track, leaves the emulsion before being brought to rest. Because of the short range of the hyperfragments, the charges have been estimated from charge balance of the secondaries, except for event 9 which is represented by Track 5 in Fig. 1.

In all calculations we have used the range-energy relation given by Baroni *et al.*[2] In the determination of the binding energies we have used the mass values for elementary particles and light nuclei given by Barkas and Hahn,[3] and 36.9 Mev for the Q value in the decay of a free Λ^0 particle.[4]

[2] Baroni, Castagnoli, Cortini, Franzinetti, and Manfredini, Bureau of Standards CERN Bulletin No. 9 (unpublished).

ACKNOWLEDGMENTS

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Electron Scattering from Nonspherical Nuclei*†

B. W. Downs,‡ D. G. Ravenhall, and D. R. Yennie
Stanford University, Stanford, California

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To explain the anomalously smooth cross sections observed in electron scattering from certain nuclei (Hf, Ta, W, U), the contributions to the cross section arising from the nonspherical character of those nuclei have been examined. Approximations are developed for the calculation of these contributions, and for a sample case, Ta, numerical results are given; the value required for the nuclear distortion in order to obtain agreement with experiment in this case is in good agreement with the spectroscopic and Coulomb-excitation values. The results suggest that in certain cases electron scattering will be a useful new method for measuring both the magnitude and the radial shape of nuclear deformations.

1. INTRODUCTION

EXPERIMENTS on high-energy electron scattering by some heavy nuclei (e.g., gold, lead, bismuth) yield differential cross sections with pronounced diffraction structure.[1] On the assumption of elastic scattering from a spherically symmetric nuclear charge distribution these cross sections have been found to indicate distributions approximately uniform in the center, with a relatively sharp surface. Another group of elements (hafnium, tantalum, tungsten, uranium) show, however, markedly different differential cross sections: as can be seen from Fig. 1, they have roughly the same slope (on a semilog plot) but show no diffraction dips. In an attempted analysis of the latter experiments it was found to be not possible to alter the radius and surface thickness of the smoothed uniform charge distribution so as to fit both the lack of diffraction structure and the rather shallow slope. In search of a reason for this strange behavior, one notices that the first-mentioned group of nuclei are at or close to closed nucleon shells, whereas the second group occupies positions in the middle of a shell. Using a collective nuclear model, Bohr and Mottelson[2] have characterized nuclei in the first group as essentially spherically symmetric, while those in the second group have equilibrium shapes far from spherical, having, as a consequence, low-lying levels corresponding to a collective rotation of the outer nucleons. The energy resolution of the electron-scattering experiments does not distinguish between elastic scattering and inelastic scattering corresponding to excitation of such levels. In an attempt to explain the observed smooth cross section, an approximate calculation has been made of differential cross sections for scattering from such a deformed nucleus. It is found that agreement can be obtained by suitable choice of the parameters involved. The method, with more extensive analysis, may possibly be used to give information about both the magnitude and radial shape of nuclear deformations.[3,4]

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‡ Present address: Laboratory of Nuclear Studies, Cornell University, Ithaca, New York.

[1] Hahn, Ravenhall, and Hofstadter, Phys. Rev. **101**, 1130 (1956). For a complete bibliography of electron scattering, see R. Hofstadter, Revs. Modern Phys. **28**, 214 (1956).

[2] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

[3] A preliminary account of this work was given at the 1954 Winter Meeting of the American Physical Society, reported in Phys. Rev. **98**, 277(A) (1954).

[4] A more detailed account of the analysis and results of this calculation is contained in the Ph. D. dissertation of one of us: B. W. Downs, Ph. D. dissertation, Stanford University, 1955 (unpublished).

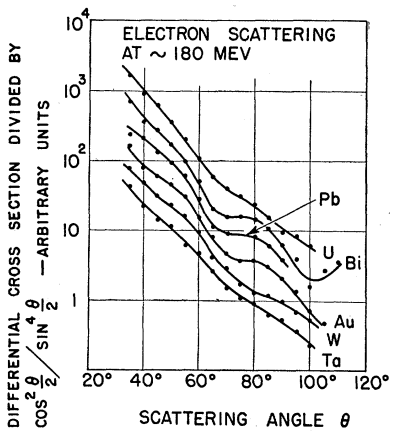

FIG. 1. Experimental cross sections for scattering of 180-Mev electrons by some heavy nuclei. To display the diffraction structure more prominently, the cross section has been divided by $[\cos^2(\frac{1}{2}\theta)/\sin^4(\frac{1}{2}\theta)]$, which is proportional to the point-scattering cross section in first Born approximation.

