tensors appear here only for operators neglected in the six-particle calculations.

$$\langle K\rangle = 5\hbar^2\alpha^2/M, \tag{A51}$$

$$\langle V_c\rangle = (2\alpha V_0/15\pi^{\frac{1}{2}}\beta) \times[15J_1(\beta/2\alpha)+J_5(\beta/2\alpha)], \tag{A52}$$

$$S_1 = (8\sqrt{2}\alpha^2T_0^2/15\pi^{\frac{1}{2}}\tau^2)(15J_2+J_6), \tag{A53}$$

$$S_2 = (16\sqrt{2}\hbar^2\alpha^4T_0^2/15\pi^{\frac{1}{2}}M\tau^2) \times(105J_2+30J_4+14J_6+J_8), \tag{A54}$$

$$V_2 = (48\sqrt{2}\alpha^4\hbar^2T_0^2/5\pi^{\frac{1}{2}}M\tau^2)J_6, \tag{A55}$$

$$S_3 = (8\sqrt{2}\alpha^2T_0^2/15\pi^{\frac{1}{2}}\tau^2)(15J_4+J_8), \tag{A56}$$

$$\langle S(t'V_ct')\rangle = (16\alpha^3V_0T_0^2/15\pi^{\frac{1}{2}}\beta\tau^2) \times[15J_3([2\tau+\beta]/2\alpha) +J_7([2\tau+\beta]/2\alpha), \tag{A57}$$

$$\langle T(t'V_ct')\rangle = (16\alpha^3V_0T_0^2/105\pi^{\frac{1}{2}}\beta\tau^2)J_7([2\tau+\beta]/2\alpha), \tag{A58}$$

$$\langle S(t'tt')\rangle = (-32\alpha^3T_0^3/15\pi^{\frac{1}{2}}\tau^3)[15J_3(3\tau/2\alpha) +J_7(3\tau/2\alpha), \tag{A59}$$

$$\langle T(t'tt')\rangle = (-32\alpha^3T_0^3/35\pi^{\frac{1}{2}}\tau^3)J_7(3\tau/2\alpha), \tag{A60}$$

$$\langle\{V_ct'\}\rangle = (-4\sqrt{2}V_0T_0/105\pi^{\frac{1}{2}}\beta\tau)J_6([\tau+\beta]/2\alpha). \tag{A61}$$

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Tensor Coupling and the Vector Shell Model*

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Because of the large tensor force contribution to the nuclear binding energy, which results in a situation of near degeneracy for the low-lying states, the tensor force may produce a large mixing of the low states even in the absence of tensor matrix elements between them. The type of coupling among the low states resulting from this near degeneracy is investigated by perturbation theory; the high-lying states, which are considered largely responsible for the effects of the tensor force, are eliminated by applying closure. An intermediate-coupling model for the low states emerges which is very similar to the customary one based on a vector force in that the effective nuclear potential for the low states is shown to consist of a central two-body force (with a spin-dependence different from that of the elementary two-body central force), plus a strong vector force, plus a tensor force which is probably weaker than the elementary two-body tensor force. The effective vector force is principally a three-body force, and hence may be expected to show a quite different "hole"-particle relationship than the one- or two-body vector forces usually assumed in the shell model. Because of the neglect in the wave function of the high-lying states which are mixed in directly by the tensor force, the model is expected to be valid for light nuclei only. The β decay of B^{12} is discussed briefly, and is shown to be compatible with the tensor force as the sole spin-orbit force in the elementary two-body interaction.

INTRODUCTION

THE success of the shell model[1] shows conclusively that there exists an effective vector-type spin-orbit term in the nuclear Hamiltonian. The notion of intermediate coupling which follows from the existence of a vector force has been especially successful in explaining the properties of light nuclei.[2,3] It has been suggested by a number of authors[4] that the apparent vector force so basic to the shell model might be a reflection of higher order effects of the tensor force. Inglis[2] has expressed the hope that the intermediate-coupling model with a vector force but no tensor force might well yield a good simulation of the tensor force effects. In the present paper we investigate the reliability of this simulation by studying the type and degree of mixing of the low-lying nuclear states produced by the tensor force. The main result is that the intermediate-coupling model is actually a fairly faithful portrait of the tensor force effects, at least for light nuclei.

