Level Density of a System of Fermi Particles*

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Formulas for the density of the energy levels of a system of independent Fermi particles are derived which are valid for all values of the excitation energy, and thus a criterion for the validity for the usual (high-degeneracy) formulas is obtained. It is shown that as the energy of the system increases, the degree to which the usual (high-degeneracy) formulas are modified, depends strongly on the distribution of the individual-particle levels. If the levels are spaced uniformly, the corrections are especially small. Applications to nuclei are discussed briefly.

1. INTRODUCTION

SOME years ago Bethe[1] derived the level density of an excited nucleus by regarding it as an almost completely degenerate Fermi-Dirac gas of Z protons and N neutrons. Thereby he explained the rapid convergence of the experimentally observed level spacing with increasing excitation energy, and he obtained quantitative results which are of the correct order of magnitude.

In an attempt to improve the agreement with experiment, Van Lier and Uhlenbeck[2] generalized Bethe's problem in two respects. First, they based their derivation of the level density formula on a general form $\rho(\epsilon)$ for the energy level density of the individual particles; however, they found that for a highly degenerate system the result depends on only one parameter, namely the value of $\rho(\epsilon_0)$ at the Fermi level. Secondly, they investigated what the influence on the level density is when the degeneracy is not almost complete. Since a calculation for arbitrary values of the degeneration parameter is rather complicated for a general $\rho(\epsilon)$, they confined their study to the case in which the individual-particle levels are equidistant. On that basis they confirmed Bethe's original idea that, insofar as the derivation of the level density formulas is concerned, a nucleus excited to about 10 Mev may be treated as an almost completely degenerate Fermi system.

We have extended the work of van Lier and Uhlenbeck by investigating the second question for a $\rho(\epsilon)$ which follows a general positive power law. We find that as the energy of the system increases the rise in the level density is less rapid than for a highly degenerate system. However, the result depends rather sensitively on the distribution of the individual-particle levels $\rho(\epsilon)$. Furthermore, the case of equidistant spacing is an exceptional one in which the effects of lack of almost complete degeneracy are atypically small compared to other distributions.

The special character of uniform spacing is exhibited in Sec. 2 for one kind of particle and in Sec. 3 for a mixture of two kinds. We shall obtain the formulas which are valid for all values of the degeneration parameter and base some illustrative numerical computations on those. However, we shall also derive some simple analytical expressions for the level density which contain the first order corrections to the formulas of Bethe, and of van Lier and Uhlenbeck. Thereby a degeneracy criterion for the validity of the zeroth order formulas is established. The physical assumptions and the general approach to the combinatorial problem is the same as that of van Lier and Uhlenbeck.

The implications of the foregoing for nuclear level densities are discussed briefly in Sec. 4. The main conclusions are these: For excitations of about 10 Mev (thermal neutron resonance experiments) Bethe's original idea is now confirmed to within a few percent rather than the small fraction of one percent obtained by van Lier and Uhlenbeck for a uniform spacing of the individual-particle levels. At 100 Mev, corresponding, for example, to the excitation of a fission fragment, the zeroth order results are affected by only a few percent for uniform spacing, whereas other distributions may give rise to reductions of as much as an order of magnitude.

2. ONE KIND OF PARTICLE

The combinatorial problem which we shall treat in this section is the one considered in Sec. 2 of the paper by van Lier and Uhlenbeck. There are N identical, noninteracting Fermi particles. Let the allowed energies of an individual particle be $\epsilon_1, \epsilon_2, \cdots$. Let these energies be expressed in terms of a unit δ which is sufficiently small so that the ϵ's are integers. Let the levels be nondegenerate, so that each level is occupied by either no particles or one particle.

Let the resulting levels of the compound system be denoted by $E_1, E_2, E_3, \cdots$. We assume with van Lier and Uhlenbeck that the level structure of the compound system has the following two properties: The levels are separated by the same small unit δ, and the number of realizations $D(E)$ of the total energy E is a slowly and smoothly varying function of energy. In that case the product of the level density $R(N,E)$ and δ is equal to $D(E)$, and is therefore given by the Darwin-

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[1] H. Bethe, Phys. Rev. **50**, 332 (1936).
[2] C. van Lier and G. E. Uhlenbeck, Physica 4, 531 (1937).