The system to be investigated is that of a very energetic electron scattered by the Coulomb field of a heavy, nonspherical nucleus, with the possibility of nuclear excitation. It has been shown previously, for the spherically symmetrical case with no excitation, that a solution accurate enough for detailed comparison with the experiments requires a complete phase-shift analysis of the Dirac equation for the system.[5] Such an analysis would be impossible for the system of interest here since the Dirac equation cannot be separated into partial waves. It has therefore been necessary to make a number of approximations. These are as follows:

(a) The interactions additional to the spherically symmetric interaction (that is, the fields of higher multipole order caused by the nonspherical charge distribution and by the nuclear excitation) are assumed small, and are treated by first-order perturbation theory.

(b) The electron wave functions used in the above perturbation theory, which describe scattering in the spherically symmetric part of the charge distribution, will be approximated by distorted plane waves (the zero-order scattering by the spherically symmetric part of the charge distribution is *not* approximated: it is calculated by the phase-shift analysis described previously).

(c) A model of the nucleus which is essentially an extension of Bohr's collective model in the strong-coupling limit will be assumed.

Approximation (a) is necessary in order to be able to do the calculation at all. We feel that this is a plausible approximation, although at present it cannot be strictly justified. We shall discuss it at the end of Sec. 2, after the formalism has been developed. Approximation (b) is not mandatory, since it is feasible to use as zero-order wave functions the exact partial-wave expansion of the elastically scattered wave. In the case of interest here it would be necessary to evaluate and combine about sixty radial matrix elements. The alternative we have chosen is to find for the wave function describing the elastic scattering a three-dimensional approximation which is reliable close to the nucleus. The complete matrix element can then be evaluated analytically. With approximation (c) the sum of elastic and inelastic scattering due to the deformed part of the charge distribution turns out to be independent of the nuclear spin, and is identical with what would be obtained from a classical, deformed charge distribution oriented at random.[6] That this should be so can be understood by uncertainty principle arguments, since the energy is not resolved. We assume, with Bohr,[2] that the radial matrix element is the same for all transitions; instead of the surface delta function, however, we use a smooth transition density approximately proportional to $\partial\rho_0/\partial r$, where ρ_0 is the spherically symmetric part of the charge distribution. The use of the Bohr model for the nuclear matrix elements thus simplifies the analysis, but it is not essential. Since, however, comparison with experiment does not allow complete determination of the parameters of even this simplified model, it would be unprofitable at this stage to consider more complex matrix elements.

2. ANALYSIS

The development of the three-dimensional approximation to the wave function describing the elastic scattering will be given in a later paper,[7] and we summarize it here. It is based on two observations concerning the behavior of the exact partial waves near the nucleus: the radial wave functions are to a good approximation spherical Bessel functions with modified argument; and the total phase shifts are connected for small j by the relation

$$\eta_j = a + bj(j+1), \tag{1}$$

where $j(j+1)\hbar^2$ is the eigenvalue of the square of the total angular momentum J. Thus the total scattering wave function $\varphi^{(\pm)}(\mathbf{r})$ [where (+), (−) refer, respectively, to outgoing and incoming scattered waves], which rigorously is an expansion in terms of the exact radial wave functions $F_j(r)$ and $G_j(r)$ and the spin-angular functions χ_j^1 and χ_j^2,

$$\varphi^{(\pm)}(\mathbf{r}) = \sum_j (kr)^{-1} i^{j-\frac{1}{2}} e^{\pm i\eta_j} [G_j\chi_j^1 + iF_j\chi_j^2], \tag{2}$$

can be approximated at small distances by

$$\varphi^{(\pm)}(\mathbf{r}) \simeq (k'/k)e^{\pm i\eta} \sum_j i^{j-\frac{1}{2}} [j_{j-\frac{1}{2}}(k'r)\chi_j^1 + ij_{j+\frac{1}{2}}(k'r)\chi_j^2]$$
$$= (k'/k)e^{\pm i\eta} e^{i\mathbf{k}'\cdot\mathbf{r}} \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \exp(i\phi) \end{pmatrix}, \tag{3}$$

where k' is the electron's wave number in the vicinity

[5] Yennie, Ravenhall and Wilson, Phys. Rev. **95**, 500 (1954).