Recent calculations[5,6] of the level structure of Li^6 and Li^7, using a mixture of central and tensor forces alone, gave level positions in qualitative agreement with both experiment and intermediate-coupling calculations based on the vector shell model. The tensor calculations were based on the use of a variational wave function of the form $\psi_0+\lambda t\psi_0$, where $t=\sum_{i<j}t_{ij}$ is the tensor force, considered as a perturbation on the central-force wave function ψ_0, λ being the variational parameter.[7] Such a

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[1] M. G. Mayer, Phys. Rev. **75**, 1969 (1949); Haxel, Jensen, and Suess, Phys. Rev. **75**, 1766 (1949).

[2] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).

[3] A. M. Lane, Proc. Phys. Soc. (London) **A66**, 977 (1953); **A68**, 189, 197 (1955); J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A229**, 536 (1955); M. Redlich, Phys. Rev. **99**, 1427 (1955); French, Halbert, and Pandya, Phys. Rev. **99**, 1387 (1955); D. Kurath, Phys. Rev. **101**, 216 (1956).

[4] A. M. Feingold and E. P. Wigner, Phys. Rev. **79**, 221 (A) (1950); J. Keilson, Phys. Rev. **82**, 759 (1951); K. H. Höcker, Z. Physik **132**, 399 (1952); E. P. Wigner, in *Symposium on New Research Techniques in Physics, July 15–29, 1952* (Academia Brasileira de Ciências, Rio de Janeiro, 1954).

[5] A. M. Feingold, Phys. Rev. **101**, 258 (1956).

[6] D. H. Lyons, Phys. Rev. **105**, 936 (1957).

[7] The calculations of references 5 and 6 were actually based on a variational function of the form $\psi_0+t'\psi_0$, where t' is t with a changed radial dependence. This modification was necessary in the variational method in order to obtain a reasonable estimate

wave function can be considered as roughly equivalent to the first-order wave function given by perturbation theory,

$$\psi_0+\sum_p{}'\frac{t_{p0}}{E_0-E_p}\psi_p\approx\psi_0+\frac{t\psi_0}{E_0-\bar{E}}, \tag{1}$$

with the variational calculation giving an estimate for $\bar{E}$, the "average" energy of the states admixed into ψ_0 by the tensor force. The calculations yielded a value of $\bar{E}$ of the order of 100–150 Mev, showing that the tensor force mixes in many states of very high excitation, states arising from core excitation in particular. The second-order energy,

$$\Delta E=\sum_p{}'\frac{t_{0p}t_{p0}}{E_0-E_p}\equiv\frac{(t^2)_{00}}{E_0-\bar{E}}, \tag{2}$$

was found to be of the order of -12 Mev.[8] Because of the large value of $\bar{E}$, the percentage admixture of excited states was quite small, 5–15% in intensity, indicating only a small departure from LS coupling.

It is clear that the deviation from LS coupling in the first-order wave function (1) is quite different from that given by intermediate coupling in the vector shell model, since different states are admixed by the tensor and vector forces in first order. In particular, the first-order tensor wave function permits no mixing of doublet states or of singlet with triplet states, both types of mixing occurring with a vector force. Talmi and Wigner[9] have used this as an argument for the existence of a true vector force, since the small ft value of the B^{12} β decay indicates a large admixture of the (110) supermultiplet into the predominantly (000) supermultiplet ground state wave function of C^{12}, an admixture not permitted by the first-order tensor wave function but allowed by the vector force in first order.

However, because of the large contribution of the tensor forces to the binding energy of nuclei, $\sim$12 Mev for Li^6 and Li^7, which is large compared to the level spacing of the low levels, one has a situation of near degeneracy, and the tensor force can lead to a large mixing of the low states even though there may be no direct tensor matrix elements between them.[10] The first-order wave function may thus be a poor approximation to the effects of the tensor force.

The effect of degeneracy can be seen most easily in the following simple example.[11] Let ψ_1 and ψ_2 be two low-lying states separated by an energy ϵ in first order, with zero tensor matrix element between them, but both states being connected to a high-lying state, ψ_3, of energy E_q, by the tensor matrix elements V_1 and V_2, respectively. The tensor-force contribution to the energy, ΔE, and the amplitudes c_1, c_2, c_3 of the wave functions ψ_1, ψ_2, and ψ_3 in the ground-state wave function, are determined by the secular equation,

$$\begin{vmatrix}-\Delta E & 0 & V_1\\ 0 & \epsilon-\Delta E & V_2\\ V_1 & V_2 & E_q-\Delta E\end{vmatrix}=0. \tag{3}$$

