Continued Fraction Approximants to the Brillouin-Wigner Perturbation Series*

R. C. YOUNG AND L. C. BIEDENHARN, *The Rice Institute, Houston, Texas*

AND

E. FEENBERG, *Washington University, St. Louis, Missouri*

(Received March 1, 1957)

The Brillouin-Wigner series for the energy is converted into a continued fraction. Refinements on the Brillouin-Wigner formulas developed in recent publications are identified with alternate ($E^{(n)}$) approximants to the continued fraction. A second sequence of approximants [$E^{(n+1/2)}$] occurs between successive terms of the $E^{(n)}$ sequence. These are useful in calculations as shown by an illustrative example, but do not possess the extremum property which is a valued characteristic of the first sequence. A general proof is given that the approximants $E^{(n)}$ are invariant under the μ transformation defined and verified for $n=1, 2$, and ∞ in an earlier publication.

INTRODUCTION

RECENTLY Goldhammer and Feenberg[1] have described a refinement of the Brillouin-Wigner perturbation scheme[2-4] that improves the accuracy and rapidity of convergence of the resulting series for the energy. It is the purpose of the present note to show that the modified formulas for the energy are approximants to a continued-fraction expansion equivalent to the original Brillouin-Wigner series. The approximants obtained from this continued fraction furnish the set of approximations given by reference 1 in a simpler form, and also give intermediate approximations between every two terms of this set.

A second result is a general proof that the modified formulas for the energy are invariant under the μ transformation defined and discussed in an earlier publication.[5]

The perturbation expansion for bound states generated by the operator $H+V$ can be developed in terms of the complete set of functions ψ_m generated by the eigenvalue equation

$$H\psi_m=E_m\psi_m, \tag{1}$$

and the corresponding set of matrix elements $V_{ab}=(a|V|b)$. The approximate trial function

$$\psi^{(n)}=\frac{1}{N}\Big[\psi_0+G_1\sum_a{}'\psi_a\frac{V_{a0}}{E-E_a}+G_2\sum_{a,b}{}'\psi_b\frac{V_{ba}V_{a0}}{(E-E_b)(E-E_a)}+\cdots+G_n\sum_{a,\cdots}{}'\psi_l\frac{V_{lk}\cdots V_{a0}}{(E-E_l)(E-E_k)\cdots(E-E_a)}\Big] \tag{2}$$

serves for both the original Brillouin-Wigner development and the modified procedures. The variational integral for the energy, $E=(\psi^{(n)}|H+V|\psi^{(n)})/(\psi^{(n)},\psi^{(n)})$ now yields, with $G_1=\cdots=G_n=1$,

$$E=E_0+V_{00}+\epsilon_2+\epsilon_3+\cdots+\epsilon_{2n+1}, \tag{3}$$

in which

$$\epsilon_2=\sum_a{}'\frac{V_{0a}V_{a0}}{E-E_a},$$

$$\epsilon_3=\sum_{a,b}{}'\frac{V_{0a}V_{ab}V_{b0}}{(E-E_a)(E-E_b)}, \tag{4}$$

etc. These are the basic formulas of the Brillouin-Wigner perturbation scheme. E in the energy denominator is identified with the approximate value of the energy given by the variational integral. The prime on the summation symbols signifies that the value 0 is excluded; the indices range through the values 1, 2, $\cdots n$, $\cdots\infty$ independently.

DERIVATION OF THE CONTINUED FRACTION APPROXIMANTS

A considerable improvement becomes possible if the G_i are retained as independent parameters.[1] The expression for the energy becomes

$$E=E_0+V_{00}+(2G_1-G_1^2)\epsilon_2+(G_1^2+2G_2-2G_1G_2)\epsilon_3+(2G_3-G_2^2+2G_1G_2-2G_1G_3)\epsilon_4+(G_2^2+2G_4+2G_1G_3-2G_1G_4-2G_2G_3)\epsilon_5+\cdots+(2G_nG_{n-1}-G_n^2)\epsilon_{2n}+G_n^2\epsilon_{2n+1}. \tag{5}$$

