega'+\omega}+\frac{1}{M}\right]. \quad (20.4)$$

It should be added that Eqs. (20.3) and (20.4) differ from Eq. (4.1) of the preceding paper in that they contain contributions to the integrals from the small P-wave scattering amplitudes, which are of course numerically negligible.

Both in the scattering relations, (20.3) and (20.4), and in the photoproduction relations, (20.1) and (20.2), orbital angular momenta of 2 units and higher have been discarded under the dispersion integrals. The legitimacy of this neglect is not clear but, with the present state of knowledge about scattering phase shifts, it seems the only practical course to take.

Notice that even when one includes $1/M$ corrections, the first and the third of the above four photoproduction relations have the same form as the first and third of the P-wave scattering relations. The simple proportionality (13.1), then, would continue to satisfy these two magnetic dipole equations. However, the second and fourth dispersion relations differ between scattering and photoproduction when $1/M$ terms are kept. Let us see how much alteration in (13.1) this difference requires one to make.

21. The dominant amplitude in both scattering and photoproduction belongs, of course, to the 33 state. By taking appropriate linear combinations of the above equations, one finds for the 33 amplitudes the relations

$$\mathrm{Re}\frac{M_{1\mu}^3(\omega)}{kq}=\frac{2}{3}\frac{f(\mu_p-\mu_n)}{\omega}+\frac{1}{\pi}\int_1^\infty d\omega'\times\left\{\frac{\mathrm{Im}M_{1\mu}^3(\omega')/k'q'}{\omega'-\omega}+\sum_{\beta=1}^{3}A_{3\beta}\frac{\mathrm{Im}M_{1\mu}^\beta(\omega')/k'q'}{\omega'+\omega}+\frac{1}{3M}\mathrm{Im}\frac{M_{1\mu}^2(\omega')+2M_{1\mu}^3(\omega')}{k'q'}\right\}, \quad (21.1)$$

$$\mathrm{Re}h_3(\omega)=\frac{4}{3}\frac{f^2}{\omega}+\frac{1}{\pi}\int_1^\infty d\omega'\left\{\frac{\mathrm{Im}h_3(\omega')}{\omega'-\omega}+\sum_{\beta=1}^{3}A_{3\beta}\frac{\mathrm{Im}h_\beta(\omega')}{\omega'+\omega}+\frac{1}{M}\mathrm{Im}h_3(\omega')\right\}, \quad (21.2)$$

where the indices 1, 2, 3 have the significance explained in Sec. (17). One may now substitute the simple trial solution

$$\frac{M_{1\mu}^3(\omega)}{kq}=\frac{\mu_p-\mu_n}{2f}h_3(\omega) \quad (21.3)$$

into the right-hand side of (21.1) to see how well it reproduces itself. The error is a constant,

$$\frac{1}{3M}\frac{1}{\pi}\int_1^\infty d\omega'\frac{M_{1\mu}^2(\omega')-M_{1\mu}^3(\omega')}{k'q'}, \quad (21.4)$$

which can be approximately evaluated, if one assumes a sharp 33 resonance at $\omega=2$, to be

$$\frac{1}{3M}\tfrac{2}{3}f(\mu_p-\mu_n). \quad (21.5)$$

The error (21.5) numerically is no larger than $1/M^2$ corrections which have systematically been dropped, so we shall assume (21.3) and thus in general (13.1) to be of adequate accuracy.

Presumably the reason that a factor $(1+\omega/M)^{-1}$ is not needed in connection with (13.1) as it was for (15.1) is that such a factor is already contained in the scattering amplitudes h_α.

F. COMPLETE PHOTOPRODUCTION AMPLITUDE TO ORDER $1/M$. DISCUSSION

22. We summarize the results of the preceding considerations by writing down complete expressions for the three fundamental amplitudes $f^{(+)}$, $f^{(-)}$, and $f^{(0)}$. To facilitate the writing, the following combinations of P-wave scattering amplitudes are introduced:

$$h^{(++)}=\tfrac{1}{3}(h_{11}+2h_{13}+2h_{31}+4h_{33}), \quad (22.1)$$

$$h^{(+-)}=\tfrac{1}{3}(h_{11}-h_{13}+2h_{31}-2h_{33}), \quad (22.2)$$

$$h^{(-+)}=\tfrac{1}{3}(h_{11}+2h_{13}-h_{31}-2h_{33}), \quad (22.3)$$

$$h^{(--)}=\tfrac{1}{3}(h_{11}-h_{13}-h_{31}+h_{33}), \quad (22.4)$$

as well as the constant

$$\lambda=(g_p-g_n)/4Mf^2, \quad \text{where} \quad g_p=1.78, \quad g_n=-1.91.$$

One then has

$$\frac{1}{ef}\mathfrak{F}^{(+)}=i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\{\tfrac{2}{3}i(\delta_1-\delta_3)F_S+\omega N^{(+)}\}+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}\{-\lambda h^{(+-)}-\tfrac{2}{3}ie^{i\delta_{33}}\sin\delta_{33}(F_Q-\tfrac{1}{3}F_M)\}+i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}\{\lambda h^{(+-)}-\tfrac{2}{3}ie^{i\delta_{33}}\sin\delta_{33}(F_Q+\tfrac{1}{3}F_M)\}+\mathbf{q}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})\{\lambda h^{(++)}+(4/9)ie^{i\delta_{33}}\sin\delta_{33}F_M\}+i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}\frac{1}{2M\omega}. \quad (22.5)$$

The dominant terms here are those containing the 33 scattering amplitude multiplied by λ. All other terms may be regarded as perturbations. Omitted from (22.5) are "small" P scattering amplitudes multiplying F_Q and F_M, which arise from (16.9) and (17.5). These could be included without difficulty but numerically they are no larger than $1/M^2$ corrections.