Fowler integral:

$$R(N,E)\delta=\frac{1}{(2\pi i)^2}\oint\oint\frac{\prod_i(1+xy^{\epsilon_i})dxdy}{x^{N+1}y^{E+1}}. \quad (1)$$

Evaluating the above by the saddle point method, one obtains

$$R(N,E)\delta=\frac{e^{f(\alpha,\beta)}}{2\pi(\det A)^{\frac{1}{2}}}, \quad (2)$$

with

$$f(\alpha,\beta)=E\beta-N\alpha+\sum_i\log(1+e^{\alpha-\beta\epsilon_i}), \quad (3)$$

and A is the symmetric matrix formed from the second derivatives,

$$A=\begin{pmatrix} f_{\alpha\alpha} & f_{\alpha\beta} \\ f_{\beta\alpha} & f_{\beta\beta}\end{pmatrix}. \quad (4)$$

α and β correspond to the location of the saddle point on the real axes of the integrand of (1) and are determined by the two equations

$$\partial f/\partial\alpha=\partial f/\partial\beta=0. \quad (5)$$

We make the usual "continuous approximation" in which the sum over discrete states is dealt with by assuming that there exists an equivalent individual-particle density function $\rho(\epsilon)$ such that

$$\sum_i\log(1+e^{\alpha-\beta\epsilon_i})=\int_0^\infty\rho(x)\log(1+e^{\alpha-\beta x})dx. \quad (6)$$

Next, we introduce the Fermi level ϵ_0—the top of the distribution in the most compact occupation of the individual-particle levels—and the energy E_0 of the ground state of the entire system according to:

$$N=\int_0^{\epsilon_0}\rho(x)dx \quad (7)$$

and

$$E_0=\int_0^{\epsilon_0}x\rho(x)dx. \quad (8)$$

We shall now restrict ourselves to a ρ which follows a general positive power law:

$$\rho_n(\epsilon)=k^n\epsilon^{n-1},\quad n\geq 1. \quad (9)$$

The power law is sufficiently general to give useful information about the dependence of the level density (of the entire system of particles) on the distribution of the energy levels of the individual particle, and at the same time the power law is simple enough to permit an exact treatment of the integral in Eq. (6). We shall find it convenient to have the constant of proportionality in the form k^n. Evidently, $n=1$ corresponds to a uniform spacing of the individual-particle levels.

We shall now show that subject only to the assumptions which we have already made, the level density of the system is given by

$$R(N,E)=\frac{\exp\{\pi[\frac{2}{3}Q\rho_n(\epsilon_0)]^{\frac{1}{2}}g_n(Q/E_0)\}}{(48)^{\frac{1}{4}}Qd_n(Q/E_0)}, \quad (10)$$

where g_n and d_n are functions of the ratio Q/E_0. Q is the excitation energy, i.e.,

$$Q=E-E_0. \quad (11)$$

We shall see that g_n and $d_n\to 1$ as $Q/E_0\to 0$ (high degeneracy), thus giving the zeroth-order formulas obtained in references 1 and 2 [our formula (44)] in the appropriate limit. The rate at which $g_n\to 1$ will depend very strongly on n; in particular, if $n=1$ (equidistant spacing) $g_1\to 1$ especially fast.

If we introduce (6) and (9) into (3) and integrate by parts, f becomes

$$f=E\beta-N\alpha+(k/\beta)^n\phi_n(\alpha), \quad (12)$$

where

$$\phi_n(\alpha)=\frac{1}{n}\int_0^\infty\frac{x^ndx}{1+e^{-\alpha+x}}. \quad (13)$$

We shall frequently suppress the subscript n. The saddle point equations (5) assume the form

$$N=(k/\beta)^n\phi', \quad (14)$$

$$kE=n(k/\beta)^{n+1}\phi. \quad (15)$$

Equations (7) and (8) can now easily be integrated, giving the relations

$$(k\epsilon_0)^n=nN, \quad (16)$$

$$(n+1)kE_0=(nN)^{i+1/n}. \quad (17)$$

Using (14) and (15), we can eliminate E and N from (3), and this leads to

$$f=(k/\beta)^n[(n+1)\phi-\alpha\phi']. \quad (18)$$

In view of (9), (14), and (16), we have[3]

$$\rho(\alpha/\beta)\equiv k\alpha^{n-1}(k/\beta)^{n-1}=\rho(\epsilon_0)\alpha^{n-1}(n\phi')^{-1+1/n}. \quad (19)$$

From (14), (15), and (17), we obtain

$$\left(\frac{k}{\beta}\right)^{n+1}=kQ\left[n\phi-\frac{(n\phi')^{1+1/n}}{n+1}\right]^{-1}. \quad (20)$$

Substitution of (19) and (20) in (18) yields

$$f=\pi[\tfrac{2}{3}Q\rho(\epsilon_0)]^{\frac{1}{2}}\bar{g}(\alpha), \quad (21)$$

in which

$$\bar{g}_n(\alpha)=\frac{1}{\pi}(\tfrac{3}{2})^{\frac{1}{4}}(n\phi')^{-\frac{1}{4}(1-1/n)}\times\left[n\phi-\frac{(n\phi')^{1+1/n}}{n+1}\right]^{-\frac{1}{2}}[(n+1)\phi-\alpha\phi']. \quad (22)$$

[3] Equation (19) is true only if $\alpha/\beta\geq 0$. If $\alpha/\beta<0$, then $\rho=k^n\times(-\alpha/\beta)^{n-1}$. However, the final formulas remain the same.