The substitution $G_i=1+K_i$ into Eq. (5) transforms it into

$$E=E_0+V_{00}+\sum_{i=2}^{2n+1}\epsilon_i+2\sum_{i=1}^{n}K_i\epsilon_{i+n+1}+S(K_1,K_2,\cdots K_n), \tag{6}$$

in which S is a homogeneous quadratic function of the K's. The condition that E take on an extreme value is now

$$2\epsilon_{i+n+1}=-\partial S/\partial K_i, \quad i=1, 2, \cdots n. \tag{7}$$

* Work done at The Rice Institute was supported in part by the U. S. Atomic Energy Commission. Work done at Washington University was supported in part by the U. S. Atomic Energy Commision, the Office of Naval Research, and the Office of Scientific Research.

[1] P. Goldhammer and E. Feenberg, Phys. Rev. **101**, 1233 (1956).
[2] E. P. Wigner, Math. u. naturw. Anz. ungar. Akad. Wiss. **53**, 475 (1935).
[3] L. Brillouin, J. Phys. **4**, 1 (1933).
[4] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Chap. 9.
[5] E. Feenberg, Phys. Rev. **103**, 1116 (1956).

This is a system of linear inhomogeneous equations for the K's, with the explicit solution

$$K_i=(\sum_1^i \Delta_k)(\Delta-\sum_1^n \Delta_k)^{-1}, \quad i=1, 2, \cdots n, \tag{8}$$
$$K_0=K_{-j}=0, \quad K_{n+j}=-1, \quad j=1, 2, \cdots.$$

Here Δ is the determinant

$$\Delta^{(n)}=\begin{vmatrix} \epsilon_2 & \epsilon_3 & \cdots & \epsilon_{n+1} \\ \epsilon_3 & \epsilon_4 & \cdots & \epsilon_{n+2} \\ \vdots & & & \\ \epsilon_{n+1} & \epsilon_{n+2} & \cdots & \epsilon_{2n} \end{vmatrix}, \quad \Delta^{(1)}=\epsilon_2, \quad \Delta^{(0)}=1, \tag{9}$$

and Δ_k is obtained from Δ by the substitution of ϵ_{n+2}, ϵ_{n+3}, $\cdots\epsilon_{2n+1}$ for the kth column of Δ. (We have modified the notation of reference 1 by introducing explicitly a superscript giving the order of the determinants.)

The denominator of the K_i can be written conveniently as a single determinant; that is,

$$\Delta^{(n)}-\sum_{i=1}^n \Delta_i{}^{(n)}\equiv D^{(n)}=\begin{vmatrix} \epsilon_2 & \epsilon_3 & \cdots & \epsilon_{n+2} \\ \vdots & & & \\ \epsilon_{n+1} & \epsilon_{n+2} & \cdots & \epsilon_{2n+1} \\ 1 & 1 & \cdots & 1 \end{vmatrix}. \tag{10}$$

We observe that

$$2\sum_{i=1}^n K_i\epsilon_{i+n+1}=-\sum_{i=1}^n K_i\frac{\partial S}{\partial K_i} \tag{11}$$
$$=-2S,$$

by Euler's theorem on homogeneous functions. Inserting (11) into (5), we get

$$E=E_0+V_{00}+E^{(n)},$$
$$E^{(n)}=\sum_{i=2}^{2n+1}\epsilon_i+\sum_{i=1}^n K_i\epsilon_{i+n+1}. \tag{12}$$

By collecting over a common denominator, $E^{(n)}$ can be expressed in the form

$$E^{(n)}=N^{(n)}/D^{(n)}; \tag{13}$$

$D^{(n)}$ is the quantity defined in (10), and the numerator may be written as the determinant

$$N^{(n)}=\begin{vmatrix} \epsilon_2 & \epsilon_3 & \cdots & \cdots & \epsilon_{n+2} \\ \vdots & & & & \\ \epsilon_{n+1} & \cdots & \cdots & \cdots & \epsilon_{2n+1} \\ 0 & \epsilon_2 & (\epsilon_2+\epsilon_3) & \cdots & \sum_{i=2}^{n+1}\epsilon_i \end{vmatrix}. \tag{14}$$