[6] L. I. Schiff, Phys. Rev. **96**, 765 (1954).

[7] D. R. Yennie and D. G. Ravenhall (to be published).

of the nucleus and η is now the operator

$$\eta=a+b[i^{-1}\mathbf{r}\times\nabla+\tfrac{1}{2}\sigma]^2. \quad (4)$$

The simple form of Eq. (3) arises from the physical situation that an incident plane wave, after traveling through the weak, slowly varying Coulomb field, arrives at the nucleus with modified argument, amplitude, and phase, and with curved wave fronts [due to the factor $\exp(ibJ^2)$]. For the case of interest here, scattering of 182-Mev electrons by tantalum ($Z=73$), b turns out to be -0.0080 which is small enough for us to expand the exponential in powers of b and retain only the first few terms.

The application of the Born approximation to this physical situation has been made by Schiff in some detail.[6] He obtains the results, mentioned in the Introduction, that the total contribution to the scattering from the intrinsic nuclear deformation is independent of the nuclear spin, and is equivalent to scattering from a classical deformed charge distribution. The corresponding arguments in the present case with the more accurate wave functions (2) are essentially the same as Schiff's, but we give them for the sake of completeness.

The essential feature of the Bohr-Mottelson model is the close connection between the static quadrupole moment and the $E2$ transition matrix elements among the rotational levels. These matrix elements of the charge density operator $\rho_{\text{op}}(\mathbf{r})$,

$$\rho_{\text{op}}(\mathbf{r})=\sum_{\text{protons}} e\delta(\mathbf{r}-\mathbf{r}_p),$$

are related as follows:

$$\begin{aligned}&\langle I'M'|\rho_{\text{op}}(\mathbf{r})|IM\rangle\\&=\delta_{II'}\delta_{MM'}\rho_0(r)+(2I+1)^{\frac{1}{2}}[\textstyle\sum_m \gamma_{II'}{}^{(2)}\\&\quad\times C_{II'}(2m;-M'M)Y_{2m}(\theta,\phi)]\rho_2(r)+\cdots. \quad (5)\end{aligned}$$

The dependence on the Clebsch-Gordan coefficients C is a consequence of rotational invariance. The factors $\gamma_{II'}{}^{(2)}$ are peculiar to the Bohr-Mottelson model; for $I\neq\frac{1}{2}$ they are given by

$$\begin{aligned}[\gamma_{II}{}^{(2)}]^2&=I(2I-1)/(I+1)(2I+3),\\[\gamma_{II+1}{}^{(2)}]^2&=3I/(I+1)(I+2),\\[\gamma_{II+2}{}^{(2)}]^2&=6/(I+2)(2I+3).\end{aligned}$$

They have the property, observed by Schiff,[6] that

$$\textstyle\sum_{I'}[\gamma_{II'}{}^{(2)}]^2=1. \quad (6)$$

In calculating the electron scattering due to these static and transition charge densities, we shall treat the ρ_0 term exactly by using the partial-wave analysis, i.e., we solve exactly the Dirac equation containing the electrostatic potential generated by $\rho_0(r)$. The potential due to ρ_2 is then treated by perturbation theory. The scattering amplitude for elastic collisions is the sum of the ρ_0 term and a part of the ρ_2 term. It is easy to show that the interference term vanishes for an unpolarized target. The total quadrupole contribution to the cross section is simplified because of (6), so that the total cross section, including all transitions, is given by

$$\sigma(\theta)=\sigma_0(\theta)+\sigma_2(\theta); \quad (7)$$

for σ_2 we have the simple expression

$$\sigma_2(\theta)=(E/2\pi\hbar^2c^2)^2\textstyle\sum_m\left|\int\rho_2(r)Y_{2m}(\theta,\phi)V(\mathbf{r})d^3r\right|^2, \quad (8)$$

where $V(\mathbf{r})$ is the potential due to the passing electron:

$$V(\mathbf{r})=e\int\frac{1}{|\mathbf{r}-\mathbf{r}'|}\varphi_{\mathbf{k}'}{}^{(-)*}(\mathbf{r}')\varphi_{\mathbf{k}}{}^{(+)}(\mathbf{r}')d^3r'. \quad (9)$$

We observe that the cross section (8) is just what could be obtained from a classical deformed charge distribution, oriented at random.

