# Derivation of Dispersion Relations for Forward Scattering*†

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The dispersion relation for forward meson-nucleon scattering is derived in the simplified case of scalar neutral particles. Use is made of the local property of the nucleon field and of certain features of the mass spectrum. In addition, it is assumed that the only singularities of certain matrix elements of the nucleon field commutator are derivatives of finite order of $\delta$ functions on the light cone. Under some further assumptions of existence, the dispersion relations for the derivatives of the scattering amplitude with respect to angle at zero angle can also be derived.

THE details of the dispersion relations for forward scattering of mesons by nucleons have been given by Goldberger[1] and Goldberger, Miyazawa, and Oehme.[2] Since these relations seem to be experimentally confirmed,[3] the question of whether they are rigorous consequences of the local meson field theory has been of considerable theoretical interest. The derivations of these relations given so far[1,4] are known to be inconclusive.

In this paper we answer this question in the affirmative and shall outline, under well-specified assumptions, the rigorous proof of these relations.[5]

To this end it will be sufficient if, for the sake of lucidity, we confine ourselves to the simple model of the coupled fields of two scalar neutral particles, a heavy one of mass $M$ and a light one of mass $\mu$. For brevity we shall refer to these particles as nucleon and meson, respectively. We shall observe the selection rule that the matrix elements of the nucleon field operator $\psi(x)$ vanish if taken between two states each having even or odd numbers of nucleons. (This is related to but not implied by the fact that nucleons can be produced only in pairs. A possible realization of this selection rule would be by the introduction of a nucleon spin.) Bound states, e.g., the deuteron, may exist. The only properties of the mass spectrum we shall actually use are that $\mu<2M$ and that any state of odd nucleon number, if not the one-nucleon state, has a rest mass greater than or, as in the case of the one-nucleon-one-meson system, at least equal to $M+\mu$.

We are using the 1, $-1$, $-1$, $-1$ metric; $T$ is the time-ordering symbol; $\overleftrightarrow{\partial}_{x0}$ is defined by

$$f(x)\overleftrightarrow{\partial}_{x0}g(x)\equiv f(x)\frac{\partial}{\partial x_0}g(x)-g(x)\frac{\partial}{\partial x_0}f(x),$$

and $O(x)$ by

$$O(x)\equiv\left(-\frac{\partial^2}{\partial x_0{}^2}+\Delta-M^2\right)\psi(x).$$

$|\ \rangle$ is the true vacuum state. $|k\rangle$, $|p\rangle$ are states of one meson or one nucleon, respectively, with the indicated four-momenta. $|pk\rangle$ is a state of an ingoing nucleon and an ingoing meson, etc. We adopt the usual convention that all one-particle states are defined to be invariant under space-time inversion. The $x$ dependence of $\psi(x)$ is given by

$$\psi(x)=\exp(i\mathcal{P}x)\psi(0)\exp(-i\mathcal{P}x), \tag{1}$$

where $\mathcal{P}$ is the total four-momentum operator, and similarly that of $O(x)$. We impose the causality condition that the nucleon field be local, i.e., that

$$[\psi(x),\psi(x')]=0 \quad \text{if } (x-x')^2<0.$$

The $S$-matrix element for scattering of a nucleon of momentum $p$ and a meson of momentum $k$ into a nucleon of momentum $p'$ and a meson of momentum $k'$ is given by[6,7]

$$\begin{aligned}\langle p'k'|S|pk\rangle &= -i\int e^{ip'x}\overleftrightarrow{\partial}_{x0}\langle k'|\psi(x)|pk\rangle d\mathbf{x}\\ &= -i\int_{x_0\to-\infty} e^{ip'x}\overleftrightarrow{\partial}_{x0}\langle k'|\psi(x)|pk\rangle d\mathbf{x}\\ &\quad +i\int e^{ip'x}\langle k'|O(x)|pk\rangle dx\\ &= \langle p'k'|pk\rangle+i(2\pi)^4\delta(p'+k'-p-k)\\ &\qquad\times\langle k'|O(0)|pk\rangle. \end{aligned}\tag{2}$$