In order to proceed further, it is convenient to introduce a few definitions. In place of the numerator and denominator used above, let

$$A_{2n}=N^{(n)}/\Delta^{(n)},$$
$$B_{2n}=D^{(n)}/\Delta^{(n)}; \tag{15}$$

the nth approximant to the energy is then

$$E^{(n)}=A_{2n}/B_{2n}. \tag{16}$$

Next one introduces the terms

$$A_{2n+1}\equiv N^{(n+1/2)}/\Delta^{(n+1/2)},$$
$$B_{2n+1}\equiv D^{(n+1/2)}/\Delta^{(n+1/2)}, \tag{17}$$

where

$$N^{(n+1/2)}\equiv\begin{vmatrix} \epsilon_3 & \epsilon_4 & \cdots & \epsilon_{n+3} \\ \epsilon_4 & & & \\ \vdots & & & \\ \epsilon_{n+2} & \epsilon_{n+3} & \cdots & \epsilon_{2n+2} \\ \epsilon_2 & (\epsilon_2+\epsilon_3) & \cdots & \sum_{i=2}^{n+2}\epsilon_i \end{vmatrix},$$

$$D^{(n+1/2)}\equiv\begin{vmatrix} \epsilon_3 & \epsilon_4 & \cdots & \epsilon_{n+3} \\ \vdots & & & \\ \epsilon_{n+2} & \epsilon_{n+3} & \cdots & \epsilon_{2n+2} \\ 1 & 1 & \cdots & 1 \end{vmatrix},$$

$$\Delta^{(n+1/2)}\equiv\begin{vmatrix} \epsilon_3 & \epsilon_4 & \cdots & \epsilon_{n+2} \\ \epsilon_4 & & & \\ \vdots & & & \\ \epsilon_{n+2} & \cdots & \cdots & \epsilon_{2n+1} \end{vmatrix}, \quad \Delta^{(1/2)}=1. \tag{18}$$

This permits an additional series of formal approximations to the energy

$$E^{(n+1/2)}=A_{2n+1}/B_{2n+1}. \tag{19}$$

whose meaning will be clear presently. Finally, one introduces

$$a_{2n}\equiv-\frac{\Delta^{(n-1)}\Delta^{(n+1/2)}}{\Delta^{(n)}\Delta^{(n-1/2)}},$$
$$a_{2n+1}\equiv-\frac{\Delta^{(n+1)}\Delta^{(n-1/2)}}{\Delta^{(n)}\Delta^{(n+1/2)}}, \quad a_1=\epsilon_2. \tag{20}$$

These definitions may be utilized now to investigate the recursion relations obeyed by the A's and B's. Consider first the A coefficients.

$$A_{2n-1}+a_{2n}A_{2n-2}=\frac{N^{(n-1/2)}}{\Delta^{(n-1/2)}}-\frac{\Delta^{(n-1)}\Delta^{(n+1/2)}}{\Delta^{(n)}\Delta^{(n-1/2)}}\cdot\frac{N^{(n-1)}}{\Delta^{(n-1)}}$$
$$=[\Delta^{(n-1/2)}\Delta^{(n)}]^{-1}\{\Delta^{(n)}N^{(n-1/2)}-\Delta^{(n+1/2)}N^{(n-1)}\}. \tag{21}$$

By Eq. (A4) of the Appendix (identifying $\alpha_i=\sum_{k=2}^{i-1}\epsilon_k$), the bracketed term is just $\Delta^{(n-1/2)}N^{(n)}$. Thus there results the recursion relation

$$A_{2n}=A_{2n-1}+a_{2n}A_{2n-2}. \tag{22}$$

In a precisely similar way, using (A4) and setting $\alpha_i=1$, one finds exactly the same recursion relation for the B's. For the intermediate case,

$$A_{2n}+a_{2n+1}A_{2n-1}=\frac{N^{(n)}}{\Delta^{(n)}}-\frac{\Delta^{(n+1)}\Delta^{(n-1/2)}}{\Delta^{(n)}\Delta^{(n+1/2)}}\frac{N^{(n-1/2)}}{\Delta^{(n-1/2)}}$$

$$=[\Delta^{(n)}\Delta^{(n+1/2)}]^{-1}\{\Delta^{(n+1/2)}N^{(n)}-\Delta^{(n+1)}N^{(n-1/2)}\}. \tag{23}$$

