Magnetoconductivity in p-Type Germanium

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(Received October 18, 1956)

Measurements of the Hall coefficient and resistivity of p-type germanium have been made as a function of magnetic field, temperature, and carrier concentration between 77°K and 300°K. An attempt is made to interpret the data quantitatively using a two-carrier model but no completely satisfactory quantitative interpretation is possible.

1. INTRODUCTION

CYCLOTRON resonance measurements at the temperature of liquid helium show that the valence band structure of germanium is complex, consisting of two valence bands having carriers of effective mass approximately $0.3\,m_0$ and $0.04\,m_0$, respectively.[1,2] Galvanomagnetic measurements made at higher temperatures confirm the general picture of a double valance band in a semiquantitive way.[3] However, because of various complications having to do with the detailed nature of the two valence bands, no very quantitive analysis of the magnetoconductivity has been made.

The studies described below were undertaken in the hope of achieving a more quantitative treatment of the galvanomagnetic properties. It was our objective to find a model of p-Ge that would permit a consistent understanding of the following characteristics:

(*a*) *Temperature dependence of lattice mobility.*—Experimentally the mobility follows roughly a $T^{-2.3}$ law.[4]

(*b*) *Temperature dependence of the Hall constant in relatively pure materials.*—This temperature dependence was apparently different for samples of different resistivity,[5] even though the samples appeared sufficiently pure that impurity scattering ought to be negligible.

(*c*) *Dependence of the Hall constant and magnetoresistance on magnetic field.*[3,6]—The field dependence at room temperature had been fitted[3] semiquantitatively with a simple two-carrier model. However, this two-carrier model required that the ratio of the number of light holes to the number of heavy holes be quite different than would be expected from low-temperature cyclotron resonance measurements, and the model was based on a treatment of scattering which was apparently not consistent with the observed temperature dependence of mobility.

In order to try to clarify the experimental situation a rather extensive set of experiments seemed necessary. A set of samples with carrier concentrations ranging from 10^{12} to 4×10^{14} was prepared.

Two kinds of experiments were done on this series of samples. One kind involved determining the dependence of the Hall constant and mobility on temperature in the temperature interval 77°K to 300°K. In these experiments we wanted to get the limiting value of Hall constant and mobility in vanishing magnetic field.

The second kind of experiment involved determining the dependence of Hall constant and resistivity on magnetic field at fixed temperature. In these experiments we wanted to get the shape of the strong field magnetoconductivity functions, so we needed to have high mobility values and to work at a well-defined temperature. Accordingly, these experiments were carried out in a liquid nitrogen bath over a magnetic field range of 0–25 000 gauss.

Generally speaking, the choice of the experiments was governed by the reasoning that from the two series we could separate that part of the temperature dependence of mobility which originates in the direct temperature dependence of the scattering process from that part which comes about because the average energy of the carriers changes with temperature. However, to make that separation in a satisfactory way we would need to be able to get a quantitative understanding of the shapes of the conductivity functions as function of both temperature and magnetic field.

Thus in addition to the experiments reported in the first part of this paper, we needed to make a careful theoretical analysis. In particular we needed to be able to make a fairly accurate treatment of the magnetoconductivity for the warped energy surfaces which occur in p-Ge. Our analysis and its results make up the second part of the paper.

2. SAMPLE PREPARATION

Single-crystal germanium was pulled in the [001] direction from a melt of zone-refined material. Indium was added to the melt to make the crystals p-type. The amount of compensation in these crystals is unknown but is believed to be small. One single crystal was made by a double-pull technique with no added impurities. Samples cut from a slice of this ingot proved to be p-type with a carrier concentration of approximately

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[1] Dresselhaus, Kip, and Kittel, Phys. Rev. **92**, 827 (1953); C. Kittel, Physica **20**, 829 (1954).
[2] Lax, Zeiger, Dexter, and Rosenblum, Phys. Rev. **93**, 1418 (1954); Lax, Zeiger, and Dexter, Physica **20**, 818 (1954).
[3] Willardson, Harman, and Beer, Phys. Rev. **96**, 1512 (1954).
[4] M. B. Prince, Phys. Rev. **92**, 681 (1953).
[5] C. Goldberg and R. E. Davis, Phys. Rev. **98**, 1192 (1955).
[6] Adams, Davis, and Goldberg, Phys. Rev. **99**, 625 (1955).

TABLE I. Carrier concentrations determined from the saturation value of the Hall coefficient. $\text{Lim}_{H\to\infty}R_H=1/pe=1/(p_l+p_h)e$.

Sample	Carrier concentration (cm^{-3})
A	8.1×10^{11}
B	9.8×10^{12}
C	1.6×10^{13}
D	2.4×10^{13}
E	4.4×10^{13}
F	1.5×10^{14}
G	3.4×10^{14}

8×10^{11} cm^{-3}. The compensation in this ingot is unknown but is probably higher than in the less pure ingots.

The samples were cut so that the magnetic field would be in a [001] direction during Hall measurements. The current direction in these samples were chosen to be either [100] or [110]. The sample surfaces were lapped with 600-mesh alundum to avoid some of the effects[7] which may exist at temperatures where an appreciable number of minority carriers are present. Leads were attached by using indium solder.

Most of the samples were cut with the usual rectangular geometry. Some samples were cut in a bridge shape[8] from the high-purity ingot. There was no significant difference in any of the measured quantities between measurements on rectangular samples and measurements on bridge-shaped samples.

Although it may be incorrect to do so because of the lack of information about compensation, the terms "carrier concentration" and "impurity concentration" will be used interchangeably in this paper. The carrier concentrations as determined from the saturation values of the Hall coefficient at 77°K are given in Table I.

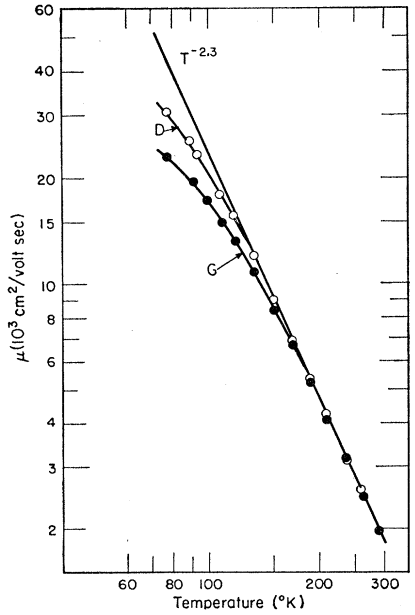

FIG. 1. Temperature dependence of mobility for samples *D* and *G*.

[7] R. Landauer and J. Swanson, Phys. Rev. **91**, 555 (1953).
[8] P. P. Debye and E. M. Conwell, Phys. Rev. **93**, 693 (1954).

3. MEASUREMENTS

A. Temperature Dependence of Lattice Mobility and Hall Coefficient

The measured mobilities of two of the samples as a function of temperature are shown in Fig. 1. These values were obtained from measurements of resistivity by assuming that the mobility at 300°K is 1820 cm^2/volt sec. (This is the value found by Prince[4] for the drift mobility at this temperature.) These data are well represented by a $T^{-2.3}$ power law at the higher temperature. At the lower temperatures the mobility is affected by impurity scattering. As the sample purity increases, the $T^{-2.3}$ region is extended to lower temperatures.