Since we assume invariance of the theory under space-

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[1] M. L. Goldberger, Phys. Rev. **99**, 979 (1955).
[2] Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955).
[3] Anderson, Davidon, Kruse, Phys. Rev. **100**, 339 (1955); G. F. Chew, *Encyclopedia of Physics* (Springer-Verlag, Berlin, to be published).
[4] R. Oehme, Phys. Rev. **100**, 1503 (1955); **102**, 1174 (1956).
[5] After this work had been completed, the author learned at the International Conference on Theoretical Physics in Seattle that N. N. Bogoliubov has proved the dispersion relations even for the case of finite scattering angle. The dispersion relations for forward scattering have also been proven by R. Jost and H. Lehmann, using a different method.
[6] E.g., Lehmann, Symanzik, Zimmermann, Nuovo cimento **1**, 205 (1955).
[7] F. E. Low, Phys. Rev. **97**, 1392 (1955).

time inversion, the scattering amplitude

$$T_{p'k',\,pk}\equiv\langle k'|O(0)|pk\rangle$$

is symmetric. (Here we may subtract the vanishing[6] quantity $\langle k'|k\rangle\langle|O(0)|p\rangle$ at will and carry through the following manipulations with its inclusion, without hereby affecting the final result.) We furthermore have

$$\begin{aligned}T_{p'k',\,pk}&=i\int_{x_0\to-\infty} e^{-ipx}\overleftrightarrow{\partial}_{x_0}\langle k'|O(0)\psi(x)|k\rangle d\mathbf{x}\\&=i\int_{x_0\to-\infty} e^{-ipx}\overleftrightarrow{\partial}_{x_0}\langle k'|T\psi(x)O(0)|k\rangle d\mathbf{x}\\&=i\int_{x_0\to+\infty} e^{-ipx}\overleftrightarrow{\partial}_{x_0}\langle k'|\psi(x)O(0)|k\rangle d\mathbf{x}\\&\quad+i\int e^{-ipx}\left(-\frac{\partial^2}{\partial x_0{}^2}+\Delta-M^2\right)\\&\qquad\times\langle k'|T\psi(x)O(0)|k\rangle dx\\&=i\int e^{-ipx}\langle k'|TO(x)O(0)|k\rangle dx\\&\quad-i\int_{x_0=0} e^{-ipx}\overleftrightarrow{\partial}_{x_0}\langle k'|[\psi(x),O(0)]|k\rangle d\mathbf{x}.\quad(3)\end{aligned}$$

Contrary to the procedure of Low[7] and Goldberger,[1] we have fixed the meson and set the nucleon free. The reason for this will be apparent later.

Formulas (2) and (3) may be expected to hold even if there are bound states. In order to show this, we remember that actually the validity of

$$|pk\rangle=\lim_{T\to\infty}T^{-1}\int_{-2T}^{-T}dx_0 i\int e^{-ipx}\overleftrightarrow{\partial}_{x_0}\psi(x)|k\rangle d\mathbf{x}\quad(4)$$

is sufficient for that derivation. By use of (1), the right-hand side of (4) is easily shown to be an eigenvector of $\mathcal{P}$ with eigenvalue $p+k$, and adopting the weak-coupling treatment of the bound state problem by Nishijima[8] one may show that the right-hand side of (4) actually represents the properly normalized state of two ingoing particles as expressed by the left-hand side.

The last integral in (3) is real. Because of the commutator condition, the matrix element and its derivative must be linear combinations of spatial $\delta$ functions and derivatives thereof of finite order. (The occurrence of derivatives of finite order only might be taken as substantial supplement to the definition of a local field.) The integral then becomes a polynomial in the components of $p$, and, by reasons of invariance, may be written as a polynomial in $p(k-k')$ and $p(k+k')$ with real $kk'$-dependent coefficients. Since $p(k-k')=kk'-\mu^2$, that variable can be omitted. The transformation $p\to-p$, $k\leftrightarrow k'$ shows that $p(k+k')$ can only appear squared. The integral may therefore be written $\mathcal{R}[(p(k+k'))^2;kk']$. In the decomposition

$$TO(x)O(0)=\tfrac{1}{2}(-1+\mathrm{sign}x_0)[O(x),O(0)]+O(x)O(0),\quad(5)$$

the last term of (5) does not contribute in (3) since intermediate states of odd nucleon number and momentum $k'-p$ do not exist. Specializing now to forward scattering, we have

$$\begin{aligned}T_{pk,\,pk}-\mathcal{R}(4(pk)^2;\mu^2)&\equiv\tilde{T}_{pk/\mu}\\&=i\int e^{ipx}\langle k|[O(x),O(0)]|k\rangle 2^{-1}(1+\mathrm{sign}x_0)dx.\quad(6)\end{aligned}$$