Next we have

$$\frac{1}{ef}\mathfrak{F}^{(-)}=\frac{1}{1+\omega/M}\left[i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}+2i\frac{\boldsymbol{\sigma}\cdot(\mathbf{k}-\mathbf{q})\mathbf{q}\cdot\boldsymbol{\varepsilon}}{(\mathbf{k}-\mathbf{q})^2+1}\right]$$
$$+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}[i(\tfrac{2}{3}\delta_1+\tfrac{1}{3}\delta_3)F_S+\omega^2N^{(-)}]$$
$$+i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\mathbf{q}\cdot\mathbf{k}[-\lambda h^{(--)}+\tfrac{1}{3}ie^{i\delta_{33}}\sin\delta_{33}(F_Q-\tfrac{1}{3}F_M)]$$
$$+i\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}[\lambda h^{(--)}+\tfrac{1}{3}ie^{i\delta_{33}}\sin\delta_{33}(F_Q+\tfrac{1}{3}F_M)]$$
$$+\mathbf{q}\cdot(\mathbf{k}\times\boldsymbol{\varepsilon})[\lambda h^{(-+)}-(2/9)ie^{i\delta_{33}}\sin\delta_{33}F_M]$$
$$-i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}\frac{1}{2M\omega}. \quad (22.6)$$

Here, in addition to terms containing λh_3, the Born part of the e amplitude is very important.

Finally there is

$$\frac{1}{ef}\mathfrak{F}^{(0)}=-i\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\frac{g_p+g_n}{2M}\omega$$
$$-i\boldsymbol{\sigma}\cdot\mathbf{q}\times(\mathbf{k}\times\boldsymbol{\varepsilon})\frac{g_p+g_n}{2M\omega}+i\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}\frac{1}{2M\omega}. \quad (22.7)$$

This amplitude has no large part at all and may be considered as a perturbation on $\mathfrak{F}^{(+)}$ and $\mathfrak{F}^{(-)}$. The chief significance of $\mathfrak{F}^{(0)}$ is that it gives rise to a difference between the cross sections for positive and negative meson production.

23. In a paper to be published later, the above amplitudes are used to calculate various experimentally measurable cross sections. It turns out that with the unknown terms $N^{(+)}$ and $N^{(-)}$ set equal to zero and $f^2=0.08$, no serious discrepancies between theory and experiment are apparent up to the resonance energy. This value for f^2 is in good agreement with determinations based on scattering experiments.

In closing, it should be emphasized again that the results given above by no means represent a definitive evaluation of the dispersion relations (8.1), even if one assumes the latter to be valid. Our central assumption has been that under dispersion integrals, the only significant contributions arise from the 33 resonance, but beyond this many further approximations have been made. The extent to which the cutoff model has been used as a guide also should not be ignored. There is as yet no real justification for the identification of the cutoff equations with dispersion relations.

The practical results of this paper seem so similar to those of reference 7 based on the cutoff model that it is worthwhile to emphasize the differences. As mentioned already in reference 8, these differences are at least partly due to the lack of Lorentz and gauge invariance in the cutoff model.

First there are trivial modification which could be guessed on the basis of plausibility considerations. These are the "phase space factor" $(1+\omega/M)^{-1}$, the imaginary parts of the electric dipole amplitude proportional to S phase shifts, and the nucleon recoil current contribution (19.1). The absence of the cutoff factor also could, of course, be guessed. Nontrivial new results are the real electric dipole terms $N^{(+)}$, $N^{(-)}$ and the corresponding term in $\mathfrak{F}^{(0)}$ which is responsible for the threshold negative to positive ratio. Only the latter is predicted quantitatively but a qualitative understanding of $N^{(+)}$ and $N^{(-)}$ has been achieved. Also the absence of magnetic moment form factors from the new results should be noted. Finally it should not be forgotten that simpler equations for calculating "secondary scattering" corrections have been achieved even though these equations are as yet unexploited.

With luck, if we have made no serious mistakes, the final amplitude written above may have an accuracy $\sim$5–10% in the subresonance region. It will certainly deteriorate rapidly above resonance. Some further improvement of the theoretical formula should be possible but a basic limitation will necessarily remain due to electromagnetic radiative effects, which among other things destroy charge independence. The fact that radiative corrections are strong enough to produce a 4% difference in mass between neutral and charged pions shows that we must already be close to the minimum error possible within the charge-independent framework of calculation.