By Eq. (A9) of the Appendix (identifying $\alpha_i=\sum_2^{i-1}\epsilon_j$), the bracketed term is just $\Delta^{(n)}N^{(n+1/2)}$, so that the desired relation is found to be

$$A_{2n+1}=A_{2n}+a_{2n+1}A_{2n-1}. \tag{24}$$

The relation for the B's is once again exactly the same.

Relations (22) and (24) with (20) suffice to prove[6] that the quantities $A_{2n}/B_{2n}=E^{(n)}$ and $A_{2n+1}/B_{2n+1}=E^{(n+1/2)}$ are successive approximants to the continued fraction

$$E^{(k/2)}=\cfrac{a_1}{1+\cfrac{a_2}{1+\cfrac{a_3}{1+\cdots a_k}}}, \tag{25}$$

where the a_i are given by Eq. (20).

The refinement proposed by Goldhammer and Feenberg[1] is therefore to be interpreted as yielding alternate approximants to a continued fraction expansion for the energy. The continued fraction is itself equivalent to the Brillouin-Wigner series for the energy.[6]

This identification is of great interest in connection with the problem of convergence. Not only does the continued fraction generally converge more rapidly, but in many cases the continued fraction expansion will converge where the formally equivalent linear series may not. (Asymptotic series often behave this way, as for example, in the case for the gamma function.[7])

The substitution $V\rightarrow\lambda V$ requires replacement of V_{00} by λV_{00} and ϵ_k by $\lambda^k\epsilon_k$. An elementary calculation employing Eqs. (9), (18), and (20) then shows that a_{2n} and a_{2n+1} ($n>0$) are just multiplied by λ. Thus λ serves

TABLE I. Successive approximations to $E_0=1.54486\cdots$.

Order	1	2	3	4	5
E_0	2.00000	1.26795	1.55505	1.54429	1.54487
Error	+0.455	−0.2769	+0.0102	−0.00057	+0.00001

[6] Oskar Perron, *Die Lehre von den Kettenbrüchen* (B. G. Teubner, Leipzig, 1929), pp. 5, 304.

[7] H. S. Wall, *Analytic Theory of Continued Fractions* (D. Van Nostrand Company, Inc., New York, 1948), p. 365.

TABLE II. Approximation to fourth order by three methods.

Method	E_0	Error
Brillouin-Wigner	1.1541	0.3908
Secular determinant	1.5412	0.0037
Continued fraction	1.5443	0.0006

as an expansion parameter in an equally direct manner in both the linear series approximations and the continued fraction approximants of Eq. (25).

To illustrate the theory, consider the Mathieu equation

$$\left(-\frac{d^2}{dx^2}+4\cos^2x\right)\psi=E\psi, \tag{26}$$

whose lowest eigenvalue is $E_0=1.54486\cdots$. Successive approximations to this value, obtained using the continued fraction expansion, are shown in Table I. The modified Brillouin-Wigner method[1] provides only the odd-numbered approximants and thus gives a set of monotonically decreasing elements. However, successive approximants of the continued fraction are seen to *oscillate* about the true value. Although probably not a general property, this oscillatory behavior is quite useful when it occurs, since it also provides a lower bound to the eigenvalue. Table II shows approximations carried to ϵ_4 or equivalent by means of the unmodified Brillouin-Wigner scheme, a secular determinant method,[4] and the continued fraction. It should be pointed out that in the case at hand the convergence is very poor in the basic series, the ϵ_n's barely decreasing in magnitude. However, the continued fraction result is quite accurate and requires little additional labor.