We write the invariant quantity $\tilde{T}_{pk/\mu}$ in the rest system of the meson. $\omega\equiv pk/\mu$ is the nucleon energy in this system. We have, replacing $x$ by $-x$,

$$\tilde{T}_\omega=\int_0^\infty F(\omega,r)dr,\quad(7)$$

with

$$\begin{aligned}F(\omega,r)&\equiv4\pi ir(\omega^2-M^2)^{-\frac{1}{2}}\sin[(\omega^2-M^2)^{\frac{1}{2}}r]\\&\quad\times\int_0^\infty e^{i\omega t}\langle\mu,0|[O(t,\mathbf{x}),O(0,0)]|\mu,0\rangle dt.\quad(8)\end{aligned}$$

Although $\tilde{T}_\omega$, as it stands in (7), is meaningful only for $\omega^2>M^2$ (because of the exponential factor if $\omega^2<M^2$), $F(\omega,r)$ is definable by (8) for all values on the real axis and in the upper half $\omega$ plane. Since the integral actually extends only from $r$ to infinity, the factor $\exp(i\omega r)$ could have been taken out and the exponential increase of the integral function $(\omega^2-M^2)^{-\frac{1}{2}}\times\sin[(\omega^2-M^2)^{\frac{1}{2}}r]$ in the upper half-plane is just compensated. We may furthermore assume that the matrix element of the commutator is continuous inside the light cone so that, as a consequence of the Riemann-Lebesgue Lemma,[9] the asymptotic behavior of $F(\omega,r)$ for $\omega\to\infty$ is governed by the singularity of the commutator on the light cone.[10] If one inserts in (8) a derivative of $\delta(t^2-r^2)$ of order $n\geqslant0$, one finds an asymptotic behavior like $\omega^{n-1}$. Since, as before, an infinite $n$ would not correspond to a local theory, the multiplication of $F(\omega,r)$ by a finite number of factors

$$[(\omega+ib_\nu)^2-a_\nu{}^2]^{-1},\quad b_\nu>0,$$

will suffice to secure the asymptotic vanishing of the

[8] K. Nishijima, Progr. Theoret. Phys. (Japan) **10**, 549 (1953); **12**, 279 (1954); **13**, 305 (1955).

[9] E. C. Titchmarsh, *Theory of Fourier Integrals* (Oxford University Press, Oxford, 1937), p. 11.

[10] E.g., H. Lehmann, Nuovo cimento **11**, 342 (1954).

product so that the Hilbert relation[11] holds:

$$\mathrm{Re}\,\frac{F(\omega_0,r)}{\prod_\nu[(\omega_0+ib_\nu)^2-a_\nu^2]}=\frac{1}{\pi}P\int_{-\infty}^{+\infty}\frac{d\omega}{\omega-\omega_0}\,\mathrm{Im}\,\frac{F(\omega,r)}{\prod_\nu[(\omega+ib_\nu)^2-a_\nu^2]},$$

or, with $b_\nu\to+0$, $a_\nu>0$, (9)

$$\mathrm{Re}\,F(\omega_0,r)=\sum_\nu \mathrm{Re}\,F(a_\nu,r)\prod_{\lambda\neq\nu}\frac{\omega_0^2-a_\lambda^2}{a_\nu^2-a_\lambda^2}+\frac{2}{\pi}\prod_\nu(\omega_0^2-a_\nu^2)P\int_0^\infty\frac{\omega d\omega}{\omega^2-\omega_0^2}\frac{\mathrm{Im}\,F(\omega,r)}{\prod_\nu(\omega^2-a_\nu^2)}.$$

Here we have used the easily established fact that Re $F(\omega,r)$ and Im $F(\omega,r)$ are even or odd functions of the real variable $\omega$, respectively.