The ratio $c_2/c_1=V_2\Delta E/V_1(\Delta E-\epsilon)$ will be of order unity if V_1 and V_2 are comparable to each other and if $-\Delta E$ is comparable to or smaller than ϵ; while the amplitude c_3 will be small provided V_1 and V_2 are large compared to ΔE, this latter condition being equivalent to the condition $E_q\gg-\Delta E$ [see Eq. (4) below]. If the state ψ_3 is considered as representing the average effect of the many high-lying states connected to the low states by the tensor force, then we have seen that E_q is of the order of 100 Mev, while ΔE, for Li^6 and Li^7, was of the order of -12 Mev. Thus both conditions are well satisfied and we can expect that the tensor force will in general produce a large mixing of low-lying states with a much smaller admixture of the highly excited states that are directly coupled to the low states. The value of ΔE given by Eq. (3) is, assuming V_1 and V_2 to be small compared to E_q,

$$\Delta E\approx-V_1^2/E_q-V_2^2/(1-\epsilon/\Delta E)E_q, \tag{4}$$

the second term resulting from the near degeneracy of ψ_2 with ψ_1.

The above shows that the low ft value for the B^{12} β decay may be quite compatible with the tensor force as the sole spin-orbit force. The (110) and (000) supermultiplets in C^{12} are separated by $\sim$15 Mev, which is roughly equal to the expected tensor-force contribution to the binding energy. Thus a 10–20% admixture of states of the (110) supermultiplet in the ground state of C^{12}, which would be sufficient to explain the B^{12} ft value, is not unreasonable.

DERIVATION OF VECTOR-COUPLING MODEL

To determine the type of coupling between the low states when many highly excited states must be considered, we shall use perturbation theory and apply closure,[12] as has been indicated in Eq. (2). The main effect of the tensor force is assumed to come from highly excited states whose average energy, $\bar{E}$, is much larger than both ΔE and the energy spread of the low states. By low states we mean those states whose energy separations are small or comparable in magnitude to ΔE, the tensor-force binding-energy contribution. The energy, $\mathcal{E}_n$, of the perturbed state, Ψ_n, where n desig-

for $\bar{E}$ [see Eq. (2)]. Since in the present paper we shall use the variational results only as a guide to the magnitude of the quantities entering into our perturbation formulas, we shall disregard the distinction between t' and t.

[8] The energy formulas in references 5 and 6 are actually somewhat different in form from Eq. (2) because of the use of t' instead of t (see footnote 7).

[9] I. Talmi and E. P. Wigner, Phys. Rev. 91, 443(A) (1953).

[10] That the tensor force could lead to large mixing of low states because of near degeneracy, was first pointed out by E. P. Wigner in an unpublished manuscript with the author on β decay (1949).

[11] This example is due to E. P. Wigner (see reference 10).

[12] An elegant method for treating the near-degeneracy effect, without making the simplifying but imprecise closure approximation used here, has been given by Feenberg [M. Bolsterli and E. Feenberg, Phys. Rev. 101, 1349 (1956)].

nates one of the low states, is given by[13]

$$\mathcal{E}_n=E_n+V_{nn}+\sum_p{}'\frac{V_{np}V_{pn}}{\mathcal{E}_n-E_p}+\sum_{pq}{}'\frac{V_{np}V_{pq}V_{qn}}{(\mathcal{E}_n-E_p)(\mathcal{E}_n-E_q)}$$
$$+\sum_{pqr}{}'\frac{V_{np}V_{pq}V_{qr}V_{rn}}{(\mathcal{E}_n-E_p)(\mathcal{E}_n-E_q)(\mathcal{E}_n-E_r)}+\cdots, \quad (5)$$

where E_p is the energy of the unperturbed state, ψ_p, with respect to the unperturbed Hamiltonian, H_0. It is customary to take $H_0=K+U$, where K is the kinetic energy operator and U is an effective harmonic oscillator potential. The perturbation potential, V, in Eq. (5) is then given by

$$V=V_C-U+t\equiv V_s+t, \quad (6)$$

where $V_C=\sum_{i<j}V_{Cij}$ and $t=\sum_{i<j}t_{ij}$, V_{Cij} and t_{ij} being the elementary two-body central and tensor potentials, respectively. We assume that the low states ψ_p are chosen to be diagonal among themselves with respect to V_s.

For odd A and even-even nuclei the low states are expected to be predominantly doublet and singlet spin states, respectively, according to the supermultiplet theory,[14] and thus for these nuclei the low states will have matrix elements only with high-lying states. The degeneracy effect thus first shows up in the fourth-order energy term in Eq. (5), where q can be a low-state label, giving a small energy denominator. For odd-odd nuclei, the states of the lowest supermultiplet will be either singlet or triplet spin states and thus there will exist tensor matrix elements between several of the low states of these nuclei.