The zero-field Hall coefficients of two of our samples are given as a function of temperature in Fig. 2

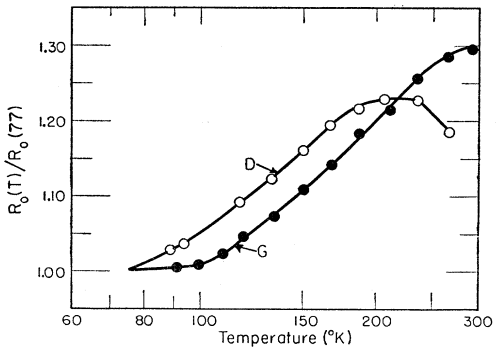

FIG. 2. Temperature dependence of the zero-field Hall coefficient for samples *D* and *G*. For ease of comparison the data are divided by the zero-field Hall coefficient at 77°K.

($R_0=\lim_{H\to0}R_H$). In order to compare the temperature dependence of the two curves, $R_0(T)/R_0(77)$ is plotted. In each case the value of $R_0(T)/R_0(77)$ at the higher temperatures decreases because of the presence of intrinsic electrons. The horizontal portion of the curve for sample *G* at low temperature, which is perhaps a minimum, is apparently due to impurity scattering since it gradually disappears as the sample purity increases. For sample *D*, Fig. 2 gives only a very slight indication that such a region is being approached.

Figure 2 shows that the Hall constant has a marked temperature dependence. No such temperature dependence is to be expected on the ordinary lattice scattering theory. Because of the temperature dependence of R_0, the zero-field Hall mobility ($\mu_H=R_0\sigma_0$) varies more nearly as $T^{-2.1}$ than as $T^{-2.3}$.

A derived quantity of theoretical interest is the mobility ratio (μ_H/μ). The most accurate way to determine this quantity is to evaluate the ratio R_0/R_∞, where R_∞ is the strong field saturation value of the Hall coefficient. This ratio is equal to the mobility

ratio μ_H/μ since $R_\infty = 1/pe$, where p is the carrier concentration, and $R_0 = (\mu_H/\mu)(1/pe)$. The field dependence of the Hall coefficient discussed later in this section indicates that at 77°K saturation has been reached at a field of 3700 gauss so that μ_H/μ at a temperature T should be equal to $R_0(T)/R_{3700}(77)$. The temperature dependence of this quantity is shown in Fig. 3. The shapes of the curves in Fig. 3 are, of course, the same as in Fig. 2.

B. Field Dependence of Resistivity and Hall Coefficient at 77°K

The field dependences of the Hall coefficient and of the resistivity at 77°K are shown in Fig. 4 for two samples that differ in carrier concentration by a factor

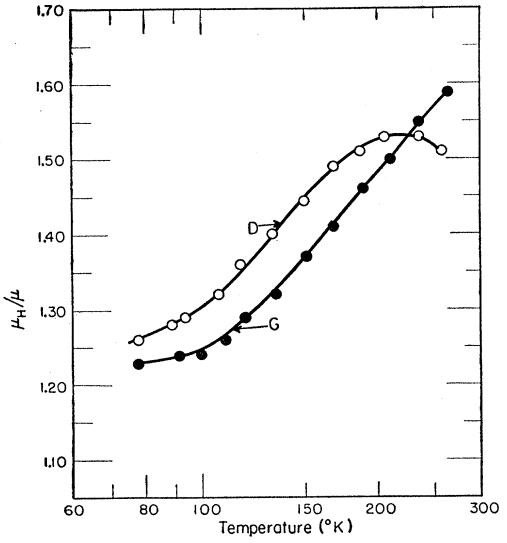

FIG. 3. Temperature dependence of the mobility ratio μ_H/μ for samples D and G. This ratio has been determined by dividing the zero-field value of the Hall coefficient by the strong-field saturation value.

of approximately 440. In both cases the magnetic field was in the [001] direction.

It can be seen that at low magnetic fields the Hall coefficient for the pure sample varies more strongly with magnetic field, but reaches the strong-field saturation value at a lower value of field than does that of the less pure sample. The field dependence of the resistivity is larger for the pure sample. In contradiction to all theories based upon scattering mechanisms independent of magnetic field, there is no indication of a strong-field saturation value for the resistivity, the resistivity being approximately linear with field above 7000 gauss.

The curves for the other samples shown in Table I have been measured also. The data for these samples are consistent with the idea that the difference in shape of the curves is an effect of increased impurity concentration.

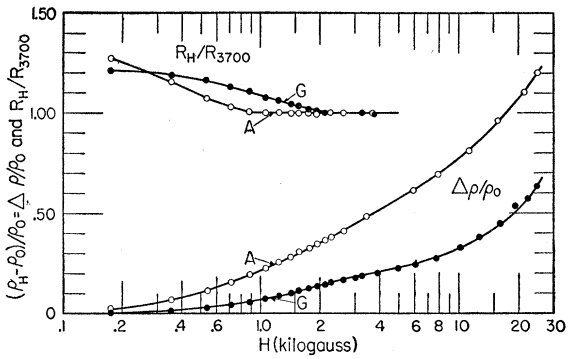

FIG. 4. Magnetic field dependence of the Hall coefficient and transverse magnetoresistance of samples A and G at 77°K.

Peterson, Swanson, and Tucker[9] have reported a minimum in the curve of the Hall coefficient as a function of magnetic field. In their experiment the Hall constant at minimum was about 0.94 times the saturation value. Our data do not show any such minimum in excess of the experimental uncertainty. However, it should be pointed out that the samples for which minima have been found[9] were cut with a different crystallographic orientation than those used here.

C. Magnetic Field Dependence of the Hall Coefficient as a Function of Temperature

Figure 5 shows the magnetic field dependence of the Hall coefficient at several temperatures in the interval between 77°K and room temperature. At 77°K the Hall coefficient is strongly field-dependent at low fields but quickly reaches a saturation value. As the temperature increases, the low-field values of the Hall coefficient appear to increase while the field dependence at low fields decreases. The data are consistent with the idea that the saturation value of the Hall constant should at each temperature be the same as it is at 77°K: Interpretation of the field dependence of the Hall coefficient will be deferred to a later section.

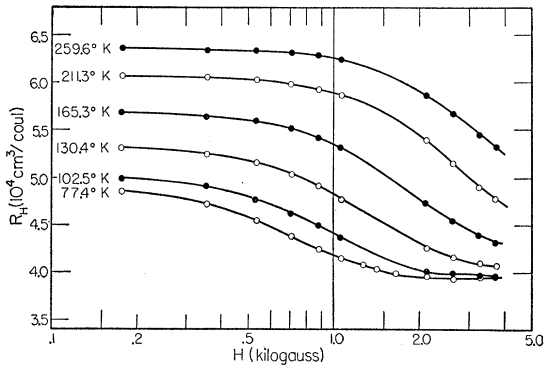

FIG. 5. Magnetic field dependence of the Hall coefficient for sample F.

[9] Peterson, Swanson, and Tucker, Bull. Am. Phys. Soc. Ser. II, 1, 117 (1956).

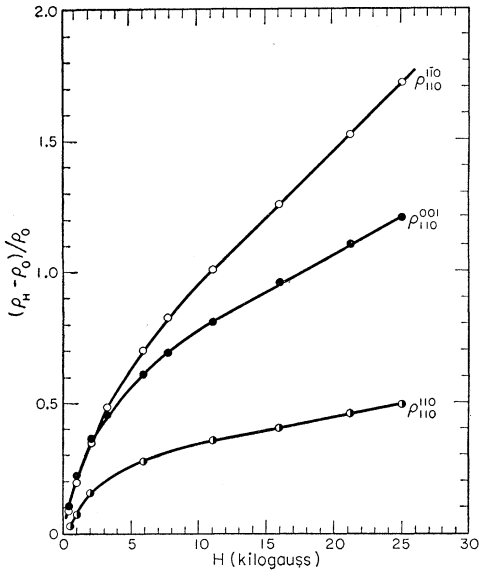

FIG. 6. Magnetic field dependence of the magnetoresistance for sample A at 77°K. The subscripts and superscripts indicate the crystallographic directions of the electric current and magnetic field, respectively.