The denominator of (2) is similarly evaluated by calculating the second derivatives on the basis of (12) and using the relation (20). The result is

$$2\pi(\det A)^{\frac{1}{2}}=(48)^{\frac{1}{2}}Q\bar{d}(\alpha), \tag{23}$$

with

$$\bar{d}_n(\alpha)=\frac{\pi}{(12)^{\frac{1}{2}}}[n(n+1)\phi\phi''-(n\phi')^2]^{\frac{1}{2}} \times\left[n\phi-\frac{(n\phi')^{1+1/n}}{n+1}\right]^{-1}. \tag{24}$$

It should be noted, in view of (2) and (23), that the quantity $R(N,Q)\delta$ is given by a dimensionless function divided by Q. Q is an integer, the excitation energy expressed in terms of the unit δ, which was introduced at the beginning in order to make the treatment in terms of the Darwin-Fowler integral possible. If δ is expressed, for example, in Mev, then $Q\delta$ is equal to the excitation energy in Mev. Therefore, formula (10) gives the level density in terms of that unit of energy in which Q is expressed. That is the justification for the practice of setting the level density equal to the Darwin-Fowler integral, an equality which holds only in the sense described (references 2 and 6).

From (14), (17), and (20) it follows that for a given n, α is determined solely by the ratio Q/E_0, *viz.*,[4]

$$\frac{Q}{E_0}=\frac{n(n+1)\phi}{(n\phi')^{1+1/n}}-1. \tag{25}$$

For that reason α is called the degeneration parameter,

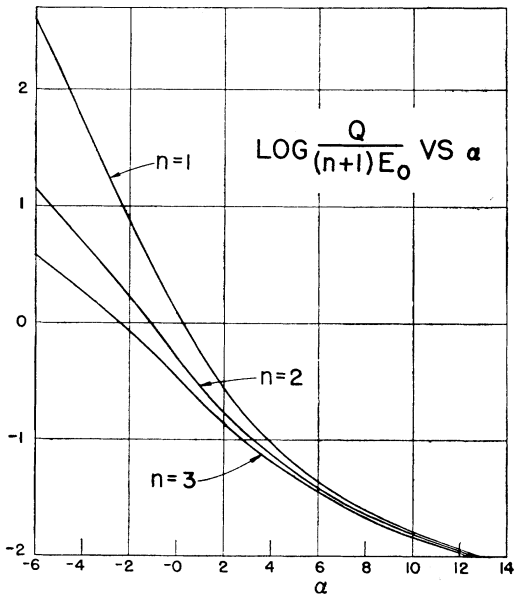

FIG. 1. $\text{Log}_{10}[Q/(n+1)E_0]$ as a function of α for $n=1$, 2, and 3. The plot illustrates that Q/E_0 is a monotonic function of α also in the region in which the asymptotic formulas (32) do not apply.

[4] Formula (25) reduces to the result obtained in reference 2 if one sets $n=1$.

and we are also justified in writing

$$\bar{g}(\alpha)=g(Q/E_0). \tag{26}$$

We shall now have to consider the functions ϕ_n and their first and second derivatives more closely.[5] If $\alpha\leq 0$, it is readily shown that ϕ_n is given exactly by the exponential series

$$\phi_n(\alpha)=\Gamma(n)S_n(\alpha), \quad \alpha\leq 0,\ n\geq 1, \tag{27}$$

with

$$S_n(\alpha)=\sum_{k=1}^{\infty}\frac{e^{-k|\alpha|}}{k^{n+1}}. \tag{28}$$

If n is a positive integer, then one can show by the method suggested in reference 2 that if $\alpha\geq 0$,

$$\phi_n(\alpha)=P_n(\alpha)+(-)^n(n-1)!S_n(\alpha), \quad \alpha\geq 0,\ n=\text{positive integer} \tag{29}$$

where P_n is a polynomial of degree $n+1$. In particular,

$$\begin{aligned} P_1(\alpha)&=\alpha^2/2+\pi^2/6,\\ P_2(\alpha)&=\alpha^3/6+\alpha\pi^2/6,\\ P_3(\alpha)&=\alpha^4/12+\alpha^2\pi^2/6+(7/5)(\pi^2/6)^2. \end{aligned} \tag{30}$$

The *leading three* terms of the polynomials of higher degree are given by

$$P_n(\alpha)=\alpha^{n+1}\left[\frac{1}{n(n+1)}+\frac{\pi^2}{6\alpha^2}+\frac{7}{10}(n-1)(n-2)\left(\frac{\pi^2}{6\alpha^2}\right)^2\right]. \tag{31}$$

If $n\geq 1$ but is not an integer and α is positive, then ϕ_n is asymptotically equal to the Sommerfeld series, the first three terms of which are also given by (31).