To obtain the vector coupling model, the second-order energy expression in Eq. (5) can be written as

$$\sum_p{}'\frac{V_{np}V_{pn}}{\mathcal{E}_n-E_p}=\sum_p\frac{V_{np}V_{pn}}{\mathcal{E}_n-\bar{E}_{nn}}-\frac{V_{nn}V_{nn}}{\mathcal{E}_n-\bar{E}_{nn}}$$
$$+\sum_p{}'\left\{\frac{V_{np}V_{pn}}{\mathcal{E}_n-E_p}-\frac{V_{np}V_{pn}}{\mathcal{E}_n-\bar{E}_{nn}}\right\}. \quad (7)$$

The last sum in (7) is then separated into $\sum_p{}'=\sum_j{}'+\sum_s$, where j indicates a sum over the low states alone, and s the sum over the high states. Applying closure to the first sum in (7) and defining $\bar{E}_{nn}$ such that

$$\sum_s\left\{\frac{V_{ns}V_{sn}}{\mathcal{E}_n-E_s}-\frac{V_{ns}V_{sn}}{\mathcal{E}_n-\bar{E}_{nn}}\right\}=0, \quad (8)$$

the second-order energy becomes

$$\frac{(V^2)_{nn}-(V_{nn})^2}{\mathcal{E}_n-\bar{E}_{nn}}+\sum_j{}'\left\{\frac{V_{nj}V_{jn}}{\mathcal{E}_n-E_j}-\frac{V_{nj}V_{jn}}{\mathcal{E}_n-\bar{E}_{nn}}\right\}. \quad (9)$$

The second term in the sum in (9) is negligible compared to the first term because of the large value of $\bar{E}$ compared to E_j; also $(V_{nn})^2$ is negligible compared to $(V^2)_{nn}$. With these approximations, the second-order energy reduces to

$$\sum_p{}'\frac{V_{np}V_{pn}}{\mathcal{E}_n-E_p}\approx\frac{(V^2)_{nn}}{\mathcal{E}_n-\bar{E}_{nn}}+\sum_j{}'\frac{V_{nj}V_{jn}}{\mathcal{E}_n-E_j}, \quad (10)$$

where the sum on the right-hand side is over the low states alone. The definition of $\bar{E}_{nn}$ in terms of the sum over high states alone, Eq. (8), has been done in order to treat all the low states on an equal basis, since only some of these, in the case of odd-odd nuclei as mentioned previously, will have matrix elements with other low states.

An approximate formula for $\bar{E}_{nn}$ can be obtained from Eq. (8) by rewriting it as

$$\sum_s\frac{V_{ns}V_{sn}(E_s-\bar{E}_{nn})}{(\mathcal{E}_n-E_s)}=0,$$

and assuming that the energy spread of the states s giving the major contribution to the numerator is small compared to $\mathcal{E}_n-\bar{E}_{nn}$. $\bar{E}_{nn}$ then becomes[15]

$$\bar{E}_{nn}\approx(VH_0V)_{nn}/(V^2)_{nn}. \quad (11)$$

With similar approximations, the third-order energy term in Eq. (5) becomes

$$\sum_{pq}{}'\frac{V_{np}V_{pq}V_{qn}}{(\mathcal{E}_n-E_p)(\mathcal{E}_n-E_q)}\approx\sum_j{}'\frac{(V^2)_{nj}V_{jn}}{(\mathcal{E}_n-\bar{E}_{nj})(\mathcal{E}_n-E_j)}$$
$$+\sum_j{}'\frac{V_{nj}(V^2)_{jn}}{(\mathcal{E}_n-E_j)(\mathcal{E}_n-\bar{E}_{jn})}+\sum_{jk}{}'\frac{V_{nj}V_{jk}V_{kn}}{(\mathcal{E}_n-E_j)(\mathcal{E}_n-E_k)}, \quad (12)$$

j and k representing sums over the low states alone, and where $\bar{E}_{jk}$ is defined by

$$\sum_s\left\{\frac{V_{js}V_{sk}}{\mathcal{E}_j-E_s}-\frac{V_{js}V_{sk}}{\mathcal{E}_j-\bar{E}_{jk}}\right\}=0, \quad (13)$$

j and k being any two low-state labels, the sum being carried only over the high states s. With the same crude approximation as used in obtaining Eq. (11), a rough formula for $\bar{E}_{jk}$ is

$$\bar{E}_{jk}\approx\bar{E}_{kj}\approx(VH_0V)_{jk}/(V^2)_{jk}. \quad (14)$$

Proceeding in a similar fashion with the higher-order energy terms, Eq. (5) finally becomes

[13] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), p. 1001 ff.