INVARIANCE UNDER THE μ TRANSFORMATION

One arbitrary element in the formulation of the Brillouin-Wigner perturbation procedure is the choice of the *zero*th-order Hamiltonian operator. Part of this freedom finds expression in the μ transformation[5] defined by the relations

$$\begin{aligned} H_0'&=H_0+(\mu-1)(H_0-E),\\ V'&=V-(\mu-1)(H_0-E). \end{aligned} \tag{27}$$

These imply

$$E-H_0'=\mu(E-H_0), \tag{28}$$

a uniform change of scale in all energy denominators. The inversion of Eq. (27) yields

$$\begin{aligned} H_0&=H_0'+(\mu'-1)(H_0-E'),\\ V&=V'-(\mu'-1)(H_0-E'),\\ \mu'&=1/\mu. \end{aligned} \tag{29}$$

The further discussion is facilitated by the notation

$$M_k(i)\equiv\sum_{lm\cdots q}{}'\frac{V_{kl}V_{lm}\cdots V_{q0}}{(E-E_k)(E-E_l)\cdots(E-E_q)} \tag{30}$$

for the ith-order coefficient of ψ_k in the Brillouin-Wigner expansion. With this notation, Eq. (2) for $\psi^{(n)}$ becomes

$$\psi^{(n)}=\psi_0+\sum_k{}'\psi_k[\sum_{i=1}^{n} G_i M_k(i)]. \tag{31}$$

In the primed system the wave function has the form

$$\psi^{(n)\prime}=\psi_0+\sum_k{}'\psi_k[\sum_{i=1}^{n} G_i' M_k'(i)], \tag{32}$$

in which

$$M_k'(i)=\frac{1}{\mu^i}\sum_{s=0}^{\infty}\binom{i-1}{s}(\mu-1)^s M_k(i-s), \tag{33}$$

and the primed amplitude coefficients G_i' are independent variable parameters.

It is clear that $\psi^{(n)}\neq\psi^{(n)\prime}$ if we return to the original Brillouin-Wigner formulation with $G_i'=G_i=1$. Then the corresponding approximate formulas for the energy are not identical and, consequently, a physically meaningful value of μ may be determined by minimizing the energy with respect to μ.

Primed and unprimed wave functions are identical if

$$\sum_{i=1}^{n} G_i' M_k'(i)=\sum_{i=1}^{n} G_i M_k(i) \tag{34}$$

for all values of k. This requires

$$G_t=\frac{1}{(\mu-1)^t}\sum_{i=1}^{n}\binom{i-1}{i-t}\left[\frac{\mu-1}{\mu}\right]^i G_i', \tag{35}$$

and

$$\begin{aligned} G_t'&=\frac{1}{(\mu'-1)^t}\sum_{i=1}^{n}\binom{i-1}{i-t}\left[\frac{\mu'-1}{\mu'}\right]^i G_i \\ &=\frac{(-1)^t\mu^t}{(\mu-1)^t}\sum_{i=1}^{n}\binom{i-1}{i-t}(\mu-1)^i(-1)^i G_i. \end{aligned} \tag{36}$$

The substitution of G_i' from Eq. (36) into the right hand member of Eq. (35) yields the consistency condition

$$\sum_{i=1}^{n}\binom{i-1}{i-t}\binom{j-1}{j-i}(-1)^i=(-1)^j\delta_{jt}, \tag{37}$$

an easily verified identity, since the left-hand member is simply

$$\binom{j-1-t}{j-t}.$$

Consider now the two expressions $E(G_1, G_2, \cdots G_n)$ and $E'(G_1', G_2', \cdots G_n')$ for the energy. If Eq. (34) holds, we have

$$E(G_1,G_2,\cdots G_n)=E'(G_1',G_2',\cdots G_n') \tag{38}$$

since the wave functions are identical. The linear relation between the primed and unprimed amplitude factors means that the supplementary conditions

$$\partial E/\partial G_i=0, \quad i=1, 2, \cdots n \tag{39}$$

and

$$\partial E'/\partial G_i'=0, \quad i=1, 2, \cdots n \tag{40}$$

are mutually dependent, either set implying the other. Equations (34) and (38) insure that the extreme values of $E(G_1, G_2, \cdots G_n)$ and $E'(G_1', G_2', \cdots G_n')$ are equal. Consequently $E^{(n)\prime}$ is actually independent of μ as surmised in reference 5 and verified there by explicit calculation for $n=1$, 2, and ∞.