D. Anisotropy of Magnetoresistance at 77°K

For sufficiently small values of magnetic field, the electric current takes the form[10]

$$\mathbf{i}=\sigma_0\mathbf{E}+\alpha\mathbf{E}\times\mathbf{H}+\beta\mathbf{E}H^2+\gamma\mathbf{H}(\mathbf{E}\cdot\mathbf{H})+\delta M\mathbf{E}. \quad (3.1)$$

Here M is a diagonal tensor with elements H_1^2, H_2^2, H_3^2 in the 11, 22, and 33 positions, respectively.

The anisotropy of magnetoresistance can be expressed by giving the values of the two quantities $a=(2\beta+\delta)/2\beta$ and $b=(2\beta+2\gamma+\delta)/2\beta$. For an isotropic material these quantities have the values 1 and 0 respectively.

Figure 6 shows the observed field dependence of resistivity at 77°K for a current in the [110] direction and three directions of the magnetic field, *viz.*, the [001], [1$\bar{1}$0], and [110] directions. The anisotropy is substantial. Extrapolating these curves to zero field, we obtain the following values for the anisotropy parameters: $a=0.90$ and $b=0.17$.

The dependence of resistance on field is shown in Fig. 7 for the case where both the current and magnetic field are in the [100] direction. The effect of the field is much smaller than that shown in Fig. 6. In Fig. 7, as in Fig. 6, there is no indication that the resistivity is saturating at high fields.

4. THEORETICAL

Lattice scattering in p-Ge can take place as a result of the interaction of the holes with either the acoustical modes of lattice vibration or the optical modes. For the states of interest to us, the modes must be of very long wavelength in either case.

[10] F. Seitz, Phys. Rev. **79**, 372 (1950).

The observed temperature dependence of mobility is $T^{-2.3}$ in the temperature interval 77°K to 300°K. Acoustical scattering alone would lead to a temperature dependence $T^{-1.5}$. Therefore, it seems likely that optical-mode scattering plays an important, perhaps even a dominant, role in the lattice scattering.[11]

If it is assumed that only optical mode scattering is important, we can estimate the effective Debye temperature for the optical mode from the temperature dependence of the mobility. We have made such an analysis and found a value of about 300°K for the optical frequency. This method of estimate gives a lower bound on the optical frequency, of course, since any acoustical scattering present will tend to lower the mean power law towards a value of -1.5.

While 300°K is somewhat lower than is expected for the optical frequency in germanium, it is not so low as to be ruled out. It is more probable, however, that the scattering receives appreciable contributions from both optical and acoustical branches, and that the Debye temperature is somewhat higher.

On the above assumptions about the relaxation process, we are led to expect that the relaxation time at 77°K will depend only weakly on energy. The exact energy dependence of the relaxation time will depend on the ratio between acoustical and optical mode scattering rates. It should be intermediate between the $\epsilon^{-\frac{1}{2}}$ expected on the former and the $(\hbar\omega_D+\epsilon)^{-\frac{1}{2}}$ expected on the latter mechanism of scattering.

The above reasoning indicates that no great error will be made by assuming τ at 77°K to be proportional to $\epsilon^{-\frac{1}{2}}$. In the first place, neither energy dependence is very strong. In the second place, the relative importance of the acoustical mode scattering should be very great at 77°K since the optical mode scattering is proportional to a factor $e^{-\theta/kT}$ as compared with a factor T/θ for the acoustical mode scattering. Accordingly, for analyzing the magnetic field dependence of magnetoconductivity

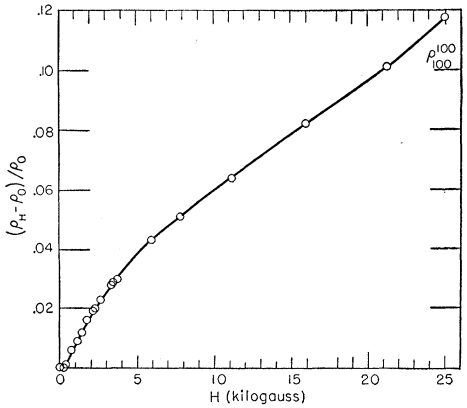

FIG. 7. Magnetic field dependence of longitudinal magnetoresistance of sample B at 77°K. The electric current and the magnetic field are in the [100] direction.

[11] H. Ehrenreich, Bull. Am. Phys. Soc. Ser. II, **1**, 48 (1956).

at 77°K, we have assumed that τ is exactly proportional to $\epsilon^{-\frac{1}{2}}$.

Clearly the two kinds of holes could have different temperature and energy dependence of relaxation time. We can justify the assumption that the two relaxation times have the same energy dependence at 77°K, however.

We remark that there are two kinds of scattering process for each kind of hole, one that we call "intraband" scattering, in which there is a simple scattering to another energy state on the same energy surface, the other which we call "conversion scattering" in which a fast hole converts to a slow one or vice versa. The energy dependence for the relaxation time associated with each of these processes is the reciprocal of the energy dependence of the density of final states, the matrix elements being approximately energy independent.

Since at 77°K we expect acoustical mode scattering to predominate, the density of final states is proportional to $\epsilon^{\frac{1}{2}}$, even for a combination process of an arbitrary sort. Thus we expect each τ at 77°K to be proportional to $\epsilon^{-\frac{1}{2}}$.

Actually, from the observed value of the mobility ratio of the two kinds of hole, we can get an idea as to which of the four processes are important. From simple theory the ratio of scattering times for *intraband* scattering is proportional to the inverse 5/2 power of the masses. Thus

$$(\tau_l{}^I/\tau_h{}^I)\sim(m_h/m_l)^{5/2}\approx 150. \tag{4.1}$$

On the other hand, from detailed balancing, the *extraband* (conversion) scattering times are in the ratios of the concentrations, so

$$(\tau_l{}^E/\tau_h{}^E)\sim p_l/p_h\approx 1/20. \tag{4.2}$$

The experimental mobility ratio is $(\mu_l/\mu_h)\approx(m_h/m_l)$, hence $\tau_h\approx\tau_l$.

From the approximate equality of the relaxation times, it seems most likely that the two important processes are conversion scattering of light holes and intraband scattering of heavy holes. Such an outcome will result provided only that the matrix elements for all four processes are of the same order of magnitude, since the density of states in the heavy hole band is about twenty times as great as the density in the light hole band.

The density of states is the origin of the energy dependence of each relaxation time τ_i. If the same density of states, viz., the density of heavy holes state, is involved for both τ_l and τ_h, then it might be expected that τ_l and τ_h should have the same dependences on energy, and perhaps on temperature. In particular it might be thought that the mobility ratio should be temperature-independent. However, this need not be the case. Although at any temperature the energy dependence of τ_l should be almost the same as that of τ_h, it is possible with a different ratio of matrix elements for optical and acoustical mode scattering to have a considerable variation of the ratio (μ_h/μ_l) with temperature in the temperature interval of changeover from acoustical- to optical-mode scattering. Such a variation probably accounts in part for the temperature dependence of the zero-field Hall coefficient.