We now perform on (9) the $r$ integration required by (7), choosing $\omega_0>M$. This integration certainly commutes with the $\omega$ integration in the region $\omega\geqslant M$. For the sake of simplicity, we choose all $a_\nu>M$ and are left with the integral from $O$ to $M$:

$$\mathrm{Re}\,\tilde{T}_{\omega_0}-\sum_\nu \mathrm{Re}\,\tilde{T}_{a_\nu}\prod_{\lambda\neq\nu}\frac{\omega_0^2-a_\lambda^2}{a_\nu^2-a_\lambda^2}-\frac{2}{\pi}\prod_\nu(\omega_0^2-a_\nu^2)P\int_M^\infty\frac{\omega d\omega}{\omega^2-\omega_0^2}\frac{\mathrm{Im}\,\tilde{T}_\omega}{\prod_\nu(\omega^2-a_\nu^2)}$$
$$=4\prod_\nu(\omega_0^2-a_\nu^2)\int_0^\infty rdr\int_0^M\frac{\omega d\omega}{\omega^2-\omega_0^2}\prod_\nu(\omega^2-a_\nu^2)^{-1}\frac{\sinh[(M^2-\omega^2)^{\frac{1}{2}}r]}{(M^2-\omega^2)^{\frac{1}{2}}}\int_{-\infty}^{+\infty}e^{i\omega t}\langle\mu,0|[O(t,\mathbf{r}),O(0,0)]|\mu,0\rangle dt. \quad (10)$$

In order to obtain the dispersion relation we are looking for, we have to evaluate the right-hand side of (10).[12] This we shall accomplish by shifting the $\omega$ path in such a fashion that we can perform the $r$ integration under the $\omega$ integral. This, of course, requires some knowledge of the commutator since an analytic continuation of the $\omega$ integrand has to be carried out. Splitting the matrix element and inserting a complete set of states which we label by $\zeta$, with energy $k_{0\zeta}$ and momentum $\mathbf{k}_\zeta$, we get for the matrix element in (10):

$$\sum_\zeta\{\exp[-i(k_{0\zeta}-\mu)t]-\exp[i(k_{0\zeta}-\mu)t]\}\times\exp(i\mathbf{k}_\zeta\cdot\mathbf{r})|\langle k_{0\zeta},\mathbf{k}_\zeta|O(0)|\mu,0\rangle|^2.$$

Carrying out the $t$ integration and remembering the selection rule, we obtain from the first exponential

$$2\pi\sum_{M_\zeta=M}\frac{\sin\{[(\omega+\mu)^2-M^2r]^{\frac{1}{2}}\}}{[(\omega+\mu)^2-M^2r]^{\frac{1}{2}}}\delta(\omega+\mu-k_{0\zeta})|\langle k_{0\zeta},\mathbf{k}_\zeta|O(0)|\mu,0\rangle|^2, \quad (11)$$

where $\mathbf{k}_\zeta^2=(\omega+\mu)^2-M^2$ and only the one-nucleon intermediate states contribute. We shall show later that the nucleon-meson matrix element

$$\mathfrak{M}(\omega)[\equiv\langle\omega,\mathbf{k}|O(0)|\mu,0\rangle \quad (12)$$

is an analytic function in the cut $\omega$ plane; the cut goes from $-\infty$ to $-M$; $\mathfrak{M}(\omega)$ is real on the real axis from $-M$ to $+\infty$ and increases at infinity at most like a finite power of $\omega$. $\mathfrak{M}(\omega)^2$, which we can insert instead of $|\mathfrak{M}(\omega)|^2$ because of the reality of $\mathfrak{M}(\omega)$ in the required range $M\leqslant\omega<M+\mu$, then has the same properties. With the normalization factor adapted to our choice of the wave functions $(2\pi)^{-3}\delta(k_\zeta^2-M^2)dk_\zeta$, i.e.,

$$(2\pi)^{-2}|\mathbf{k}_\zeta|\delta[|\mathbf{k}_\zeta|-(k_{0\zeta}^2-M^2)^{\frac{1}{2}}]d|\mathbf{k}_{0\zeta}|dk_{0\zeta},$$

and from (11) and (12) we obtain for the right-hand side of (10):

$$\frac{2}{\pi}\prod_\nu(\omega_0^2-a_\nu^2)\int_0^\infty dr\int_{M-\mu}^M\frac{\omega d\omega}{\omega^2-\omega_0^2}\frac{[\mathfrak{M}(\omega+\mu)]^2}{\prod_\nu(\omega^2-a_\nu^2)}\frac{\sinh[(M^2-\omega^2)^{\frac{1}{2}}r]}{(M^2-\omega^2)^{\frac{1}{2}}}\sin\{[(\omega+\mu)^2-M^2]^{\frac{1}{2}}r\}.$$