Thus, regardless of whether n is an integer or not, one obtains the following asymptotic formulas for Q/E_0:

$$\begin{aligned} Q/(n+1)E_0&\to\pi^2/6\alpha^2 \quad \text{as} \quad \alpha\to\infty\\ &\to[\Gamma(n+1)]^{-1/n}e^{-\alpha/n} \quad \text{as} \quad \alpha\to-\infty. \end{aligned} \tag{32}$$

The variation of Q/E_0 with α is monotonic between the regions of asymptotic behavior, and to illustrate that we have plotted $\log_{10}[Q/(n+1)E_0]$ against α for $n=1$, 2 and 3. (See Fig. 1.) The exact expressions (27) and (29) were used in the numerical computations, regarding which we shall make some further remarks below.

We are now in a position to discuss the behavior of $\bar{g}(\alpha)$ and $\bar{d}(\alpha)$ at high degeneracies (large positive α). If we make the asymptotic approximation

$$\phi_n(\alpha)\simeq\alpha^{n+1}/n(n+1), \tag{33}$$

then $\bar{g}(\alpha)=0$, and that approximation is evidently inadequate. Approximating by two terms, *viz.*:

$$\phi_n(\alpha)\simeq\alpha^{n+1}\left[\frac{1}{n(n+1)}+\frac{\pi^2}{6\alpha^2}\right], \tag{34}$$

[5] See A. Sommerfeld, Z. Physik 47, 1 (1928).

has the following consequences: If $n=1$ (equidistant spacing), both $\bar{g}_1$ and $\bar{d}_1$ reduce to unity for all α, and furthermore P_1 contains no terms beyond (34). Therefore, in order to obtain a first-order correction one must include the first term of the exponential series (28). As we shall see, the behavior is quite different if the spacing is not uniform ($n\neq 1$).

If $n\neq 1$, a straightforward application of the two-term approximation (34) would lead to an infinite power series in $1/\alpha^2$ with the leading term equal to unity. It can readily be verified, however, that because of a cancellation in the factor $n(n+1)\phi-(n\phi')^{1+1/n}$, which occurs in both $\bar{g}$ and $\bar{d}$, the third term in the asymptotic series

$$\phi_n(\alpha)\simeq\alpha^{n+1}\left[\frac{1}{n(n+1)}+\frac{\pi^2}{6\alpha^2}+\frac{7}{10}(n-1)(n-2)\left(\frac{\pi^2}{6\alpha^2}\right)^2+\cdots\right] \quad (35)$$

TABLE I. Pertinent values of g_n and d_n for $Q/E_0=0.01$ and 0.1. The values of 10^3 (g_n-1) and d_n are listed in odd and even rows, respectively.

n \ Q/E_0	0.01	0.1
1	0.000	−1.005
	1.000	0.999
2	−0.837	−8.718
	0.999	0.991
3	−0.749	−7.298
	0.999	0.993

also contributes to the coefficient of $1/\alpha^2$. Incidentally, that fact suggests strongly that in the derivation of the zeroth-order formulas for a *general* ρ, as given in reference 2 and more recently also by Bloch,[6] the additional assumptions such as "$|\epsilon-\epsilon_0|$ sufficiently small" are not really separate assumptions, but that they are already implied by the two-term Sommerfeld approximation. However, $n=2$ (also $n=1$) clearly forms an exception to the rule since the entire asymptotic series consists of only two terms, namely P_2.

Applying (35), we obtain the following results:

$$\bar{g}_n(\alpha)\simeq 1+(n-1)(2n-9)(\pi^2/120\alpha^2), \quad (36)$$

and

$$\bar{d}_n(\alpha)\simeq 1-(n-1)(\pi^2/12\alpha^2). \quad (37)$$

However, if $n=1$ we must set

$$\phi_1\simeq\alpha^2/2+\pi^2/6-e^{-\alpha}, \quad (38)$$

and that leads to

$$\bar{g}_1(\alpha)\simeq 1-e^{-\alpha}(3/\pi^2), \quad (39)$$

[6] C. Bloch, Phys. Rev. **93**, 1094 (1954).