The μ invariance suggests the possibility of approximate invariance or insensitivity of $E^{(n)}$ under other transformations. A uniform displacement of the *zero*th-order eigenvalues comes to mind.[8] The suggested insensitivity has in fact been observed in a sample calculation of a generalized $E^{(1)}$ type.[9]

APPENDIX

(a) Let $X^{(n)}$ be defined as

$$X^{(n)}\equiv\begin{vmatrix} \epsilon_2 & \epsilon_3 & \cdots & \epsilon_{n+2} \\ \epsilon_3 & \epsilon_4 & \cdots & \epsilon_{n+3} \\ \vdots & & & \\ \epsilon_{n+1} & \epsilon_{n+2} & \cdots & \epsilon_{2n+1} \\ \alpha_2 & \alpha_3 & \cdots & \alpha_{n+2} \end{vmatrix}, \tag{A1}$$

where the ϵ_i are given in (4) and the α_i are arbitrary. One may write the product, $\Delta^{(n-1/2)}X^{(n)}$, as a single determinant

$$\Delta^{(n-1/2)}X^{(n)} = \begin{vmatrix} \epsilon_3 & \epsilon_4 & \cdots & \epsilon_{n+1} & 0 & \cdots & \cdots & 0 \\ \epsilon_4 & \cdots & \cdots & \epsilon_{n+2} & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \epsilon_{n+1} & \cdots & \cdots & \epsilon_{2n-1} & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & \epsilon_2 & \epsilon_3 & \cdots & \epsilon_{n+2} \\ 0 & \cdots & \cdots & 0 & \epsilon_3 & \cdots & \cdots & \epsilon_{n+3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 0 & \epsilon_{n+1} & \epsilon_{n+2} & \cdots & \epsilon_{2n+1} \\ \alpha_3 & \alpha_4 & \cdots & \alpha_{n+1} & \alpha_2 & \alpha_3 & \cdots & \alpha_{n+2} \end{vmatrix}. \tag{A2}$$

In this determinant, add row n to row 1, row $n+1$ to row 2, $\cdots$, row $2n-2$ to row $n-1$. This, of course, leaves the value unchanged. The result is

$$\Delta^{(n-1/2)}X^{(n)} = \begin{vmatrix} \epsilon_3 & \cdots & \epsilon_{n+1} & \epsilon_2 & \cdots & \epsilon_{n+2} \\ \vdots & & & & & \\ \epsilon_{n+1} & \cdots & \epsilon_{2n-1} & \epsilon_n & \cdots & \epsilon_{2n} \\ 0 & \cdots & 0 & \epsilon_2 & \cdots & \epsilon_{n+2} \\ \vdots & & & & & \\ 0 & \cdots & 0 & \epsilon_{n+1} & \cdots & \epsilon_{2n+1} \\ \alpha_3 & \cdots & \alpha_{n+1} & \alpha_2 & \cdots & \alpha_{n+2} \end{vmatrix}. \tag{A3}$$

[8] M. Bolsterli and E. Feenberg, Phys. Rev. **101**, 1349 (1955).
[9] E. Feenberg and P. Goldhammer, Phys. Rev. **105**, 750 (1957).

Now make a Laplace expansion using rows 1 through $n-1$ and row $2n$ for one set of determinants. It will be seen that, because of either null or identical columns, only two terms enter. That is,

$$\Delta^{(n-1/2)}X^{(n)}=X^{(n-1/2)}\Delta^{(n)}-X^{(n-1)}\Delta^{(n+1/2)}, \quad \text{(A4)}$$

with

$$X^{(n-1/2)}\equiv\begin{vmatrix}\epsilon_3 & \cdots & \epsilon_{n+2}\\ \vdots & & \\ \epsilon_{n+1} & \cdots & \epsilon_{2n}\\ \alpha_3 & \cdots & \alpha_{n+2}\end{vmatrix}, \quad \text{(A5)}$$

in which $X^{(k)}$ represents either $N^{(k)}$ or $D^{(k)}$.