The exponentially decreasing part of the hyperbolic sine can be integrated at once to give

$$-\frac{1}{\pi}\prod_\nu(\omega_0^2-a_\nu^2)\int_{M-\mu}^M\frac{\omega d\omega}{\omega^2-\omega_0^2}\frac{[\mathfrak{M}(\omega+\mu)]^2}{\prod_\nu(\omega^2-a_\nu^2)}\frac{[(\omega+\mu)^2-M^2]^{\frac{1}{2}}}{2\omega\mu+\mu^2}\frac{1}{(M^2-\omega^2)^{\frac{1}{2}}}. \quad (13)$$

[11] See Chap. V of reference 9.

[12] Equation (10) and the method we have used to derive it are due to R. Oehme, Nuovo cimento (to be published). The author is indebted to Dr. Oehme for communicating these results to him prior to publication.

For the remaining integral, we write

$$\frac{1}{\pi}\prod_\nu(\omega_0^2-a_\nu^2)\int_0^\infty dr\,\mathrm{Re}\int_{M-\mu}^{M}\frac{\omega d\omega}{\omega^2-\omega_0^2}\frac{\exp\{i(\omega^2-M^2)^{\frac{1}{2}}r-i[(\omega+\mu)^2-M^2]^{\frac{1}{2}}r\}}{(\omega^2-M^2)^{\frac{1}{2}}\prod_\nu(\omega^2-a_\nu^2)}[\mathfrak{M}(\omega+\mu)]^2. \tag{14}$$

Here we have chosen the cuts in such a way (Fig. 1) that the square roots behave at infinity like $\omega$ and $\omega+\mu$, respectively. The original integration path $AB$ is indicated in the figure. We deform this path into the path $ACDEB$, where $C$ lies arbitrarily between $\omega=-\mu/2$ and $\omega=-M$ on the real axis and the quadrant $DE$ is removed to infinity. $CD$ and $EB$ lie at $\mathrm{Re}\,\omega<-\mu/2$

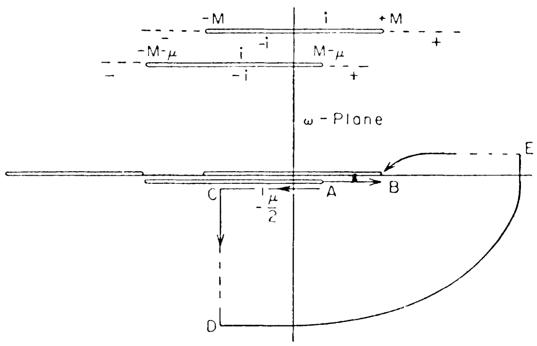


FIG. 1. Integration path for (14).

or $\mathrm{Im}\omega>0$, respectively. The residues at $\omega_0$ and $a_\nu$ have to be written separately. We now observe that along the segment $AC$ of the real axis the integrand in (14) is imaginary so that this part can be omitted. This is the decisive point of our method. (Note that there are no poles on $AC$.) Along $CDEB$ we can interchange the $r$ and the $\omega$ integration. In order to show this, we consider the $r$ integral as the limit as $R\rightarrow\infty$ of the integral with the finite upper limit $R$. On $CD$ and $EB$ the integrand is an exponentially decreasing function of $r$. On $DE$ the integrand is exponentially increasing. But there the $r$ integration gives the factor

$$\exp\{-i[(\omega+\mu)^2-M^2]^{\frac{1}{2}}R+i[\omega^2-M^2]^{\frac{1}{2}}R\}-1$$
$$=\exp[-i\mu R-\tfrac{1}{2}\mu M^2R\omega^{-2}-\cdots]-1,$$

and by shifting the quadrant $DE$ to values $|\omega|\gtrsim M(\mu R)^{\frac{1}{2}}$, without hereby altering the value of the integral, this factor is made bounded so that the contribution from $DE$ vanishes if we have introduced so many auxiliary denominators that

$$\overline{\lim_{|\omega|\rightarrow\infty}}\prod_\nu|\omega^{-2}|\,|\mathfrak{M}(\omega+\mu)|^2<\infty.$$