We have not discussed the detailed nature of the scattering process and could not do so quantitatively without an elaborate theory of the scattering mechanism. We shall assume in our subsequent analysis that τ is a function of energy only, although this assumption may well be invalid.

5. SIMPLE TWO-CARRIER MODEL

In this section, the magnetoconductivity data for sample A will be analyzed on the basis of a simple two-carrier model. Our calculation assumes that each of the two energy surfaces is spherical, and that the relaxation time is proportional to the inverse square root of the energy (acoustical lattice scattering). As the parameters of fit we can take the light hole mobility μ_l, the heavy hole mobility μ_h, and the ratio r of the number of light holes to the number of heavy holes.

The conductivity tensor $\mathbf{S}(\mathbf{H})$ is defined by the relation between the current $\mathbf{i}$ and the electric field $\mathbf{E}$:

$$\mathbf{i}=\mathbf{S}\cdot\mathbf{E}. \tag{5.1}$$

If the magnetic field is in the [001] direction, the isotropy of the energy surfaces results in the following symmetry relations among the elements of $\mathbf{S}$:

$$S_{11}=S_{22},\quad S_{12}=-S_{21},\quad S_{23}=S_{13}=0. \tag{5.2}$$

The relations (5.2) together imply that the components of the conductivity tensor are invariant against an arbitrary rotation of the coordinate system about the magnetic field (3 axis).

The conductivity of the material is the sum of the individual conductivities of the two kinds of holes. Thus it is necessary in the analysis to attempt to decompose the observed conductivity into its two parts before any simple interpretation can be made. Furthermore, the usual experiments do not determine the conductivity directly, but instead determine certain derived quantities, *viz.*, the longitudinal resistivity ρ_L, the transverse resistivity ρ_H, and the Hall constant R_H and the experimental conductivities must be deduced from the measured values of these.

In virtue of the third of Eqs. (5.2), the longitudinal resistivity is the reciprocal of the conductivity component S_{33}. The other two independent elements of S are determined from somewhat more complicated relations. We have

$$S_{11}=\rho_H/(\rho_H{}^2+H^2R_H{}^2), \tag{5.3a}$$

$$S_{12}=HR_H/(\rho_H{}^2+H^2R_H{}^2), \tag{5.3b}$$

$$S_{33}=1/\rho_L. \tag{5.3c}$$

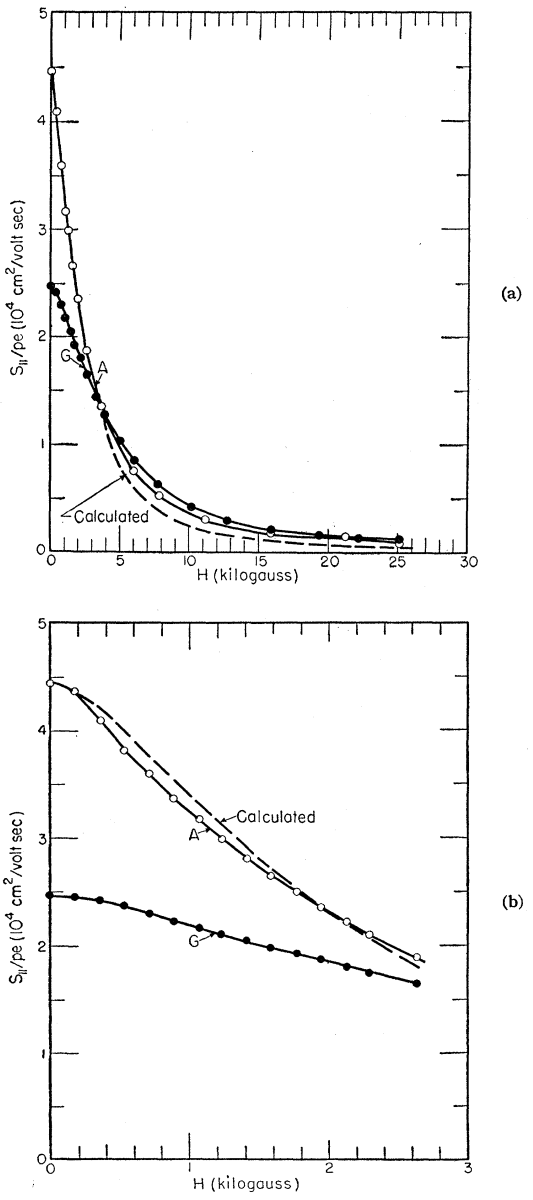

FIG. 8. Magnetic field dependence of S_{11} for samples A and G at 77°K. The dashed line represents a curve calculated by the method outlined in Sec. 6. (8b) gives the weak-field data of (8a) on an expanded scale.

On the simple two-carrier model, we can at once write down a theoretical expression for the magnetoconductivity. We write

$$S_{11}(H)=\rho_h e\mu_h(K_h+K_l rs), \tag{5.4a}$$

$$S_{12}(H)/H=(3\pi/8)\rho_h e\mu_h^2(L_h+L_l rs^2), \tag{5.4b}$$

$$S_{33}=\rho_h e\mu_h(1+rs); \tag{5.4c}$$

the subscripts h and l refer to the light and heavy holes, respectively; $K_i=K(9\pi\mu_i^2H^2/16)$ and $L_i=L(9\pi\mu_i^2H^2/16)$ are the field-dependent functions defined by Willardson, Harman, and Beer,[3] r is the ratio (p_l/p_h) of the hole concentrations, and s is the ratio (μ_l/μ_h) of the mobilities at zero magnetic field. Each of the functions K and L is unity at $H=0$ and is a monotonically decreasing function of μ^2H^2.

In our problem $s>1$, so K_l and L_l decrease more rapidly than K_h and L_h as the field increases. If $s\gg1$ the light hole can make important contributions to the low-field values of S_{11} and S_{12}/H, even though the concentration ratio r is only a few percent. The fractional contribution of the light hole to S_{12}/H will, of course, be s times greater than the contribution to S_{11} at zero magnetic field.

Figures 8 and 9 show the field dependence of S_{11}/pe and S_{12}/peH for samples A and G at 77°K calculated from Hall and resistivity data by means of Eqs. (5.3). The quantity $1/pe=1/(p_h+p_l)e$ was calculated from strong-field measurements of the Hall coefficient.

The curves for sample A are easily understood qualitatively on the basis of the simple two-carrier model with a very few light holes of very high mobility.[3] At low fields there is a large contribution to S_{12}/Hpe from light hole conduction because of the large value of rs^2 due to the high mobility of the light hole. However, L decreases rapidly with increasing field, again because of the high light-hole mobility, and the fractional contribution of the light hole to S_{12}/Hpe becomes small at about 1000 gauss.

The fractional contribution of the light hole to S_{11}/pe is always fairly small, of the order of 20%. Further, the light hole contribution to S_{11} decreases with increasing field just as rapidly as its contribution to S_{12}/H.

The functions S_{11} and S_{12}/H as function of H for samples A and G have different shape, so have to be fitted by somewhat different parameters of the two-carrier models. Qualitatively the difference indicates that the increased impurity concentration of sample G, the sample with the larger number of carriers, has caused a large decrease in the light-hole mobility and a smaller decrease in the heavy-hole mobility. The field dependence of S_{11}/pe and S_{12}/pHe has been measured for all seven of the samples listed in Table I. The results indicate that, in each case, as the impurity concentration is increased both μ_h and $s=\mu_l/\mu_h$ decrease.