(b) The product $\Delta^{(n+1)}X^{(n-1/2)}$ may be written as the determinant,

$$\Delta^{(n+1)}X^{(n-1/2)}=\begin{vmatrix}\epsilon_2 & \epsilon_3 & \cdots & \epsilon_{n+2} & 0 & \cdots & 0\\ \epsilon_3 & \epsilon_4 & \cdots & \epsilon_{n+3} & 0 & \cdots & 0\\ \vdots & & & & & & \\ \epsilon_{n+2} & \cdots & \cdots & \epsilon_{2n+2} & 0 & \cdots & 0\\ 0 & \cdots & \cdots & 0 & \epsilon_3 & \cdots & \epsilon_{n+2}\\ \vdots & & & & \vdots & & \\ 0 & \cdots & \cdots & 0 & \epsilon_{n+1} & \cdots & \epsilon_{2n}\\ \alpha_3 & \cdots & \cdots & \alpha_{n+3} & \alpha_3 & \cdots & \alpha_{n+2}\end{vmatrix}. \quad \text{(A6)}$$

Subtract column 1 from column $n+2$, column 2 from column $n+3$, $\cdots$, column n from column $2n+1$, to obtain

$$\Delta^{(n+1)}X^{(n-1/2)}=\begin{vmatrix}\epsilon_2 & \cdots & \epsilon_{n+2} & -\epsilon_2 & \cdots & -\epsilon_{n+1}\\ \epsilon_3 & \cdots & \epsilon_{n+3} & -\epsilon_3 & \cdots & -\epsilon_{n+2}\\ \vdots & & & & & \\ \epsilon_{n+2} & \cdots & \epsilon_{2n+2} & -\epsilon_{n+2} & \cdots & -\epsilon_{2n+1}\\ 0 & \cdots & 0 & \epsilon_3 & \cdots & \epsilon_{n+2}\\ \vdots & & & & & \\ 0 & \cdots & 0 & \epsilon_{n+1} & \cdots & \epsilon_{2n}\\ \alpha_3 & \cdots & \alpha_{n+3} & 0 & \cdots & 0\end{vmatrix}. \quad \text{(A7)}$$

Next make a Laplace expansion using columns 1 through $n+1$ for one set of determinants. As before, only two terms enter so that

$$\Delta^{(n+1)}X^{(n-1/2)}=\Delta^{(n+1/2)}\begin{vmatrix}\epsilon_2 & \epsilon_3 & \cdots & \epsilon_{n+2}\\ \epsilon_3 & \cdots & \cdots & \epsilon_{n+3}\\ \vdots & & & \\ \epsilon_{n+1} & \cdots & \cdots & \epsilon_{2n+1}\\ \alpha_3 & \cdots & \cdots & \alpha_{n+3}\end{vmatrix}-\Delta^{(n)}X^{(n+1/2)}. \quad \text{(A8)}$$

This is not quite the desired result, however, since the last row in the determinant written out in (A8) is not correct for $X^{(n)}$ in (A1). For the two special cases we desire, namely $\alpha_i=1$ and $\alpha_i=\sum_2{}^{i-1}\epsilon_j$, the result nevertheless follows. It is obviously so for $\alpha_i=1$, but for $\alpha_i=\sum_2{}^{i-1}\epsilon_j$ we need only subtract row 1 from row $n+1$ to obtain the desired result,

$$\Delta^{(n)}X^{(n+1/2)}=\Delta^{(n+1/2)}X^{(n)}-\Delta^{(n+1)}X^{(n-1/2)}, \quad \text{(A9)}$$

in which $X^{(k)}$ is again either $N^{(k)}$ or $D^{(k)}$.