Flow of Helium II in Narrow Slits*

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The isothermal flow properties of helium II have been studied at pressure heads between 1 and 16 cm of helium and in slits with an average width between 2 and 5 microns. The results can be partially described in terms of the mutual friction theory of Gorter and Mellink. The prediction of the theory that the flow rate should be proportional to the one-third power of the pressure head was verified for the entire pressure range investigated. The agreement is not complete, since our work is consistent only with a mutual friction constant, A, which is a function of temperature and of the slit dimensions. There was no evidence of critical velocity effects in the velocity range investigated (10–30 cm/sec). Interpretation of the experimental work of Allen and Reekie and of Hollis-Hallet indicates that there is a small superfluid friction proportional to the square of the superfluid velocity in addition to the mutual friction. This friction was not observed in the velocity range of our experiments.

INTRODUCTION

In a previous paper,[1] we reported some results of experiments on the flow properties of helium II. Since then, the flow measurements have been refined and extended. The object of these experiments has been the study of the dissipative processes that take place in pure superfluid flow. The flow measurements were made isothermally in slits of a few microns width. These widths are small enough to nearly remove the contribution to the flow by the normal fluid, yet large enough to give easily measurable rates for the flow of superfluid.

Previous experiments on the flow in wide capillaries,[2] the nonstationary fountain effect,[3] and the damping of an oscillating disk,[4] have shown that there are dissipative forces acting in liquid helium other than the viscous forces of the normal fluid. The "mutual friction" theory of Gorter and Mellink[5] describes these forces with some success, but Atkins has shown in a review article[6] that this theory does not describe all experiments adequately. We have studied the isothermal flow properties of the superfluid in an attempt to test the predictions of the Gorter-Mellink theory and to see if the forces acting could be described by mutual friction forces, or if some other forces were necessary. Measurements have been made for several slit widths between 2 and 5 microns and at pressure heads corresponding to 16 cm of liquid helium.

EXPERIMENTAL METHOD

The apparatus and methods used in the study of the flow properties were described previously.[1] The flow channel was formed by pressing a ground glass plate against a ground flange at the bottom of a glass reservoir. Optical examination revealed that the slits formed in this way were not uniform. The deviations from uniformity were sometimes as much as 40%. The average slit widths were determined from the flow rate of helium I and its known viscosity of 30 micropoise[7] and should have been accurate to within approximately 5%. All slit widths quoted are averages obtained in this way.

Flow rates were determined from plots of reservoir height h *vs* t (see Fig. 1). The time intervals for given increments of the reservoir height were recorded by energizing with a hand switch the movement of an Esterline-Angus recording milliammeter while observing the reservoir level visually. A roll speed of 3 inches/minute allowed the measurement of time intervals to within 1/5 of a second.

Flow measurements were made for flow into the reservoir and flow out of the reservoir. The results for inflow and outflow were identical in almost all cases.

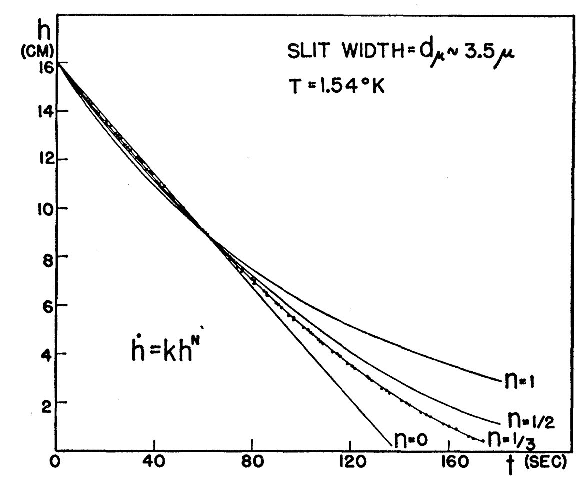

FIG. 1. Primary experimental observations of reservoir level h *vs* time t. The points are from several separate measurements. The solid curves give the dependence of h on t for various cases: $n=1$, Poiseuille flow; $n=\frac{1}{2}$, turbulent or Torricelli flow; $n=0$, flow at constant velocity.