A natural way to introduce this classical deformation is to write

$$\rho_d(\mathbf{r})=\rho(r[1-\alpha P_2(\cos\gamma)]), \quad (10)$$

where α is a deformation parameter, and γ is the angle measured from the symmetry axis. Surfaces of constant charge density are give by the relation

$$r[1-\alpha P_2(\cos\gamma)]=\text{constant},$$

i.e., they are concentric and roughly spheroidal. A multipole expansion of (10) gives the result, accurate for small α, that

$$\rho_d(\mathbf{r})=\rho_0(r)+\rho_2(r)P_2(\cos\gamma)+\cdots, \quad (11)$$

where

$$\begin{aligned}\rho_0(r)&=\rho(r)+(1/10)\alpha^2r^2\rho''(r)+\cdots,\\\rho_2(r)&=-\alpha r\rho'(r)+(1/7)\alpha^2r^2\rho''(r)+\cdots. \quad (12)\end{aligned}$$

If for $\rho(r)$ we use any "smoothed uniform" distribution with radius c (distance to the half-point) and surface thickness t (the 90% to 10% distance), then $\rho_0(r)$ is also a smoothed uniform distribution with parameters c_0, t_0 given by

$$\begin{aligned}c_0&\simeq c(1+\alpha^2/5),\\t_0{}^2&\simeq t^2(1+3\alpha^2/5)+\lambda^2\alpha^2c^2. \quad (13)\end{aligned}$$

For the Fermi distribution,

$$\rho(r)=\rho(0)\{1+\exp[(r-c)/0.228t]\}^{-1}, \quad (14)$$

the parameter λ in (13) is 1.08. A result of this plausible assumption about the form of the deformation is that $\rho_2(r)$ is not independent, but is given by $\rho(r)$. It is in fact a smooth function with a maximum at $r\simeq c$, and width of order t. The intrinsic quadrupole moment Q_0, as defined by Bohr and Mottelson,[2] is given by

$$eQ_0=(8\pi/5)\int_0^\infty\rho_2(r)r^4dr. \quad (15)$$

With a deformation of the type (10) this relation can be

simplified to

$$Q_0=2Z\langle r^2\rangle_d[1+6\alpha/7+O(\alpha^3)],$$

where $\langle r^2\rangle_d$ is the mean-square radius of $\rho_d(r)$; for the particular shape (14), $\langle r^2\rangle_d$ can be expressed in terms of c_0 and t_0 by the use of Eqs. (13) and the relations

$$\langle r^2\rangle_d=\langle r^2\rangle[1+9\alpha^2/5+\cdots],$$
$$\langle r^2\rangle=\tfrac{3}{5}c^2[1+1.70(t/c)^2+0.61(t/c)^4]/[1+0.58(t/c)^2]. \quad (16)$$

The remaining problem is to evaluate (9) and (8), using for the eigenstates of the ρ_0 scattering which appear in (9) the approximation (3). To do this it is necessary to expand $\exp(ibJ^2)$ in powers of b. The justification for this is that the j values that are important in the matrix element are of order $k'c$, which at this energy is about 6.4. Thus the exponent will be of order 0.4, so that an expansion which keeps terms up to b^2 should be reasonably accurate. At this point it should also be remembered that the representation (1) for η is itself an approximation; for the present situation it is accurate up to j values of about 6 or 7, however. In the expansion in powers of b, the first term, which is independent of b, results in an expression for (8) which is closely similar to the Born approximation, the only differences being in the modified amplitude and wave number. To this approximation $\sigma_2(\theta)$ still has the zeros typical of the first Born approximation. Because the term linear in b in the matrix element is $\pi/2$ out of phase with the Born-approximation term, the next contribution to $\sigma_2(\theta)$ is of order b^2. We calculate the b^2 term only at the zeros, since at these angles only the b term in the matrix element contributes to the cross section. The evaluation of this term is simplified by noting that

$$\mathbf{r}\times\nabla e^{i\mathbf{k}'\cdot\mathbf{r}}=-\mathbf{k}'\times\nabla_{k'}e^{i\mathbf{k}'\cdot\mathbf{r}}.$$

It can be expressed in terms of derivatives of the Born-approximation matrix element.