TABLE II. Values of $C(n,N,Q/E_0)$ to illustrate the modification of the level density formula at intermediate degeneracy.

n	N \ Q/E_0	0.01	0.1
1	50	1.000	0.973
2	50	0.988	0.671
3	50	0.986	0.646
1	100	1.000	0.945
2	100	0.976	0.446
3	100	0.972	0.414
1	200	1.000	0.892
2	200	0.952	0.197
3	200	0.945	0.171

and

$$\bar{d}_1(\alpha)\simeq 1-e^{-\alpha}(\alpha^2/2-\alpha+\pi^2/6-1)(3/\pi^2). \quad (40)$$

The above relations together with the asymptotic formula

$$Q/E_0=(n+1)\pi^2/6\alpha^2, \quad (41)$$

which holds also for $n=1$, clearly exhibit the exceptional character of uniform spacing ($n=1$). They also provide a simple degeneracy criterion for the validity of the zeroth-order level density formula which reads: If

$$Q/E_0\ll 1, \quad (42)$$

and

$$|\pi[\tfrac{2}{3}Q\rho(\epsilon_0)]^{\frac{1}{2}}(1-g)|\ll 1, \quad (43)$$

then

$$R(N,Q)\simeq R_0(N,Q)=\frac{\exp\{\pi[\frac{2}{3}Q\rho(\epsilon_0)]^{\frac{1}{2}}\}}{(48)^{\frac{1}{2}}Q}. \quad (44)$$

We have also made some numerical computations to illustrate the strong dependence of the level density on the form of ρ as Q/E_0 increases from zero. Calculations were done for $n=1$, 2, and 3 using the *exact* expressions (29) for $\phi_1\phi_2\phi_3$. The first and second derivatives could readily be evaluated from the identities:

$$\begin{aligned}&\phi_1=\phi_2'=\tfrac{1}{2}\phi_3'',\\ &\phi_1'=\phi_2''=\log(1+e^{\alpha}),\\ &\phi_3'=2\phi_2,\\ &\phi_1''=1/(1+e^{-\alpha}).\end{aligned} \quad (45)$$

We have listed some pertinent values of g_n and d_n in Table I. The effect on the zeroth-order formula $R_0(N,Q)$ can be expressed by a factor C which is defined through

$$R(N,Q)=R_0(N,Q)C(n,N,Q/E_0). \quad (46)$$

For high degeneracies, C will be close to unity, and it may then be regarded as a correction factor. However, (46) is valid for all values of Q/E_0 (for which the saddle point method and the continuous approximation are good approximations). The exceptional character of uniform spacing is clearly exhibited in the numerical results for C which are listed in Table II. It should also be noted that C is less than unity in all cases. Thus, as

the energy of the system increases, the rise in the level density is less rapid than for a highly degenerate system.

3. TWO KINDS OF PARTICLES

In this section we shall show that the case of uniform spacing of the levels of an individual particle forms an exception also when the system consists of two kinds of particles. Let there be N Fermi particles of the first kind (neutrons) with levels $\epsilon_1, \epsilon_2, \cdots$ and P particles of the second kind (protons) with levels $\eta_1, \eta_2, \cdots$ E again denotes the total energy of the combined system. The level density of the system is again given by the appropriate Darwin-Fowler integral,[2] and the saddle point integration yields

$$R(N,P,E)=\frac{e^{f(\alpha,\beta,\gamma)}}{(2\pi)^{\frac{3}{2}}(\det A)^{\frac{1}{2}}}, \tag{47}$$

with

$$f(\alpha,\beta,\gamma)=-\alpha N+\beta E-\gamma P+\sum_i \log(1+e^{-\alpha+\beta\epsilon_i}) +\sum_j \log(1+e^{-\gamma+\beta\eta_j}), \tag{48}$$

and

$$A=\begin{bmatrix} f_{\alpha\alpha} & f_{\alpha\beta} & 0 \\ f_{\beta\alpha} & f_{\beta\beta} & f_{\beta\gamma} \\ 0 & f_{\gamma\beta} & f_{\gamma\gamma} \end{bmatrix}. \tag{49}$$

The values of α, β, γ are determined by the equations for the saddle point:

$$\partial f/\partial\alpha=\partial f/\partial\beta=\partial f/\partial\gamma=0. \tag{50}$$