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1 R. T. Swim and H. E. Rorschach, Phys. Rev. **97**, 25 (1955).
2 K. R. Atkins, Proc. Phys. Soc. (London) **A64**, 833 (1951).
3 Hung, Hunt, and Winkel, Physica **18**, 629 (1952).
4 A. C. Hollis-Hallet, Proc. Roy. Soc. (London) **A210**, 404 (1952).
5 C. J. Gorter and J. H. Mellink, Physica **15**, 285 (1949).
6 K. R. Atkins, Advances in Phys. **1**, 169 (1952).
7 R. Bowers and K. Mendelssohn, Proc. Roy. Soc. (London) **A204**, 366 (1950).

The vapor pressure was controlled by an automatic regulator with a precision of 0.1 mm Hg. Temperatures were determined from the 1949 Agreed Temperature Scale.

EXPERIMENTAL RESULTS

The volume flow rate as a function of pressure head could be determined from the experimental data in two ways. One method is illustrated in Fig. 1. The expression $\dot{h}\propto(\Delta P)^n$ has been used to fit the primary data over the whole range of pressure heads. The experimental points fit the expression with $n=1/3$ quite well. Another method involves the direct determination of the velocity from the graphical time record and the expression $\dot{h}=(\Delta h/\Delta t)_{\text{average}}$. These velocities, $\dot{h}$, are shown in Fig. 2. The solid line is the velocity calculated from the curve which was fitted to the points of Fig. 1 over the entire pressure range. Although there is considerable scatter at the highest pressures, the time-measuring method gives consistent results. In the previously reported work,[1] the behavior of the flow rate at high pressure heads was uncertain. The results shown in Fig. 2 seem to show that no saturation effects occur at high pressure heads, and that $\dot{h}=C(\Delta P)^{\frac{1}{3}}$ is a satisfactory description of the flow at all pressures investigated. This interpretation is consistent with other recently reported flow measurements at high pressure heads.[8] Attempts to extend our measurements to higher pressures by a direct application of helium gas to the reservoir have been unsuccessful. The condensation of the gas in the reservoir always produced a thermomechanical effect and a net inflow. The flow rates obtained with this thermomechanical effect were always consistent with the gravitational rates under the corresponding pressure heads.

The temperature dependence of the flow rate is shown in Fig. 3. The solid curve is ρ_s/ρ as measured by Andronikashvili.[9] The points represent $\dot{h}$ determined at a pressure head of 13.0 cm He, and normalized to ρ_s/ρ at 1.81°K. The dashed curve shows the temperature dependence expected from the Gorter-Mellink mutual friction theory. The results of three separate runs show that the temperature dependence of the flow is accurately given by ρ_s/ρ.

The rate of change of the reservoir level as a function of average slit width is shown in Fig. 4. The rates have been determined at a pressure head of 13.0 cm He and normalized to 0°K by the factor ρ_s/ρ. The filled circles represent the previous results.[1] The solid curve is $\dot{h}=0.0167\ (d_\mu)^{5/3}$ where d_μ is the average slit width in microns. It seems likely that there was an error of approximately 30% in the 4.3-micron average slit width determination previously reported. The slit appears to have been approximately 3.3 microns in width. This does not change the general character of the previous results but it does change some of the detailed comparison with theory.

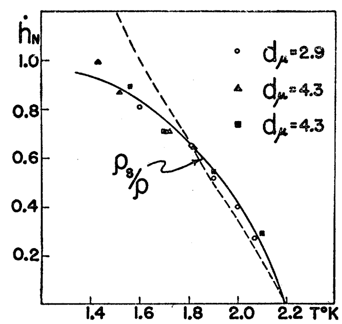

FIG. 3. Rate of change of reservoir level *vs* temperature. The rates were determined at a pressure head of 13.0 cm helium and have been scaled to the ρ_s/ρ curve at 1.81°K. The dashed curve is 2.45 $\rho_s\rho_n^{-\frac{1}{3}}$.

The side-tube behavior previously reported[1] was again observed. However, in view of the nonuniformity of the slits, we have not tried to include this point in our description of the results.

All of our flow results can be rather closely represented by:

$$\dot{V}=Bd^m(\rho_s/\rho)(\Delta P)^{\frac{1}{3}},$$

where $\dot{V}$ is the volume rate of flow in cm³/sec, d is the slit width in cm, $m=5/3\pm1/3$, and $B\simeq12\times10^2$ cgs units for $m=5/3$.