Then also the limits of the contributions from $CD$ and $EB$ exist. The residues at $\omega_0$ and $a_\nu$ are purely oscillatory in $r$ and have to be evaluated for instance in the sense of an Abelian limit. (This is justified if we keep in mind that our earlier interchanging the $t$, $r$ integration and the summation over intermediate states necessitates such a precaution. The use of the Abelian limit would also simplify the discussion of the contribution from $DE$ but there it seems to be less motivated.) The result of this limiting process is, apart from the contributions from the residues, the integral

$$\frac{1}{\pi}\prod_\nu(\omega_0^2-a_\nu^2)\,\mathrm{Re}\int_{CDEB}\frac{\omega d\omega}{\omega^2-\omega_0^2}\frac{[\mathfrak{M}(\omega+\mu)]^2}{(\omega^2-M^2)^{\frac{1}{2}}}\times\frac{(\omega^2-M^2)^{\frac{1}{2}}+[(\omega+\mu)^2-M^2]^{\frac{1}{2}}}{i(2\omega\mu+\mu^2)\prod_\nu(\omega^2-a_\nu^2)}.$$

We now deform the path $CDEB$ back into the path $CAB$. The contributions from the poles at $\omega_0$ and $a_\nu$ cancel the separately calculated terms. The contribution from $AB$ is immediately seen to be just the opposite of (13). Along $CA$ the integrand is imaginary so that only a small half-circle in the lower half-plane around the pole at $\omega=-\mu/2$ contributes. This half-circle then gives for the right-hand side of (10) the result

$$-\frac{1}{2}\prod_\nu\frac{\omega_0^2-a_\nu^2}{\frac{1}{4}\mu^2-a_\nu^2}\,\frac{[\mathfrak{M}(\frac{1}{2}\mu)]^2}{\frac{1}{4}\mu^2-a_\nu^2}.$$

Inserting this into (10), introducing the meson energy in the laboratory system $\omega'=\omega\mu/M$, defining $[\mathfrak{M}(\mu/2)]^2=g^2M\mu$, using (6), and adding $\mathrm{Im}\,T_{\omega_0'}$, we obtain

$$T_{\omega_0'}=\mathfrak{R}(4M^2\omega_0'^2;\mu^2)-\sum_\nu\mathfrak{R}(4M^2a_\nu'^2;\mu^2)\prod_{\lambda\neq\nu}\frac{\omega_0'^2-a_\lambda'^2}{a_\nu'^2-a_\lambda'^2}+\sum_\nu\mathrm{Re}\,T_{a_\nu'}\prod_{\lambda\neq\nu}\frac{\omega_0'^2-a\lambda'^2}{\omega_\nu'^2-a_\lambda'^2}-g^2\frac{\mu^3/(2M)}{(\mu^4/4M^2)-\omega_0'^2}$$
$$\times\prod_\nu\frac{\omega_0'^2-a_\nu'^2}{(\mu^4/4M^2)-a_\nu'^2}+\frac{2}{\pi}\prod_\nu(\omega_0'^2-a_\nu'^2)P\int_\mu^\infty\frac{\omega'd\omega'}{\omega'^2-(\omega_0'+i\epsilon)^2}\frac{\mathrm{Im}\,T_{\omega'}}{\prod_\nu(\omega'^2-a_\nu'^2)}, \tag{15}$$