For a purely qualitative purpose, then, the simple two-carrier model permits us to understand the gross trends of the Hall coefficient and resistivity curves in Fig. 4. The value of R_0/R_{3700} is higher for the purer sample because the mobility ratio is higher for this sample and the resistivity and Hall coefficient have a stronger field dependence at low fields because of the higher mobilities of the two holes. The temperature dependence of R_0/R_{3700} may also result to a large extent from the changing importance of impurity scattering of fast holes. The nonsaturation of resistivity at high magnetic fields is, however, not explained by the simple model.

We shall now attempt to make a more quantitative treatment of the data. From the values of $S_{11}(0)/pe$ and $(S_{12}/Hpe)_0=\lim_{H\to 0}[S_{12}(H)/Hpe]$, we can determine two of the three parameters r, s, and μ_h. Since we are interested in lattice scattering primarily, we do this for sample A, the purest sample.

The values of s and μ for sample A for several different assumed values of the concentration ratio r are shown in Table II. The concentration ratio of 0.02 is that found by Willardson *et al.*[3] from Hall coefficient and resistivity measurements at 205°K and room temperature. The concentration ratio 0.042 is that found from the parameters of the energy surfaces determined by cyclotron resonance experiments[12] at 4.2°K. It is to be noted that, assuming the number ratio 0.02 as found by Willardson *et al.*, we find a mobility ratio s or 4.1 which is appreciably smaller than that found by Willardson, Harman, and Beer (7.5 at 205°K and 8.0 at room temperature).

A calculation of the field dependence of S_{11}/pe and S_{12}/Hpe using Eqs. (5.4) and the parameters in Table II gives curves which have too weak a field dependence at low fields. The dashed lines in Figs. 8 and 9 show the curves calculated using $r=0.02$. It is to be noted that the field dependence of the calculated curves is much weaker than that of the observed curves. The situation is about the same if the curves are calculated from the mobility parameters obtained using other values of r.

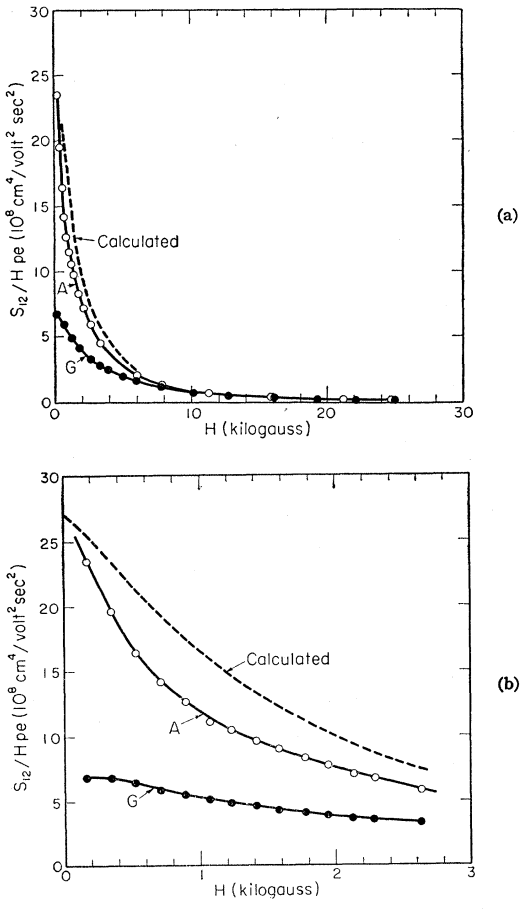

FIG. 9. Magnetic field dependence of S_{12}/H for samples A and G at 77°K. The dashed line represents a curve calculated by the method outlined in Sec. 6. (9b) gives the weak-field data of (9a) on an expanded scale.

[12] B. Lax and J. G. Mavroides, Phys. Rev. **100**, 1650 (1955).

TABLE II. Mobilities of the two holes determined from the zero-field values of S_{11}/pe and S_{12}/Hpe for sample A, with the concentration ratio as a variable parameter.

r	s	μ_h (cm²/volt sec)
0.01	5.1	4.3×10^4
0.02	4.1	4.2×10^4
0.042	3.5	4.1×10^4
0.06	2.9	4.0×10^4

From the above it is clear that the simple two-carrier model is inadequate to give a quantitative fit of the field dependence data obtained at 77°K. The difficulty seems to be that the light-hole mobility obtained from the zero-field data is always too low. To account for the observed field dependence of S_{12}/Hpe at low fields, the light-hole mobility would need to be about twice as great.

The calculated values of S_{11}/pe at very strong fields are always considerably lower than the measured values. The difference is related to the nonsaturation of the resistivity at strong fields. The simple Boltzmann model could not possibly account for nonsaturation, so we must regard the nonsaturation as a separate phenomenon. Thus the failure of our theory to yield a good fit at very high fields is probably not due to oversimplifications in the treatment of, for example, the shape of the energy surfaces.

In summary, the simple two-carrier model permits a qualitative understanding of the dependence of S_{11} and S_{12} upon magnetic field and impurity concentration. When used to get a quantitative treatment of the 77°K data it does not permit a consistent description of any two of the following: zero-field resistivity and Hall constant, strong-field ($0<\mu H<1$) magnetoconductivity, very-strong-field magnetoconductivity ($1<\mu H$).

6. WARPED ENERGY SURFACES

The preceding discussion was conducted on the basis of a spherical energy surface model for each of the two kinds of holes. Such a model is incapable of giving any anisotropy of magnetoresistance such as is observed. Actually, we know that the energy surfaces are not perfect spheres but are warped. For the heavy holes these warped surfaces are really quite different from spheres.

In the following paragraphs we will examine the way in which the results obtained on our simple two-carrier model must be modified to take account of the warping of the energy surfaces. For this discussion we will use

the energy surface parameters obtained from cyclotron resonance measurements at 4.2°K, although it is possible that different parameters would be appropriate at 77°K.

The energy surfaces for holes in germanium can be expressed as a function of the wave vector $\mathbf{k}$ by the equation[1]

$$\epsilon=(\hbar^2/2m_0)\{Ak^2\pm[B^2k^4+C^2(k_x^2k_y^2+k_y^2k_z^2+k_z^2k_x^2)]^{\frac{1}{2}}\}, \quad (6.1)$$

where the k coordinate system is coincident with the cubic axes, m_0 is the mass of a free electron, and the values of the constants obtained from cyclotron resonance measurements[12] are

$$A=13, \quad B=8.7, \quad C=11.4. \quad (6.2)$$

The plus sign in (6.1) is to be associated with the holes of small effective mass and the minus sign with the heavier holes.

Mavroides and Lax[12] have calculated the coefficients occurring in Eq. (3.3) for the case of the energy surfaces represented by (6.1) where the relaxation times are functions of the energy only. We have combined the results of their calculations and the energy surface parameters given by (6.2) with our zero-field experimental values of S_{11}/pe and S_{12}/Hpe. In this way we find the set of values $r=0.042$, $s=4.34$, and $\mu_h=3.92\times10^4$ cm²/volt sec, for sample A at 77°K on the assumption that both τ_l and τ_h are proportional to the inverse square root of the energy. This gives a light-hole mobility higher than that found in Sec. 5 where spherical energy surfaces were assumed. This light-hole mobility is still too low, however, to account for the rapid decrease of S_{12}/Hpe with increasing field.