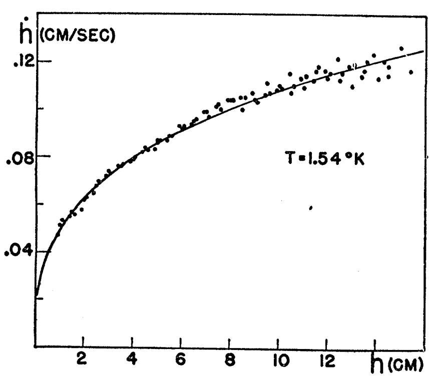

FIG. 2. Rate of change of the reservoir level *vs* the reservoir level. The points were obtained directly from the primary date (see text). The curve represents the rate of change calculated from the equation fitted to Fig. 1 with $n=\frac{1}{3}$.

DISCUSSION OF RESULTS

The thermohydrodynamic equations of motion for liquid helium may be written in the following form[6]:

$$\rho_s\frac{d\mathbf{v}_s}{dt}=-(\rho_s/\rho)\nabla P+\rho_sS\nabla T-\mathbf{F}_{sn}-\mathbf{F}_s,$$

$$\rho_n\frac{d\mathbf{v}_n}{dt}=-(\rho_n/\rho)\nabla P-\rho_sS\nabla T+\eta_n\nabla^2\mathbf{v}_n+\tfrac{1}{3}\nabla(\nabla\cdot\mathbf{v}_n)+\mathbf{F}_{sn}-\mathbf{F}_n,$$

where $\mathbf{F}_n$, $\mathbf{F}_s$, and $\mathbf{F}_{sn}$ represent frictional forces. For isothermal flow in narrow slits, the second equation will

[8] Wansink, Taconis, Staas, and Reuss, Physica **21**, 596 (1955).
[9] E. L. Andronikashvili, J. Exptl. Theoret. Phys. (U.S.S.R.) **18**, 424 (1948).

be satisfied for $\mathbf{v}_n \rightarrow 0$, and we therefore neglect the flow of the normal fluid. The first equation can be solved, since the acceleration and Bernoulli terms may be neglected. The solution for the pressure gradient is:

$$\nabla P = -(\rho/\rho_s)(\mathbf{F}_{sn}+\mathbf{F}_s);$$

and if we assume that $\mathbf{F}_{sn}$ is given by the Gorter-Mellink expression,[5] $\mathbf{F}_{sn}=A\rho_s\rho_n(\mathbf{v}_s-\mathbf{v}_n)^3$, and that $\mathbf{F}_s=0$, then we obtain

$$\nabla P = -A\rho_n\rho\mathbf{v}_s^3.$$

Integrating over the slit length, we get the total pressure drop:

$$\Delta P = 0.026 A\rho_n\mathbf{v}_s^3(R_i),$$

or

$$v_s(R_i)=3.38(A\rho_n)^{-\frac{1}{3}}(\Delta P)^{\frac{1}{3}},$$

where R_i is the inner radius of the annular slit and $v_s(R_i)$ is the velocity of the superfluid at R_i. The average flow velocity is then given by:

$$\bar{v}=(\rho_s\bar{v}_s)/\rho,$$

where the bar represents an average over the cross section of the slit. To relate the average velocity to the experimental results, we use:

$$\dot{V}=2\pi R_i d\bar{v}(R_i)=2\pi R_i d(\rho_s/\rho)3.38(A\rho_n)^{-\frac{1}{3}}(\Delta P)^{\frac{1}{3}},$$

where $\bar{v}(R_i)$ is the average fluid velocity at R_i and d is the average slit width.