[14] E. P. Wigner, Phys. Rev. **51**, 106 (1937); E. P. Wigner and E. Feenberg, Repts. Progr. Phys. **8**, 274 (1941).

[15] The formula for $\bar{E}_{nn}$ given by Eq. (11) is identical with that given by the variational method of reference 5 for the special case of $t'=t$ (see footnote 7). As discussed in reference 5, Eq. (11) can be expected to be a fair approximation to $\bar{E}_{nn}$ only if the potential is reasonably smooth. For the singular Yukawa potential, $\bar{E}_{nn}$ as given by Eq. (11) actually diverges.

$$\mathcal{E}_n=E_n+V_{nn}+\frac{(V^2)_{nn}}{\mathcal{E}_n-\bar{E}_{nn}}+\sum_p{}'\frac{V_{np}V_{pn}}{\mathcal{E}_n-E_p}$$
$$+\sum_p{}'\frac{V_{np}(V^2)_{pn}}{(\mathcal{E}_n-E_p)(\mathcal{E}_n-\bar{E}_{pn})}+\sum_p{}'\frac{(V^2)_{np}V_{pn}}{(\mathcal{E}_n-\bar{E}_{np})(\mathcal{E}_n-E_p)}$$
$$+\sum_p{}'\frac{(V^2)_{np}(V^2)_{pn}}{(\mathcal{E}_n-\bar{E}_{np})(\mathcal{E}_n-E_p)(\mathcal{E}_n-\bar{E}_{pn})}+\cdots, \quad (15)$$

where the summation is now understood to extend over the *low* states alone. Replacing $(\mathcal{E}_n-\bar{E}_{jk})$ by $(\mathcal{E}_0-\bar{E}_{jk})$, where $\mathcal{E}_0$ is the average energy of the low states—the spread in energy among the low states being assumed to be negligible compared to $(\mathcal{E}_0-\bar{E}_{jk})$, Eq. (15) can be written in the form

$$\mathcal{E}_n=E_n+\mathcal{U}_{nn}+\sum_p{}'\frac{\mathcal{U}_{np}\mathcal{U}_{pn}}{\mathcal{E}_n-E_p}+\cdots, \quad (16)$$

where

$$\mathcal{U}=V+V^2/(\mathcal{E}_0-\bar{E}), \quad (17)$$

the matrix elements of $\mathcal{U}$ being understood to be given by

$$\mathcal{U}_{jk}=V_{jk}+(V^2)_{jk}/(\mathcal{E}_0-\bar{E}_{jk}). \quad (18)$$

The terms neglected in Eq. (16) are of the order $(\Delta E)^2/(\mathcal{E}_0-\bar{E}_{nn})$.

Similarly, the perturbed wave function,

$$\Psi_n=\psi_n+\sum_p{}'\frac{V_{pn}\psi_p}{\mathcal{E}_n-E_p}+\sum_{pq}{}'\frac{V_{pq}V_{qn}\psi_p}{(\mathcal{E}_n-E_p)(\mathcal{E}_n-E_q)}+\cdots, \quad (19)$$

becomes, subject to the same approximations made in the energy expression,

$$\Psi_n=\psi_n+\sum_p{}'\frac{\mathcal{U}_{pn}\psi_p}{\mathcal{E}_n-E_p}+\cdots, \quad (20)$$

where again the summation extends over the *low* states alone. The most significant term neglected in Eq. (20) is $\sum_p V_{pn}\psi_p/(\mathcal{E}_n-E_p)$, where the sum is now over the *high* states. This term, which is just the first-order term of Eq. (1), is of intensity $\sim\Delta E/(\mathcal{E}_0-\bar{E})$, and amounts to 5–15% in Li^6 and Li^7.[5,6]

Equations (16) and (20) will be recognized as just the perturbation solution of the secular equation formed from the low states alone, subject to the effective perturbation potential $\mathcal{U}$ of Eq. (17).[16] The second term in the formula for $\mathcal{U}$ represents the additional potential due to the interaction with the many high states. We shall call this model, consisting of the low states alone, subject to the perturbation potential $\mathcal{U}$, the "reduced tensor coupling" model. It can be expected to give reasonably good results provided $\Delta E/(\mathcal{E}_0-\bar{E})$ is small.

[16] This result could have been derived more rigorously by using the Feenberg perturbation formula (see P. M. Morse and H. Feshbach, reference 13, p. 1010 ff), which is related directly to the expansion of the secular determinant.