If the heavy-hole energy surface were more anisotropic than is indicated by (6.2), then the mobility ratio calculated from the experimental zero-field values of S_{11}/pe and S_{12}/Hpe would be higher. It is hard to judge how probable is a large change in A, B, C between 4.2°K and 77°K. Further, the calculations of Mavroides and Lax do not indicate the nature of the field dependence of S_{12}/Hpe. Thus in any case the strong-field data cannot be used in conjunction with their calculations to indicate uniquely what the correct parameters are for Eq. (6.1) without the use of additional theory. A tentative theory of the field dependence of S_{11} and S_{12}/H for warped surfaces will be discussed below.

The calculations of Mavroides and Lax[13] can also be used to determine the ratios $a=(2\beta+\delta)/2\beta$ and $b=(2\beta+2\gamma+\delta)/\beta$ if assumptions are made concerning the exact energy dependence of $\tau_l(\epsilon)$ and $\tau_h(\epsilon)$. Without making these assumptions about the form of $\tau_l(\epsilon)$ and $\tau_h(\epsilon)$ it can be shown that for the parameters given by (6.2), $b=1.03\ (1-a)$. As indicated in Sec. 3, $a=0.90$ so that the calculated value of b is 0.10, considerably smaller than the experimental value of 0.17. This finding may be interpreted as indicating that the energy surfaces are more anisotropic at 77°K than at 4.2°K.

In summary, the zero-field anisotropy calculated from the energy surfaces obtained from cyclotron resonance on simple assumptions about the scattering does not agree well with our observed values. This disagreement is substantial. However, we are reluctant to draw any far-reaching conclusions from this fact, since (a) our extrapolated zero-field values have a considerable experimental uncertainty, and (b) the theoretical value, which is based on a simplified treatment of scattering, could be changed a good deal if it turned out that the scattering mechanism were appropriately pathological.

7. TRANSVERSE MAGNETOCONDUCTIVITY IN STRONG FIELDS

We shall now discuss more fully the field dependence of S_{11} and S_{12} for the heavy holes. McClure[14] has given a straightforward method of getting the field dependence of magnetoconductivity in the case of an arbitrary energy surface. This method is general, with the important exception that the relaxation time must be taken to be a function of energy alone.

We could use McClure's method for the analysis of the p-Ge data, but it would be very tedious to do so for the actual warped surfaces. Instead, we have studied a simpler model of the energy surface and have made a treatment which is, as a consequence, only semiquantitative.

We have used a model according to which the light-hole surface is taken to be sphere and the heavy-hole surface a cube. This model actually gives a pretty fair representation of the actual warped surface, as has been remarked by Kittel.[15] The special advantage of the model is that it permits an exact calculation of the field dependence of the magnetoconductivity for all values of the field.[16]

In McClure's formula the magnetoconductivity functions S_{11} and S_{12} are expressed in terms of the Fourier components of the velocity. The Fourier series represents the magnetoconductivity as an expansion in harmonics of the periodic time of the carrier on the hodograph which the carrier momentum executes under the action of the magnetic forces.

The hodograph of the motion of a particle in a magnetic field is the curve of intersection in momentum space of a plane of constant P_z (the momentum along the magnetic field) with an energy surface. For a spherical energy surface the hodograph is a circle and the particle moves around this circle uniformly with the cyclotron frequency. Any transverse velocity component is exactly sinusoidal, so has only a single Fourier coefficient different from zero. For a carrier with a

[13] J. G. Mavroides and B. Lax (private communication).

[14] J. W. McClure, Phys. Rev. **101**, 1642 (1956).

[15] C. Kittel (private communication).

[16] J. A. Swanson (private communication).

spherical energy surface, the magnetoconductivity has the familiar forms

$$S_{11}=(pe^2/m)\langle\tau/(1+x^2)\rangle, \tag{7.1}$$

$$S_{12}=(pe^2/m)\langle x\tau/(1+x^2)\rangle. \tag{7.2}$$

Here the angular brackets denote a Boltzmann average over the energy and x the quantity $\omega\tau$, where ω is the angular frequency of the motion along the hodograph.

We now want to discuss the forms which replace (7.1) and (7.2) when the energy surface is cubical instead of spherical. In our problem, the magnetic field was in a [100] direction, so the hodograph is a square. Further, in the exactly cubical approximation all hodographs associated with a given energy surface are exactly alike.

The velocity of the particle on the square hodograph is always perpendicular to the hodograph. It is of constant magnitude v at all points along the hodograph but its direction changes as the momentum point passes each corner of the square.

For a hodograph of edge $2P$ the magnitude of v, the velocity on the hodograph, is P/m_{100} with m_{100} the effective mass on the [100] axis in P-space. The x component of velocity as a function of time along the hodograph is shown in Fig. 10. The rate of precession along the hodograph is obtained from the equation

$$dP/dt=evH/c. \tag{7.3}$$

From (7.3) we obtain for the angular frequency the relation

$$\omega=2\pi/T=\pi eH/4m_{100}c. \tag{7.4}$$

Assuming that the relaxation time is a function of energy alone, McClure has shown that the magnetoconductivity functions for this case take the form

$$S_{11}=\left\langle(2pe^2\tau/kT)\sum_{M=0}^{\infty}v_{Nx}^2/(1+N^2x^2)\right\rangle, \tag{7.5}$$

$$S_{12}=\left\langle(2pe^2\tau/kT)\sum_{M=0}^{\infty}v_{Nx}^2(-)^MNx/(1+N^2x^2)\right\rangle, \tag{7.6}$$

in which N is $2M+1$ and v_{Nx} is the amplitude of the Nth harmonic. The formulas (7.5) and (7.6) have been specialized to the case of fourfold symmetry of the hodograph.

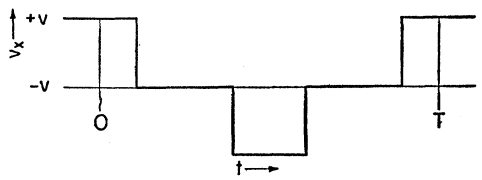

FIG. 10. x-component of velocity as a function of time along the square hodograph. T is the period of cyclotron resonance. *Note added in proof*—The horizontal axis of Fig. 10 should be labeled $v_x=0$ instead of $v_x=-v$.

For the particular motion shown in Fig. 10, the Fourier analysis yields

$$v_{Nx}=2v\sin(N\pi/4)/N\pi \quad (N=2M+1). \tag{7.7}$$

The formula (7.7) for these Fourier coefficients permit us to calculate the strong-field magnetoconductivity functions on our model, provided we know the energy dependence of τ.

The formula (7.7) gives the Fourier coefficients for those hodographs which pass all around the sides of the cube. However, $\frac{1}{3}$ of the carriers are in states which are in the end surfaces of the cube. For these states the velocity is along the magnetic field, so there is no magnetic force and no precession around the hodograph for such states. Since these states also carry no current transverse to the magnetic field, they contribute nothing to the conductivity components S_{11} and S_{12}. Accordingly, we must omit them when we use the Fourier components (7.7) in Eqs. (7.5) and (7.6), which we do by multiplying p by a factor $\frac{2}{3}$.

We find for the heavy-hole conductivity the following

$$S_{11}=\left(\frac{64pe}{3\pi^3}\right)\left\langle\left(\frac{\epsilon\mu_\omega}{kT}\right)\sum_{M=0}^{\infty}\frac{1}{N^2(1+N^2x^2)}\right\rangle, \tag{7.8}$$

$$S_{12}=\left(\frac{64pe}{3\pi^3}\right)\left\langle\left(\frac{\epsilon\mu_\omega}{kT}\right)\sum_{M=0}^{\infty}(-)^M\frac{x}{N(1+N^2x^2)}\right\rangle. \tag{7.9}$$

In the above we have used

$$\mu_\omega=e\tau\pi/4m. \tag{7.10}$$

Here μ_ω is defined so that in all cases $\omega\tau=\mu_\omega H=x$.