The calculation can also be carried the other way. We can determine the form of the "mutual friction" which would yield the experimental results. This calculation leads to a mutual friction term:

$$\mathbf{F}_{sn}=(C/d^k)(\rho_s/\rho)\bar{\mathbf{v}}_s^3,$$

with $K=2\pm1$ and $C\simeq 49\times10^{-9}$ cgs units for $K=2$. If we compare the experimental results with the predictions of the mutual friction theory, we note the following:

1. The dependence on the pressure head agrees with the prediction of the mutual-friction theory. There is no particular evidence of saturation at higher pressures. Some recent work has indicated that there is a critical velocity below which the mutual friction vanishes.[10] No critical velocity effects were observed in our work. However, the critical velocities expected are of the order of 10 cm/sec. Our minimum velocities, $\mathbf{v}_s$, at pressure heads of 1 cm He were approximately 10 cm/sec. In addition, the nonuniformity of the slit would tend to obscure any critical velocity effects.

TABLE I. Values of $A\rho_n$ computed from $A\rho_n=C/\rho d^2$.

Diameter d (microns)	$A\rho_n$ (cgs units)	T(°K)	Reference
81	1.5	1.22–1.52	2
5	2.0	1.72	3
4	1.8	1.72	11
1.5	3.0	1.72	10
4.3	1.8	1.4–2.1	Present
3.8	2.3	1.4–2.1	Present
2.9	4.0	1.4–2.1	Present
2.4	5.8	1.4–2.1	Present

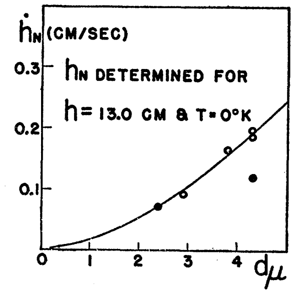

FIG. 4. The rate of change of reservoir level *vs* the average slit width. The rates were determined at a pressure head of 13.0 cm helium and scaled to 0°K. The two filled circles at 2.4 microns and 4.3 microns represent previously obtained results.[1]

2. The observed temperature dependence does not agree with the theory. The Gorter-Mellink theory predicts that the temperature dependence of the volume rate of flow should be given by $\rho_s\rho_n^{-\frac{1}{3}}$. This function, normalized to the experimental results, is plotted as the dashed line in Fig. 3. Other isothermal experiments[2,4] have given results consistent with a mutual friction whose temperature dependence is given by ρ_s/ρ. Atkins' results in wide capillaries yield values of $A\rho_n$ which are more nearly constant than the values of A. The results of Hollis-Hallet with a pile of disks can be fitted as to the temperature dependence by a mutual friction proportional to ρ_s/ρ.

3. Values of $A\rho_n$ may be calculated from our measurements for comparison with other work. If we set $A\rho_n=C/\rho d^2$ we obtain the results shown in Table I. The agreement is satisfactory as regards order of magnitude.

4. The results of Fig. 4 indicate that the mutual friction term, $\mathbf{F}_{sn}$, depends on the slit dimensions. This possibility was not anticipated in the original formulation, since F_{sn} was considered a body force. Our work indicates that the mutual friction is proportional to d^{-2}. However, the exponent is not accurately determined, owing to the small number of experimental points in Fig. 4 and the fact that the flow rate depends on the cube root of the friction force. Other recent results[3,11] have also shown that $\mathbf{F}_{sn}$ is a function of the slit dimensions. These workers report that the dependence on the slit width is given as $\mathbf{F}_{sn}\propto d^{-\frac{1}{2}}$. The difference between the two exponents may possibly be due to the nonuniformity of our slits. It should be noted, however, that the results of Atkins do not fit easily into this picture unless the dependence on d becomes less as d increases. In any case, the dependence on d indicates that the form given the mutual friction term is not correct. $\mathbf{F}_{sn}$ ought to involve differential

[10] Winkel, Delsing, and Poll, Physica **21**, 331 (1955); Winkel, Broese van Groenou, and Gorter, Physica **21**, 345 (1955).
[11] Winkel, Delsing, and Gorter, Physica **21**, 312 (1955).

operators, and the slit dimensions would then be introduced through the boundary conditions.