DISCUSSION

The type of coupling between the low states implied by the model can be seen by an examination of the form of $\mathcal{U}$. The dependence of $\bar{E}_{jk}$ on j and k affects substantially the quantitative predictions of the model, but not the main qualitative features. Therefore, for simplicity, we shall at first ignore this dependence and shall thus consider $\bar{E}$ in Eq. (17) to be a constant rather than an operator. The effect of the dependence of $\bar{E}_{jk}$ on j and k will be discussed at the end of this section. We rewrite $\mathcal{U}$ as

$$\mathcal{U}=V_s+t+(V_s^2+V_st+tV_s+t^2)/(\mathcal{E}_0-\bar{E}), \quad (21)$$

and then decompose $\mathcal{U}$ into parts which transform under separate space or spin-space rotations as scalars, vectors, tensors, etc.:

$$\mathcal{U}=\mathcal{U}_s+\mathcal{U}_v+\mathcal{U}_t+\cdots, \quad (22a)$$

where

$$\mathcal{U}_s=V_s+V_s^2/(\mathcal{E}_0-\bar{E})+(t^2)_s/(\mathcal{E}_0-\bar{E}),$$
$$\mathcal{U}_v=(t^2)_v/(\mathcal{E}_0-\bar{E}), \quad (22b)$$
$$\mathcal{U}_t=t+\{V_st+tV_s+(t^2)_t\}/(\mathcal{E}_0-\bar{E}),$$

where the subscripts s, v, t refer to the scalar, vector, and tensor parts, respectively, of the various operators. The terms in $\mathcal{U}$ that transform as irreducible tensors of the third or higher rank may be neglected since they will not contribute to the matrix elements between the states of the lowest supermultiplets.

The scalar part of $\mathcal{U}$, $\mathcal{U}_s$ affects only the diagonal elements of the energy matrix if the zeroth order states are chosen properly. The first term, V_s, represents the first-order effect of the two-body central forces in removing the independent-particle degeneracy and its effect is known for a variety of light nuclei.[17] The second term in $\mathcal{U}_s$ represents the second-order effect of the central forces and is known to contribute significantly to the binding energy and relative positions of the low levels.[18] The third term in $\mathcal{U}_s$ represents the main effect of the tensor force on the diagonal matrix elements. While $\mathcal{U}_s$ will not separate the states belonging to a given LS multiplet, it will through its spin-exchange nature separate singlet and triplet states. The two-particle part of $(t^2)_s$, which is expected to dominate over the three- and four-particle parts of $(t^2)_s$ in light nuclei, is[5]

$$(t^2)_s=\sum_{i<j}f(r_{ij})^2(3+\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\chi_{ij}^2, \quad (23)$$

[17] E. Feenberg and E. P. Wigner, Phys. Rev. **51**, 95 (1937); E. Feenberg and M. Phillips, Phys. Rev. **51**, 597 (1937).

[18] D. R. Inglis, Phys. Rev. **51**, 531 (1937); H. Margenau and K. G. Carroll, Phys. Rev. **54**, 705 (1938); K. G. Carroll, Phys. Rev. **57**, 791 (1940).

where $f(r_{ij})$ is the radial dependence of the tensor force, *viz.*,

$$t_{ij}=f(r_{ij})\{3(\mathbf{r}_{ij}\cdot\boldsymbol{\sigma}_i)(\mathbf{r}_{ij}\cdot\boldsymbol{\sigma}_j)/r_{ij}^2-(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\}\chi_{ij}, \quad (24)$$

χ_{ij} representing the exchange nature of t_{ij}, i.e., $\chi_{ij}=[g+(1-g)P_{ij}]$, P_{ij} being the Majorana space exchange operators. Thus (23) vanishes in the singlet state of the two nucleons as does the tensor force. Even if the original two-body central force is assumed to be spin-independent, the spin dependence of $(t^2)_s$ will split singlet and triplet states,[5,6] as it may be considered to do in the deuteron, for example. The effective scalar (central) potential may thus be quite different in spin dependence from the elementary two-body central potential.