For illustrative purposes we will now work out the conductivity for the case τ independent of energy. Then the Boltzmann average over energy can be performed at once and we find

$$S_{11}=\left(\frac{32pe\mu_\omega}{\pi^3}\right)\sum_{M=0}^{\infty}\frac{1}{N^2(1+N^2x^2)}, \tag{7.11}$$

$$S_{12}=\left(\frac{32pe\mu_\omega}{\pi^3}\right)\sum_{M=0}^{\infty}(-)^M\frac{x}{N(1+N^2x^2)}. \tag{7.12}$$

From the zero-field value of S_{11} we get

$$\mu=4\mu_\omega/\pi. \tag{7.13}$$

Thus the conduction mobility is about 30% larger than the mobility as determined from the field dependence of S_{11}. Similarly the Hall constant at zero field is

$$R_0=1/2pec \tag{7.14}$$

so the ratio of Hall mobility to drift mobility is $\mu_H/\mu=\frac{1}{2}$.

A value of μ_H/μ for the warped surfaces in p-Ge can be obtained from the formulas of Mavroides and Lax,[12] and is given by $\mu_H/\mu=0.77$. Thus we see that the cubical energy surface model does not permit a quantitative

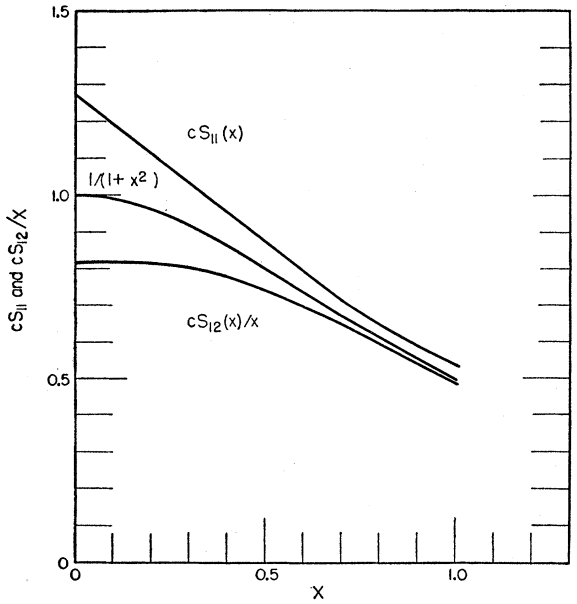

FIG. 11. Dependence of cS_{11} and cS_{12}/x upon $x=\omega\tau=\mu_\omega H$ for the case of cubical energy surfaces and an energy independent relaxation time where $c=\pi^3/(32pe\mu_\omega)$. For spherical energy surfaces these quantities both vary as $1/(1+x^2)$. This latter quantity is shown for comparison.

treatment of magnetoconductivity by carriers on the warped surface. The failure of the model undoubtedly has to do with the fact that the high harmonics of the velocity are much stronger than they would be if the velocity fell to zero more smoothly near the corners of the hodograph.

Even though the conductivities (7.11) and (7.12) cannot be used to make a quantitative treatment of the conductivity at high magnetic field, they serve to indicate what qualitative effect the warping of the energy surface has on the transverse magnetoconductivity. In Fig. 11 we show as a function of $\mu_\omega H=\omega\tau=x$ a comparison of the cubical surface functions S_{11} and S_{12} and the spherical surface functions as given by (7.1) and (7.2) for an energy-independent relaxation time. It is to be noted that for the case of the cubical surface S_{11} is higher at low fields and S_{12} lower than would be the case if the surface were spherical.

From the plots of S_{11} and $S_{12}/\omega\tau$ as functions of $\omega\tau$, it can be seen that the quantity μ_ω defined above could be obtained rather well by fitting the magnetoconductivity data in the field interval for which $\omega\tau>1$. In this interval the two functions S_{11} and S_{12}/ω should be nearly equal, so we can get μ_ω from the experimental quantity

$$\mu_\omega=S_{12}/HS_{11} \quad (H\to\infty). \tag{7.15}$$

Since this method works in the cubical case, it should be valid for the actual warped surface in Ge which is less warped than a cube.

For the actual warped surfaces the strength of the harmonics of the velocity is less than for a cube. We have studied several models in attempting to find out how the Fourier coefficients will behave in a more realistic case. While we shall not discuss the work in detail, we wish to indicate the direction in which the Fourier coefficients for the warped surface case must differ from those found when the cubical model is used. Primarily the difference is that for more rounded hodographs fewer harmonics are important and all of the higher harmonics are weaker. Our rough calculations suggest that for a value $(\mu_H/\mu)\sim 0.77$ it may well require only third and fifth harmonics to represent the velocity variation rather well.

We have made a semiempirical theory of magnetoconductivity using only third and fifth harmonics. The strengths of these are determined by requiring that the zero-field values of the two functions S_{11} and S_{12}/H be the same as those given by the formulas of Mavroides and Lax. The functions obtained in this way are given by

$$\begin{aligned} S_{11}&=pe\mu_\omega[1.02/(1+x^2)+0.06/(1+9x^2)\\ &\qquad +0.02/(1+25x^2)],\\ S_{12}&=pe\mu_\omega x[1.02/(1+x^2)-0.18/(1+9x^2)\\ &\qquad +0.10/(1+25x^2)]. \end{aligned} \tag{7.16}$$

We have tried using the functions (7.16) as a basis for an empirical fit of our data on the field dependence of the conductivity functions. In order to make such a fit, we found the values of fast- and slow-hole mobilities such that our zero-field conductivity and extrapolated zero-field Hall constant were correctly given; we used for the ratio of the concentrations the value 0.042 as found by Mavroides and Lax. These empirical mobility values were than used to compute a "theoretical" field dependence of the conductivity functions.

The "theoretical" field dependence function as obtained in this way agreed very poorly with the measured conductivity functions. Indeed the observed functions are much more like what we would compute for a cubical energy surface model and a relaxation time τ varying approximately as $\epsilon^{-\frac{1}{2}}$. It seems that a fairly good fit of the data could be made by means of such a model, but we have not undertaken to do so since no particular significance could be attributed to the mobility parameters so determined. In any case, the fit would fail for fields above a few thousand gauss since the nonsaturation behavior of the magnetoresistance cannot be accounted for on any such model as we are using.

In summary, we are unable to account for the shape of the magnetoconductivity functions as a function of magnetic field using simple assumptions about the relaxation time and assuming the warped energy surfaces obtained from cyclotron resonance. No matter what refinements we have introduced we cannot get the observed shape in the interval $0<H<1000$ gauss and also get the right shape in the interval 1000 gauss

$<H<4000$ gauss. The experimental curves depart from what we expected in such a way as to permit the interpretation that the anisotropy of the energy surfaces at 77°K is appreciably greater than at 4.2°K.

8. LONGITUDINAL MAGNETOCONDUCTIVITY IN STRONG FIELDS

From our data it is seen that in the [100] direction p-Ge exhibits a very considerable longitudinal magnetoresistance, of the order of 12% at 25 000 gauss. This magnetoresistance is different from what McClure's theory predicts in two significant ways: (a) the magnitude of the magnetoresistance is greater than is to be expected for p-Ge and (b) the field dependence is anomalous. We wish to discuss these points in some detail.