Some investigations have indicated that forces act on the superfluid in addition to the mutual friction. The rotating viscosimeter experiments of Hollis-Hallet[12] show that mutual friction is inadequate, although this conclusion may be modified if the mutual friction turns out to be dependent on boundary conditions. The results of Allen and Reekie[13] on the fountain pressure in wide capillaries also indicate that $\mathbf{F}_s \neq 0$ is necessary if we wish to use the two-fluid hydrodynamic equations to describe the flow of helium II. If we suppose that $\mathbf{F}_n = 0$ and add the two equations of motion for the normal and superfluid, we obtain, in the steady state,

$$\nabla P = \eta_n \nabla^2 \mathbf{v}_n - \mathbf{F}_s.$$

If we assume that the normal fluid velocity profile is not changed, then

$$\mathbf{v}_n = (6/d^2)[(d^2/4) - x^2]\bar{\mathbf{v}}_n.$$

If in addition we take

$$\mathbf{F}_s = D|\bar{\mathbf{v}}_s|\bar{\mathbf{v}}_s,$$

then

$$\nabla P = -(12\eta_n \bar{\mathbf{v}}_n / d^2) - D|\bar{\mathbf{v}}_s|\bar{\mathbf{v}}_s.$$

For steady-state heat conduction, $\rho_s \bar{\mathbf{v}}_s + \rho_n \bar{\mathbf{v}}_n = 0$ and therefore

$$\nabla P = -(12\eta_n/d^2)\bar{\mathbf{v}}_n + D(\rho_n/\rho_s)^2|\bar{\mathbf{v}}_n|\bar{\mathbf{v}}_n.$$

The heat current is given by

$$\bar{\mathbf{W}} = \rho S T \bar{\mathbf{v}}_n.$$

Thus for small heat currents, the heat current is proportional to the fountain pressure, and we obtain the rule of Allen and Reekie.[14] For large heat currents, the fountain pressure must change sign. The maximum in the fountain pressure occurs at a value of $\bar{\mathbf{W}}$ which is 1/2 the value at which the fountain pressure is zero. This prediction, which depends on the fact that $\mathbf{F}_s \propto \bar{\mathbf{v}}_s^2$, is roughly confirmed by the results of Allen and Reekie.[13,†]

The above considerations suggest that the hydrodynamic equations must be modified by adding a term $F_s = D|\bar{\mathbf{v}}_s|\bar{\mathbf{v}}_s$ in addition to the mutual friction. Our isothermal flow results, however, do not indicate a friction term proportional to $\bar{\mathbf{v}}_s^2$ of any magnitude. Of course, this term is likely to be small, as is indicated by the success of the mutual friction theory in describing the flow in small channels. It is also possible that in sufficiently small slits $\mathbf{F}_s$ depends on a higher power of $\mathbf{v}_s$. In any event, the results of the present experiments do not preclude the existence of a small superfluid friction proportional to $\bar{\mathbf{v}}_s^2$ which appears at low velocities and in wide channels.

CONCLUSIONS

The results of the present work show that the isothermal flow of helium II in narrow slits can be described roughly by the mutual friction theory. There are some discrepancies, especially as regards the temperature dependence of the flow and the dependence on the slit dimensions. The experimental results are consistent with a "mutual friction" having the same temperature dependence as ρ_s/ρ. Such a temperature dependence is in agreement with results reported by Atkins and by Hollis-Hallet. A consideration of the results of experiments by Allen and Reekie and by Hollis-Hallet indicates that a small superfluid friction $\mathbf{F}_s \propto \bar{\mathbf{v}}_s^2$ must be added to the hydrodynamic equations in addition to the mutual friction, $\mathbf{F}_{sn}$. It is likely that both $\mathbf{F}_s$ and $\mathbf{F}_{sn}$ involve differential operators and that $\mathbf{v}_s$ and $\mathbf{v}_s - \mathbf{v}_n$ are subject to some boundary conditions. A detailed comparison of the results from different flow geometries would be difficult unless these conditions were known. This friction term, $\mathbf{F}_s$, did not appear at the high velocities and in the narrow slits that we used for our observations on isothermal flow.

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[12] A. C. Hollis-Hallet, Proc. Cambridge Phil. Soc. **49**, 717 (1953).
[13] J. F. Allen and J. Reekie, Nature **144**, 475 (1939).
[14] J. F. Allen and J. Reekie, Proc. Cambridge Phil. Soc. **35**, 114 (1939).

† We have attempted to repeat the experiments of reference 13. Our results differ from those of reference 13 and indicate that the above interpretation is probably not applicable, since the conduction of heat occurs along two paths. However, the above analysis ought to apply to an experiment in which the heat is conducted along a single path.