We return to the question of the validity of approximation (*a*), that the quadrupole contributions can be treated by first-order perturbation theory. Some measure of the reliability of this approximation is given by the ratio of quadrupole to monopole potentials. The maximum value of this ratio, attained at the surface, is $\epsilon=\frac{3}{5}\alpha[1-0.8(t/c)+\cdots]$. For the values of α needed to fit the experiments, ϵ is at most about 0.1. In fact ϵ probably overestimates the importance of the quadrupole effects, because the quadrupole potential has a short range, whereas the central potential has a long range. In any case, the smallness of ϵ implies that higher-order contributions in ϵ can be neglected. However, there are two contributions to the cross section of order ϵ^2, of which we include only σ_2. The other term of this order in ϵ^2 is the dispersion contribution to the monopole scattering, arising from virtual excitation of the rotational levels. To order ϵ^2 the monopole scattering amplitude is actually of the form

$$F_0(q)+\epsilon^2F_{\text{disp}}(q).$$

We have not calculated $F_{\text{disp}}(q)$, but Schiff's closure estimate of this term[8] for the general case of nuclear excitation suggests that it is of the same order of magnitude as $F_0(q)$, so that the resulting contribution to the monopole cross section is only ϵ^2 of the part we use, and thus can safely be neglected.

3. RESULTS

The results of calculations for a typical case are illustrated in Fig. 2.[4] It has been made for tantalum ($Z=73$), at an energy of 182 Mev. It is seen that the quadrupole scattering does tend to fill in the diffraction minima of the ρ_0 scattering. That this should be so is clear from the simple Born-approximation argument that the form factors for the two contributions vary with q as $j_2(qc)$ and $(3/qc)j_1(qc)$, respectively, and j_1 and j_2 are $\pi/2$ out of phase. Figure 2 also illustrates the rather crude way in which we have estimated the quadrupole cross section. We have calculated the b^2 term only at the diffraction minimum of $\sigma_2(\theta)$ and have assumed that it is negligible at the diffraction maximum. From these points and from the term independent of b^2, which is easier to calculate, we have sketched in an estimated $\sigma_2(\theta)$. To the accuracy with which we wish to

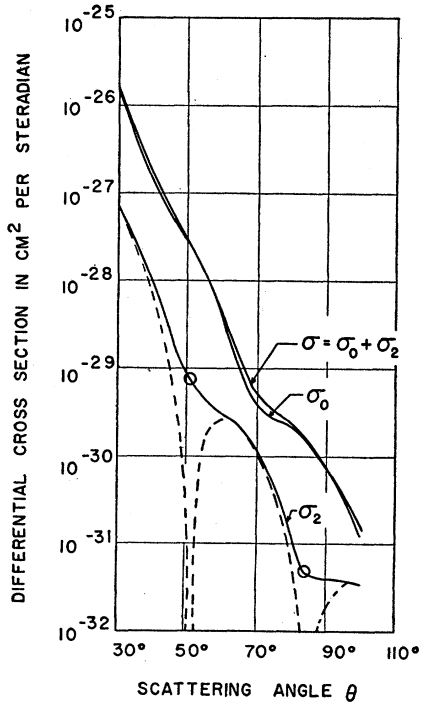

FIG. 2. A plot of the separate contributions $\sigma_0(\theta)$ and $\sigma_2(\theta)$ and the total differential cross section $\sigma(\theta)$, for electron scattering from tantalum ($Z=73$) at 182 Mev.

[8] L. I. Schiff, Phys. Rev. **98**, 756 (1955). See also B. W. Downs, Phys. Rev. **101**, 820 (1956).

compare the calculations with the experiments, this crudeness does not affect our conclusions appreciably. For a more precise comparison it is possible to calculate the b^2 term at all angles, but the evaluation is rather lengthy.