where $\sigma(\omega')=\frac{1}{2}\mu^{-1}(\omega'^2-M^2)^{-\frac{1}{2}}\,\mathrm{Im}\,T_{\omega'}$ is the total cross section. In (15) the first two terms on the right-hand side cancel each other as soon as the number of pairs of auxiliary denominators is greater than the degree of the real polynomial $\mathcal{R}(4M^2\omega_0'^2;\mu^2)$ of $\omega_0'^2$. Equation (15) is just the Goldberger dispersion relation for our model. It shows that the scattering amplitude $T_{\omega'}$ may be analytically continued from arguments on the real axis $\omega'\geqslant\mu$ onto the entire cut $\omega'$ plane with the cuts from $-\infty$ to $-\mu$ and from $+\mu$ to $+\infty$. It possesses a pair of poles at $\omega'=\pm\mu^2/(2M)$, increases in infinity at most like a finite power of $\omega'^2$, and the values in the left and right half-planes are connected by $T_{\omega'}=(T_{-\omega'^*})^*$. This latter property establishes the crossing theorem which, without reference to the analytical continuation, would be a meaningless statement.

We finally have to prove our assertions below (12) about the analytical properties of $\langle\omega,\mathbf{k}|O(0)|\mu,0\rangle$ which have been of importance in our method. To this end we write, in analogy to the treatment of the scattering amplitude in (3) to (8),

$$\mathfrak{M}(\omega)=-i\int_{x_0\to+\infty}e^{i\omega x_0-i\mathbf{k}\cdot\mathbf{x}}\overleftrightarrow{\partial}_{x_0}\langle|T\psi(x)O(0)|\mu,0\rangle d\mathbf{x}$$

$$=i\int e^{i\omega t-i\mathbf{k}\cdot\mathbf{r}}\langle|[O(t,\mathbf{r}),O(0,0)]|\mu,0\rangle\tfrac{1}{2}(1+\mathrm{sign}t)dtd\mathbf{r}-i\int_{t=0}e^{i\omega t-i\mathbf{k}\cdot\mathbf{r}}\overleftrightarrow{\partial}\langle|[\psi(t,\mathbf{r}),O(0,0)]|\mu,0\rangle d\mathbf{r}.$$

The last integral again is a real polynomial in $\omega$. The other integral may be written

$$\int_0^\infty rdr\frac{\sin[(\omega^2-M^2)^{\frac{1}{2}}r]}{(\omega^2-M^2)^{\frac{1}{2}}}\int_0^\infty ie^{i\omega t}\langle|[O(t,\mathbf{r}),O(0,0)]|\mu,0\rangle dt.$$

The imaginary part of the $t$ integral turns out to be

$$-\pi\sum_\zeta\frac{\sin|\mathbf{k}_\zeta|r}{|\mathbf{k}_\zeta|r}[\delta(k_{0\zeta}-\omega)-\delta(k_{0\zeta}-\mu+\omega)]\langle|O(0)|k_{0\zeta},\mathbf{k}_\zeta\rangle\langle k_{0\zeta},\mathbf{k}_\zeta|O(0)|\mu,0\rangle. \tag{16}$$

Because of our selection rule, this expression certainly vanishes in the region $-M<\omega<(M+\mu)$ since the matrix element of $O(x)$ between the vacuum and a one-nucleon state vanishes.[6] Consequently we can at once perform the $r$ integration on the equation which corresponds to (9) since the critical region $-M<\omega<+M$ is completely empty. Furthermore, the resulting $\delta[|\mathbf{k}_\zeta|-(\omega^2-M^2)^{\frac{1}{2}}]$ rules out contributions from the first $\delta$ function in (16) and our assertions about $\mathfrak{M}(\omega)$ immediately follow from the relation for this quantity which we obtain instead of (15). (Such a simplification does not occur if we analyze the matrix element of the meson field operator between two one-nucleon states. This is the reason why we fixed the meson in our treatment and not the nucleon as in Goldberger's.[1])

We shall add a few remarks on the derivation of the dispersion relations for the derivatives of the scattering amplitude with respect to angle at angle zero. These relations cover all dispersion formulas which have been made use of so far in meson-nucleon scattering.