Let us first discuss the case of uniform spacing. As in Sec. 2, we make the continuous approximation and write for the neutrons

$$\rho_N(\epsilon)=k, \tag{51}$$

$$\sum_i \log(1+e^{-\alpha+\beta\epsilon_i})\simeq k\int_0^\infty \log(1+e^{-\alpha+\beta x})dx, \tag{52}$$

and for the protons

$$\rho_P(\eta)=l, \tag{53}$$

$$\sum_i \log(1+e^{-\gamma+\beta\eta_i})\simeq l\int_0^\infty \log(1+e^{-\gamma+\beta x})dx. \tag{54}$$

Denoting the Fermi levels of the two distributions by ϵ_0 and η_0, one obtains

$$N=k\epsilon_0, \tag{55}$$

$$P=l\eta_0, \tag{56}$$

and

$$2E_0=k\epsilon_0^2+l\eta_0^2. \tag{57}$$

The equations which determine the saddle point become

$$\beta N=k\phi'(\alpha), \tag{58}$$

$$\beta^2 E=k\phi+l\psi, \tag{59}$$

and

$$\beta P=l\psi'(\gamma), \tag{60}$$

in which

$$\phi(\alpha)=\int_0^\infty \frac{x\,dx}{1+e^{-\alpha+x}}, \tag{60'}$$

and $\psi(\gamma)$ is the same function of γ as ϕ is of α. As in Sec. 2, we have suppressed the subscript 1 which properly belongs to ϕ_1, ψ_1, and their derivatives. Using the relation

$$Q=E-E_0, \tag{61}$$

it is readily shown by means of algebraic manipulations which are similar to those employed in Sec. 2 that

$$f(\alpha,\beta,\gamma)=\frac{2Q^{\frac{1}{2}}[k(2\phi-\alpha\phi')+l(2\psi-\gamma\psi')]}{k(2\phi-\phi'^2)+l(2\psi-\psi'^2)}, \tag{62}$$

$$(2\pi)^{\frac{3}{2}}(\det A)^{\frac{1}{2}} =\frac{Q^{5/4}[k^2l\psi''(2\phi\phi''-\phi'^2)+kl^2\phi''(2\psi\psi''-\psi'^2)]^{\frac{1}{2}}}{[k(\phi-\frac{1}{2}\phi'^2)+l(\psi-\frac{1}{2}\psi'^2)]^{5/4}}, \tag{63}$$

and

$$\frac{Q}{E_0}=\frac{k(2\phi-\phi'^2)+l(2\psi-\psi'^2)}{k\phi'^2+l\psi'^2}. \tag{64}$$

Next we wish to discuss the case of high degeneracy, i.e.,

$$Q/E_0\ll 1. \tag{65}$$

It should be noted that the condition (65) does not necessarily imply that both α and γ are large and positive. That can easily be seen by considering the example in which α is a large positive and γ is a large negative number. Then (64) is dominated by α and is given by

$$Q/E_0=\pi^2/6\alpha^2, \tag{66}$$

which can be made as small as one pleases. Thus, the situation is in general more complicated than for one kind of particle. However, if we make the *assumption*, which is a reasonable one for nuclei, that *roughly* (say within 20%)

$$N/k\simeq P/l, \tag{67}$$

then $\alpha\simeq\gamma$ in view of (58) and (60). In that case ϕ and ψ may be represented by the same number of terms in their series expansions (27), (29), or (31). From that it follows that $Q/E_0\ll 1$ requires large positive values for both α and γ. Using the two-term approximation P_1, Eq. (30), for both ϕ and ψ, one obtains

$$\frac{Q}{E_0}=\frac{\pi^2(k+l)}{6(k\alpha^2+l\gamma^2)}. \tag{68}$$

As in Sec. 2, the α and γ dependence drops out of (62) and (63) in the two-term approximation. This leads to the zeroth order result for the level density, namely,

$$f(\alpha,\beta,\gamma)\simeq f_0=\pi[\tfrac{2}{3}(k+l)Q]^{\frac{1}{2}}, \tag{69}$$

$$(2\pi)^{\frac{3}{2}}(\det A)^{\frac{1}{2}}\simeq(2\pi)^{\frac{3}{2}}(\det A_0)^{\frac{1}{2}} =4[216Q^5k^2l^2/(k+l)^3]^{\frac{1}{4}}, \tag{70}$$

and

$$R(N,P,Q)\simeq R_0=\frac{e^{f_0}}{(2\pi)^{\frac{3}{2}}(\det A_0)^{\frac{1}{2}}}. \tag{71}$$