With the limitation on $\rho_2(r)$ imposed by (10), the model contains three parameters, c, t, and α, or, as has proved more convenient, c_0, t_0, and α. It is *a priori* unlikely that we can do better here than was possible with the spherically symmetric nuclei, where the pronounced diffraction structure enabled us to determine c and t. We first observe that it is not possible to fit these experimental results with $\alpha=0$ and a distribution ρ of the type exemplified by (14). Although a distribution of this type can, with large enough t, yield a cross section with little diffraction structure, the slope of the cross section is then much too steep. We next admit that since the experimental results contain very little diffraction structure, it is not possible to determine c_0 with any accuracy; we assume a value for c_0 scaled down from the result obtained for the spherically-symmetric nuclei, lead and gold, by the relation $c=1.09A^{\frac{1}{3}}\times10^{-13}$ cm, and increased a little to allow for the α^2 term in Eq. (13). If now a value of t_0 is assumed, α can be chosen to give a smooth cross section, but this cross section will not in general have the correct slope. Thus by fitting the slope also we can determine both α and t_0. Figure 3 illustrates our results for various values of α, with $c_0=6.38\times10^{-13}$ cm and $t_0=2.80\times10^{-13}$ cm. The experimental values are for Ta^{181}, at 182 Mev.[1] We estimate that the curve for $\alpha=0.19$ is smooth and is a reasonable fit to the experimental points. The intrinsic surface thickness t is from Eq. (13) equal to 2.5×10^{-13} cm, which is close to the value obtained for the spherically symmetric nuclei.[1] With the assumed radius, this value of α leads to an intrinsic quadrupole moment Q_0 of 10×10^{-24} cm^2, which is roughly midway between the spectroscopic and Coulomb excitation values.[9]

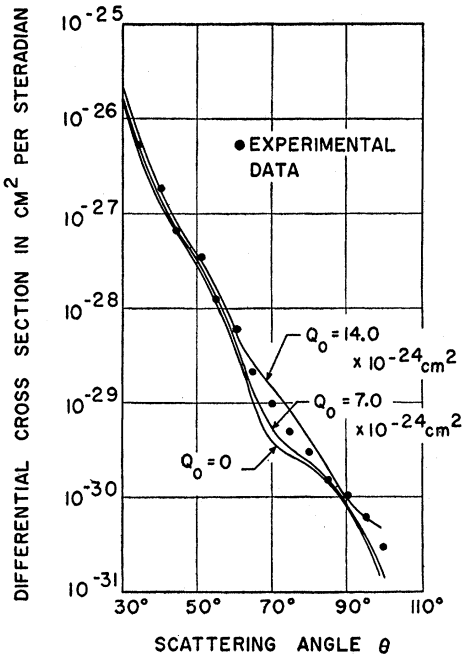

FIG. 3. Theoretical cross sections for various values of Q_0 compared with the experimental values, all for tantalum ($Z=73$) at 182 Mev.

SUMMARY

An approximate calculation has been made of quadrupole contributions to electron scattering from heavy, distorted nuclei. The general features of the results are qualitatively reliable, we feel, although inability to calculate the errors involved in the approximations does not allow us to quote accurate numerical results. The nuclear model employed contains three parameters, of which one, the radius c, cannot be determined. Assuming a reasonable value for c, comparison of the theoretical cross sections with the experiments then determines the parameter t, the surface thickness, and α, the distortion parameter. For the case of Ta^{181}, at 182 Mev, a rough comparison with experiment of our approximate calculations indicates values $t=2.5\times10^{-13}$ cm and $\alpha=0.19$. The intrinsic quadrupole moment Q_0 is then roughly 10 barns.

It is practicable, although tedious, to improve the theoretical analysis to the stage where reliable numerical values can be obtained for t and α, and consequently Q_0. Information about the shape of ρ_2 may also be found by considering it as independent of ρ_0.[4] Considerably more information could be extracted from the very difficult experiment of electron scattering from aligned tantalum nuclei, since there would then be interference between the quadrupole and monopole scattering amplitudes.

ACKNOWLEDGMENTS

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[9] McClelland, Mark, and Goodman, Phys. Rev. **97**, 1191 (1955). These authors give Q_0 (spectroscopic) = 12.9 barns, Q_0 (Coulomb excitation yield) = 6 barns.