The longitudinal magnetoresistance for a simple energy surface arises from the variation of v_z around the hodograph. It is possible to show that a useful estimate of $\Delta\rho_L/\rho_L=(\rho_L-\rho_0)/\rho_L$ is given by

$$\Delta\rho_L/\rho_L\sim[\langle(v_z^2)_{Av}\rangle-\langle[(v_z)_{Av}]^2\rangle]/\langle[(v_z)_{Av}]^2\rangle. \quad (8.1)$$

In (8.1), v_z is the carrier velocity along the magnetic field and $(\ )_{Av}$ indicates average over the hodograph.

The estimate (8.1) suggests that the longitudinal magnetoresistance should be very small for a [100] direction in p-Ge. The reason for this is that in the case of the heavy holes, the carriers which carry most of the current in the z direction are those near the (nearly flat) [100] end surface of the (nearly cubical) energy surface, and for most of these carriers the variation around the hodograph of v_z is nearly zero. A few percent would seem to be about all of the magnetoresistance which could originate in this way.

For the case of the fast holes, which carry ~20% of the current, the longitudinal magnetoresistance should be even smaller than for the heavy hole, since the energy surfaces are nearly spherical. Thus we would expect the conductivity to be constant to within a few percent at all fields.

McClure's theory of the field dependence gives for the conductivity in a [100] direction the form

$$\sigma=\sigma_0+\sum_{M=1}^{\infty}\frac{C_M}{1+(4Mx)^2}. \quad (8.2)$$

Equation (8.2) shows that the field dependence of the magnetoconductivity must manifest itself at fields smaller than that for which $\omega\tau\sim1$. Thus on the basis of the classical Boltzmann theory we would expect that whatever change of conductivity occurs with increasing H, the conductivity would approach a limiting value as H approached the value for which $\omega\tau\sim1$.

Our experiments on p-Ge show a qualitatively different behavior than that described above. To be sure, there is a small longitudinal magnetoresistance at fields of a few thousand gauss just as we might expect. However, for fields of 5000 gauss or more the resistivity increases again and is not yet saturated in the presence of a field of 25 000 gauss.

We have been able to conceive of several sources of the high magnetic field anomaly, all consisting of effects not included in McClure's theory. Such quantum effects arise because in high magnetic fields quantization of the carrier orbits becomes important.

The simplest of these effects comes about because of the zero-point energy of the motion in the magnetic field. The zero-point energy is proportional to the reciprocal mass, and so is much larger for the light hole than for the heavy hole. Thus, as the field is increased, the degeneracy of the two bands is effectively removed and there is a transfer of carriers from the light-hole band to the heavy-hole band. The relative number of light holes will satisfy

$$(p_l/p_h)=(p_{l0}/p_{h0})(\omega_l/\omega_h)[\sinh(\hbar\omega_h/2kT)/\sinh(\hbar\omega_l/2kT)]. \quad (8.3)$$

Taking the cyclotron resonance values of 0.04 m_0 and 0.3 m_0 for the light- and heavy-hole masses, respectively, we can estimate the ω values, and hence the value of p_l/p_h as a function of field. Our estimate suggests that at our highest field of 25 000 gauss the number of fast holes is only about 80% of what it is at zero field. Taking account of the small fraction of the current carried by the fast holes, we estimate a longitudinal magnetoresistance of 3 or 4% originating in the transfer effect.

Another quantum effect is that discussed by Titeica[17] and more recently by Argyres and Adams.[18] This is a reduction in the carrier mobility because of the direct effect of the orbital quantization on the scattering of carriers. While no exact formula applicable to our experiment is available, it was possible to estimate that this kind of quantum effect could give a longitudinal magnetoresistance of as much as 5 or 6% at the highest fields we used.

Still a third quantum effect is a warping of the energy surface in the presence of the strong-magnetic field. While we have not tried to calculate the warping, we can see qualitatively that it should be strongest for a nearly degenerate band structure like that of p-Ge. It is not at all inconceivable that a significant effect on the mobility of the fast hole could follow from such a warping.

Under the circumstances we see no way to make a quantitative interpretation of the longitudinal magnetoresistance. However, we are inclined to think that these quantum effects are responsible for the large value of longitudinal magnetoresistance which we observe.

9. SUMMARY AND COMMENTS

Our experiments confirm in a qualitative way the features of the two-carrier model of p-Ge found by

[17] V. S. Titeica, Ann. Physik (5) **22**, 129 (1935).
[18] P. N. Argyres and E. N. Adams, Phys. Rev. **104**, 900 (1956).

others. The ratio of mobilities and the dependence of the Hall constant and resistivity on magnetic field, on temperature, and on impurity concentration all agree well with the predictions of a model assuming a large number of heavy carriers and a small number of light carriers conducting simultaneously. Further, the deviations from a simple two-carrier model are qualitatively what would be expected to result from the warping of the energy surfaces.

However, our experiments have not yielded an entirely satisfactory theoretical model of lattice-mobility-limited conduction in p-Ge. Some discrepancies which remain quantitatively unaccounted for are these:

(a) The low-field data give values for the mobilities of the two kinds of carriers which are incompatible with the observed field dependence of the magnetoconductivity.

(b) The anisotropy of magnetoresistance is appreciably different from that expected on the warped-surface theory of Mavroides and Lax.

(c) The variation of the magnetoconductivity functions with magnetic field in the region of intermediate field strength does not appear to conform very well with what would be expected from the warped surfaces inferred from cyclotron resonance experiments.

(d) The resistivity does not saturate at high fields as would be expected according to classical transport theory.

Assuming no major systematic error in our data, one would suspect from the first three discrepancies that either (1) the energy surface parameters at 77°K are significantly different from what they are at 4.2°K, or (2) the treatment of scattering which we have used is seriously wrong. The second possibility would mean that the scattering of carriers is quite anisotropic over an energy surface, in which case the correct formulas for the various magnetoconductivity functions would need to be generalized and might be considerably changed. Since in the case of large anisotropy no relaxation time would be expected to exist, it is very difficult to know what kind of generalization would be possible or to estimate the effects on the strong-field magnetoconductivity functions.

The nonsaturation of resistivity at large values of magnetic field is very probably due to a combination of the three quantum effects mentioned in the text. It is unfortunate that no saturation of resistivity occurs, since the saturation value of the resistivity would yield a transport integral which is very useful for determining the heavy-hole mobility and the energy dependence of the relaxation time.

The failure of the high-magnetic-field data to conform to a simple transport model prevents us from getting independent evidence on the mechanism of lattice scattering in p-Ge. As we stated in Sec. IV, the most plausible assumption is that optical mode scattering together with appreciable acoustical mode scattering is responsible for the "compromise" power law $T^{-2.3}$.

The principal findings of our work are the suggestions that the warping of the energy surface changes between 4.2°K and 77°K and the suggestion of quantum effect on the strong-field conductivity. Neither of these effects has been demonstrated unequivocally in the present work, however, and a more clear-cut demonstration would be desirable.

ACKNOWLEDGMENTS

The authors would like to acknowledge helpful conversations on the present subject with many people, in particular with A. C. Beer, H. Brooks, W. P. Dumke, H. Ehrenreich, C. Herring, E. O. Kane, W. Kohn, B. Lax, M. Lax, J. W. McClure, and H. Zeiger. We wish also to express our thanks to J. G. Mavroides and B. Lax and to J. A. Swanson for communicating their results prior to publication.