The departure from LS coupling in this "reduced tensor coupling" model is due principally to $\mathcal{U}_v$. The effective vector force, $\mathcal{U}_v$, is, however, quite different in form from the one- or two-particle vector forces customarily used in the shell model. $\mathcal{U}_v$ contains no one- or two-particle terms, but consists solely of three- and four-particle terms.[19] (This statement must be modified when the state dependence of $\bar{E}_{jk}$ is considered —see below.) The three-particle terms of $\mathcal{U}_v$, which are expected to dominate for light nuclei, are of the form[19,20]:

$$\mathcal{U}_v(1,23)=(9/2)\chi_{12}f(r_{12})f(r_{13})\times[(\mathbf{r}_{12}\cdot\mathbf{r}_{13})(\mathbf{r}_{12}\times\mathbf{r}_{13})/r_{12}^2r_{13}^2]\cdot\{(\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3)+\tfrac{1}{5}i[4\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3)-\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_3)-\boldsymbol{\sigma}_3(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_1)]\}\chi_{13}. \quad (25)$$

For a nonexchange tensor operator, the terms with i as a factor may be omitted since they drop out in the sum, $\mathcal{U}_v(1,23)+\mathcal{U}_v(1,32)$; otherwise they must be retained. For the highly symmetric states of the lowest supermultiplets, the exchange and nonexchange forms can be expected to yield essentially equivalent results.

The three-particle nature of $\mathcal{U}_v$ implies that the "hole"-particle relationship will be of a more complicated nature than for a one-particle or a two-particle vector force. In particular, the core wave-function must be included in the analysis since, at least in Li^6 and Li^7, most of the effect comes from interaction with the core. Of course, for the case of a few particles outside a large closed core, integration over the core will result in a large effective one-body vector force which will dominate over the effective two-body force connecting the outer particles with the core and the three-body vector forces among the extra-core particles. In this case there will be only a slight difference between the results predicted by one-, two-, or three-body vector forces. However, for the case of many particles outside a small closed core, as occurs near the end of the p shell, the predicted level structure and wave functions may depend strongly upon the assumed nature of the vector force.

The tensor term $\mathcal{U}_t$ of Eq. (22), which can be expected to play a significant role only in odd-odd nuclei, is seen to consist of the original two-body tensor force, t, plus an additional tensor term. This second term will tend to cancel t. This can be seen by rearranging the second term to read

$$\{(Kt+tK)+(V_Ct+tV_C)-(E_n+E_m)t+(t^2)_t\}/(\mathcal{E}_0-\bar{E}), \quad (26)$$

where we have eliminated the harmonic potential U, and where n and m are the labels of the states involved in the matrix element. If the wave functions, ψ_p, were solutions of the original central-force potential, then V_s would vanish and $\mathcal{U}_t$ would reduce to $t+(t^2)_t/(\mathcal{E}_0-\bar{E})$. In practice, however, the ψ's are taken as solutions of the harmonic potential, and then V_s is by no means negligible. Lyons[6] has shown that for the ^3D states of Li^6 the first term in (26), $(Kt+tK)/(\mathcal{E}_0-\bar{E})$, is then by far the dominant term, as might be expected from the very large kinetic energy of the excited states, which is comparable to $\bar{E}$. Hence one may expect this term to be comparable to t itself, but with an opposite sign since K is positive while $(\mathcal{E}_0-\bar{E})$ is negative. Because of the approximate cancellation of the tensor parts of two terms of Eq. (17), which may be called the first- and second-order potentials, respectively, it is necessary to examine the importance of higher order terms which have been neglected in obtaining Eq. (17). These terms are roughly of the form $V^3/(\mathcal{E}_0-\bar{E})^2+V^4/(\mathcal{E}_0-\bar{E})^3+\cdots$. The main tensor term in the third-order potential is $(K^2t+KtK+tK^2)/(\mathcal{E}_0-\bar{E})$, which has the same sign as t. The contribution of this term to the diagonal matrix elements of the ^3D states of Li^6 may be readily computed for the Gaussian and Yukawa potentials of references 5 and 6, and is (0.3–0.5) times the contribution of t itself. The fourth-order tensor potential term will tend to cancel this somewhat. Thus the convergence of the tensor terms is slow and it is therefore difficult to estimate the strength, or even to be sure of the sign, of the total effective tensor potential as compared to that of the nucleon-nucleon tensor potential; though it seems plausible that the strength of the former would be less (and perhaps much less) than the strength of the latter potential.

The decomposition of $\mathcal{U}$ into scalar, vector, and tensor parts as given in Eq. (22b) is not quite correct because of the dependence of $\bar{E}_{jk}$ on the states j and k. The effect of this dependence can easily be discussed if we assume the approximation of Eq. (14) for $\bar{E}_{jk}$. The operator VH_0V can be decomposed into scalar, vector, and tensor parts, as we have done for the operator V^2 in Eq. (22b). Let us denote by S_1, V_1, T_1 the matrix elements between the states j and k of the scalar, vector, and tensor parts, respectively, of the operator V^2, and by S_2, V_2, T_2 the matrix elements of the corresponding parts of VH_0V.