The above is in agreement with formula (10) of van Lier and Uhlenbeck. The three-term approximation (38) leads to the result that f_0 is to be multiplied by $\bar{g}_1(\alpha,\gamma)$,

$$\bar{g}_1(\alpha,\gamma)=1-\frac{3}{\pi^2(k+l)}[ke^{-\alpha}+le^{-\gamma}], \tag{72}$$

and the denominator of (71) is to be multiplied by $\bar{h}_1(\alpha,\gamma)$,

$$\bar{h}_1(\alpha,\gamma)=1-\frac{3}{2\pi^2(k+l)}\left[ke^{-\alpha}\left(\alpha^2-3\alpha+\frac{\pi^2}{3}-3+\frac{\pi^2}{3}\frac{l}{k}\right)+le^{-\gamma}\left(\gamma^2-3\gamma+\frac{\pi^2}{3}-3+\frac{\pi^2}{3}\frac{k}{l}\right)\right]. \tag{73}$$

Thus, as in the case of one kind of particle, the first-order corrections for uniform spacing lead to the exponential function and $\bar{g}_1$ will approach unity especially fast with decreasing Q/E_0 in comparison with other distributions which we shall discuss next.

Let us consider, for example, a mixture having the same number of neutrons and protons and let the level density of each kind of particle follow the same power law; thus,

$$N=P, \tag{74}$$

and

$$\rho_N(x)=\rho_P(x)=k^n x^{n-1}. \tag{75}$$

Then f assumes the form

$$f(\alpha,\beta,\gamma)=-N(\alpha+\gamma)+(k/\beta)^n[\phi(\alpha)+\psi(\gamma)], \tag{76}$$

and the saddle point equations become

$$N=(k/\beta)^n\phi'(\alpha), \tag{77}$$

$$\beta E=n(k/\beta)^n(\phi+\psi), \tag{78}$$

$$P=(k/\beta)^n\psi'(\gamma). \tag{79}$$

In view of (74), (77), and (79) it is clear that at the saddle point, $\alpha=\gamma$. Therefore, f assumes the much simpler form

$$f=-2N\alpha+(k/\beta)^n 2\phi. \tag{80}$$

By manipulations which are almost identical with those of Sec. 2, it is readily shown that

$$f=\pi[(4/3)Q\rho(\epsilon_0)]^{\frac{1}{2}}\bar{g}(\alpha), \tag{81}$$

where ϵ_0 is the Fermi level of the N neutrons and also of the N protons.

For a given n in the power law (75), the degeneration parameter is again determined solely by the ratio Q/E_0, the relationship being exactly the same as (25). The denominator of (47) may be evaluated in terms of α by means of algebraic operations which are similar, though somewhat more complicated than those of Sec. 2. It helps to obtain the result first on the basis of the two-term Sommerfeld approximation (34); this leads to the *zeroth* order approximation for the denominator, namely

$$(2\pi)^{\frac{3}{2}}(\det A_0)=2[432Q^5\rho(\epsilon_0)]^{\frac{1}{4}} \tag{82}$$

in agreement with Eq. (10) of reference 2. For *any* value of α, the result is

$$(2\pi)^{\frac{3}{2}}(\det A)^{\frac{1}{2}}=(2\pi)^{\frac{3}{2}}(\det A_0)^{\frac{1}{2}}\bar{h}(\alpha), \tag{83}$$

in which

$$\bar{h}(\alpha)=\pi\bar{d}(\alpha)\left[\frac{\bar{g}(\alpha)\phi''(\alpha)}{3[(n+1)\phi-\alpha\phi']}\right]^{\frac{1}{2}}=h(Q/E_0). \tag{84}$$

$\bar{g}(\alpha)$ and $\bar{d}(\alpha)$ are given by (22) and (24) of Sec. 2, and ϕ is, of course, the integral defined by (13). $\bar{h}(\alpha)$ becomes unity in the two-term approximation for ϕ. The result for the three-term approximation is readily obtained if we make use of (31), (36), and (37):

$$\bar{h}_n(\alpha)=1-\frac{\pi^2}{240\alpha^2}(n-1)(6n+13). \tag{85}$$

Equation (85) does not give a first-order correction if $n=1$ (uniform spacing), and the situation is quite similar to that encountered in Sec. 2. For $n=1$, the three-term approximation yields

$$\bar{h}_1(\alpha)=1-\frac{3e^{-\alpha}}{2\pi^2}\left(\alpha^2-3\alpha+\frac{2\pi^2}{3}-3\right). \tag{86}$$

We have also made some computations for $n=1$, 2, and 3 to illustrate numerically the effect on the zeroth order results for the level density. As in Sec. 2, we can define a factor $C'(n,2N,Q/E_0)$ such that

$$R(N,N,Q)=R_0C'. \tag{87}$$

R_0 is given by

$$R_0=\frac{\exp\{\pi[(4/3)Q\rho(\epsilon_0)]^{\frac{1}{2}}\}}{2[432Q^5\rho(\epsilon_0)]^{\frac{1}{4}}}. \tag{87'}$$

It can easily be shown that C' is related to the factor C of Sec. 2 through

$$C'=(d/h)C(n,2N,Q/E_0). \tag{88}$$

We have calculated some values of h_n based on the exact expressions (27) and (29) for ϕ_n and the relations (15). They are listed in Table III. For $Q/E_0\leq 0.1$, d/h

TABLE III. Values of $h_n(Q/E_0)$ to illustrate the effect on the denominator of the level density formula for intermediate degeneracy.

n \ Q/E_0	0.01	0.1
1	1.000	0.990
2	0.998	0.977
3	0.996	0.962

is equal to unity within 2%, and therefore the results for C' are essentially the same as those listed for C in Table II.

For the more general case in which $N \neq P$, and in which ρ_N and ρ_P follow different power laws neither of which corresponds to uniform spacing, the results are qualitatively the same as above, although the detailed discussion is considerably more involved. It is clear, however, that since the three-term asymptotic approximation for the integrals will be of the form (35) rather than (38), the corrections to the zeroth order formulas will be qualitatively like (81) and (84) rather than (72) and (73).

4. APPLICATION TO NUCLEI

In this section we shall briefly discuss the implication of the foregoing for nuclei. Our treatment of the degeneracy question as given in the preceding two sections is entirely satisfactory for the class of Fermi systems which have the two characteristics that (1) the system consists of independent particles and (2) the level structure of each particle can be adequately represented by a continuous function $\rho(\epsilon)$. It is well known, however, that actual nuclei can be described in those terms only to a limited extent.[7] One expects, therefore, that our simple physical model will predict some general trends but no fine details. In this respect the situation is the same as for the excitation energies corresponding to the almost completely degenerate nuclear systems which were considered by Bethe and also by van Lier and Uhlenbeck.

Let us indicate, for example, the application of our results for the system which is characterized by relations (74) and (75). In addition to the total number of particles $A=2N$, there are only two independent parameters which may be taken to be any two of k, n, ϵ_0, $\rho(\epsilon_0)$, or E_0. These must be determined from experiment and/or from further assumptions about the nuclear model.

The most abundant experimental data which give information about the level density of nuclei are obtained from low-energy neutron experiments which correspond to excitation energies of about 6–10 Mev. It is well known[8] that those data have been represented, in a rough way, by zeroth order formulas of type (87′). Thus, the low-energy data determine *only one* of the parameters of our theory, namely $\rho(\epsilon_0)$. Two ways suggest themselves for determining a second parameter which is needed to make predictions for intermediate degeneracies. One approach would be to analyze, in terms of our formulas, such data as exist[9] for higher excitation. The second method is to continue the discussion of the physical model itself. Thus, the individual-particle levels ϵ_i (see Sec. 3) must presumably be regarded as the eigenvalues of a suitable potential well of depth V. If the binding energy of the last nucleon (the one which occupies the Fermi level ϵ_0) is denoted by ϵ' ($\sim$8 Mev) then it can easily be shown from the definitions that

$$E_0=[n/(n+1)](V-\epsilon')A, \tag{89}$$

and in any case

$$2\rho(\epsilon_0)=n^2A^2/(n+1)E_0. \tag{90}$$

Thus the low-energy data determine the quantity

$$2\rho(\epsilon_0)=nA/(V-\epsilon'). \tag{91}$$

If one combines that with a value of $V-\epsilon'$,[10] we shall have the necessary information.

As a numerical illustration, we may choose $\rho(\epsilon_0)$ to approximate very roughly the results given on page 372 of reference 8; then for medium heavy nuclei, $\rho(\epsilon_0)=0.02A$ per Mev. If we take $V-\epsilon'\simeq 40$ Mev corresponding to a depth which was recently used by Sokoloff and Hamermesh[10] in the analysis of neutron scattering data, we obtain $n\simeq 1.6$ and $E_0\simeq 25A$ Mev. These values can be used in formulas (25), (81), and (83) for *any* value of Q. The first-order formula can readily be obtained from (36) and (41):

$$R\simeq\frac{0.3}{Q^{5/4}A^{\frac{1}{4}}}\exp\{0.5(AQ)^{\frac{1}{2}}[1-0.0026Q/A]\}. \tag{92}$$

The correction resulting from $\bar{h}(\alpha)$ is negligibly small in the region in which (92) is valid, and has been omitted. The absolute value of (92) is probably inaccurate; however, we may expect that the following rough criterion will be valid: As $0.001Q^{\frac{3}{2}}A^{-\frac{1}{2}}$ approaches unity, the high-degeneracy formula (87′) for the level density is modified in accordance with the expressions (81) and (84).

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