For off-diagonal matrix elements between singlet and triplet states, or between doublet states having

[19] S. M. Dancoff, Phys. Rev. **58**, 326 (1940).
[20] A. M. Feingold, Ph.D. thesis, Princeton, 1952 (unpublished).

different L values, only the vector parts of the operators V^2 and VH_0V can give nonvanishing contributions. In this case, $\mathcal{V}_{nm}$ reduces to

$$\mathcal{V}_{nm}\approx V_1/(\mathcal{E}_0-V_2/V_1). \quad (27)$$

For the diagonal matrix elements the scalar terms will in general be much larger than the vector and tensor terms,[5,6] and thus we can expand the denominator of Eq. (18) as a power series, and, retaining only the leading terms, we obtain

$$\mathcal{V}_{nn}\approx V_{nn}-S_1^2/S-V_1(2S_1/S-\mathcal{E}_0S_1^2/S^2)+V_2S_1^2/S^2 -T_1(2S_1/S-\mathcal{E}_0S_1^2/S^2)+T_2S_1^2/S^2, \quad (28)$$

where $S=S_2-\mathcal{E}_0S_1$. The splitting of the members of a given LS multiplet is thus due in "first" order to the vector and tensor terms in Eq. (28), which in this approximation are all seen to act independently. The term containing V_1 is essentially identical with the matrix element of $\mathcal{V}_v$ defined in Eq. (22b), but is now multiplied by an additional factor of 2. Of particular interest now is the additional vector term $V_2S_1^2/S^2$. The vector part of the operator VH_0V, which gives the matrix element V_2, has a 2-particle part[19] (which presumably is more important than its 3- or more-particle terms), arising from the term tKt, and is of the form

$$(9i\hbar^2/M)[f(r_{12})^2/r_{12}^2](\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{r}_{12}\times(\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_2)\chi_{12}^2. \quad (29)$$

This is of the same form as the 2-body vector spin-orbit force commonly taken as the basis of the shell model.[21] However the 2-particle vector force of Eq. (29) has, for any choice of χ, a sign opposite to that demanded by the shell model, i.e., by itself it would predict that for a given multiplet the state of lowest j should lie lowest, in contradiction with experiment. Fortunately, according to the calculations of references 5 and 6, the term in Eq. (28) containing V_1 is, for a reasonable choice of potential and nuclear radius, considerably larger than the term containing V_2, and in addition has the proper sign demanded by the shell model. Thus if the tensor force is mainly responsible for the vector shell model, the effective vector force must be principally of the 3-body type of Eq. (25). As mentioned earlier, integration over a large core will result in a strong effective one-body vector force, with consequent jj coupling.

[21] J. Hughes and K. T. LeCouteur, Proc. Phys. Soc. (London) **A63**, 1219 (1950); J. P. Elliott and A. M. Lane, Phys. Rev. **96**, 1160 (1954).

CONCLUDING REMARKS

The "reduced tensor coupling" model as derived in this paper is valid only if the contribution to the first-order wave function of the high-lying states is small. This will be true only if ΔE, the tensor force contribution to the binding energy, is small compared to $(\mathcal{E}_0-\bar{E})$. ΔE almost certainly increases steadily with A, so that if $\Delta E/(\mathcal{E}_0-\bar{E})$ also increases with A, as seems likely, then a point will be reached in the periodic table beyond which the model is invalid. Even before this occurs we can expect large supermultiplet and configurational mixing when ΔE becomes large compared to supermultiplet and configuration spacings. It seems unlikely, therefore, that the model could be useful much beyond, say, $A=50$.

The neglect of the high-lying states in the wave function implies that the model cannot give accurate magnetic or quadrupole moments. Thus, for example, the deuteron quadrupole moment, which comes from high-lying D states, would be ignored by the model, and also the deviation of the magnetic moment of the deuteron from the sum of the intrinsic nucleon moments.

Recently, Jancovici and Talmi,[22] Visscher and Ferrell,[23] and Elliott[24] have explained the β decay of C^{14} on the basis of a mixture of vector and tensor forces. In the "reduced tensor coupling" model, the 2- and 3-particle vector forces of Eqs. (28) and (29) should be used, together with a presumably weakened tensor force. Calculations on the C^{14} β decay using the "reduced tensor coupling" model are in progress.

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[23] W. M. Visscher and R. A. Ferrell, Phys. Rev. **99**, 649(A) (1955).
[24] J. P. Elliott, Phil. Mag. **1**, 503 (1956).