We first rewrite (3) in the center-of-gravity-system of the ingoing and the outgoing meson. In this system the meson energy is $E=\frac{1}{2}[(k+k')^2]^{\frac{1}{2}}$ and the spatial meson momentum $\mathbf{k}$ and $-\mathbf{k}$, respectively. The initial and final nucleon energy is $\omega=p(k+k')/(2E)$. We furthermore introduce cylindrical coordinates $r$, $z$, $\varphi$ with respect to the axis $\mathbf{k}$. With earlier mentioned simplifications (3) takes the form

$$T_{\omega,E}-\mathcal{R}(4E^2\omega^2;2E^2-\mu^2)\equiv\tilde{T}_{\omega,E}=\int_0^\infty rdrF(\omega),E,r), \tag{17}$$

with

$$F(\omega,E,r)\equiv2\pi iJ_0[r(\omega^2-M^2-E^2+\mu^2)^{\frac{1}{2}}]$$

$$\times\int_0^\infty dte^{i\omega t}\int_{-\infty}^{+\infty}dz\langle E,-\mathbf{k}|\left[O\left(\frac{t}{2},\frac{z|\mathbf{k}|^{-1}\mathbf{k}+r\mathbf{n}}{2}\right),O\left(-\frac{t}{2},-\frac{z|\mathbf{k}|^{-1}\mathbf{k}+r\mathbf{n}}{2}\right)\right]|E,\mathbf{k}\rangle, \tag{18}$$

where $\mathbf{n}$ is a unit vector perpendicular to $\mathbf{k}$. As a consequence of the commutator condition, $F(\omega,E,r)$ is analytic in the upper half $\omega$ plane and increases in infinity not stronger than a finite power of $\omega$. So we get instead of (10) the equation

$$
\begin{aligned}
&\mathrm{Re}\,\tilde{T}_{\omega,E}-\sum_\nu \mathrm{Re}\,\tilde{T}_{a_\nu,E}\prod_{\lambda\neq\nu}\frac{\omega_0^2-a_\lambda^2}{a_\nu^2-a_\lambda^2}-\frac{2}{\pi}\prod_\nu(\omega_0^2-a_\nu^2)\int_{(E^2+M^2-\mu^2)^{\frac12}}^{\infty}\frac{\omega d\omega}{\omega^2-\omega_0^2}\frac{\mathrm{Im}T_{\omega,E}}{\prod_\nu(\omega^2-a_\nu^2)}\\
&=2\prod_\nu(\omega_0^2-a_\nu^2)\int_0^\infty rdr\int_0^{(E^2+M^2-\mu^2)^{\frac12}}\frac{\omega d\omega}{\omega^2-\omega_0^2}\frac{I_0[r(E^2-\mu^2+M^2-\omega^2)^{\frac12}]}{\prod_\nu(\omega^2-a_\nu^2)}\int_{-\infty}^{+\infty}e^{i\omega t}dt\\
&\qquad\times\int_{-\infty}^{+\infty}dz\left\langle E,-\mathbf{k}\left|\left[O\left(\frac{t}{2},\frac{z|\mathbf{k}|^{-1}\mathbf{k}+r\mathbf{n}}{2}\right),O\left(-\frac{t}{2},-\frac{z|\mathbf{k}|^{-1}\mathbf{k}+r\mathbf{n}}{2}\right)\right]\right|E,\mathbf{k}\right\rangle\\
&=8\pi^2\prod_\nu(\omega_0^2-a_\nu^2)\int_0^\infty rdr\int_{-(E^2+M^2-\mu^2)^{\frac12}}^{(E^2+M^2-\mu^2)^{\frac12}}\frac{\omega d\omega}{\omega^2-\omega_0^2}\frac{I_0[r(E^2-\mu^2+M^2-\omega^2)^{\frac12}]}{\prod_\nu(\omega^2-a_\nu^2)}\sum_\zeta\delta(\omega+E-k_{0\zeta})\delta(|\mathbf{k}|^{-1}\mathbf{k}\cdot\mathbf{k}_\zeta)\\
&\qquad\times J_0[r(k_{0\zeta}^2-M_\zeta^2)^{\frac12}]\langle E,-\mathbf{k}|O(0)|k_{0\zeta},\mathbf{k}_\zeta\rangle\langle k_{0\zeta},\mathbf{k}_\zeta|O(0)|E,\mathbf{k}\rangle,
\end{aligned}
\tag{19}
$$

where we have introduced the usual decomposition of the matrix element, made use of (1) and carried out the $t$ and $z$ integrations. $M_\zeta$ is the rest mass of the intermediate states.

For an easy survey of the contributions to the right-hand side of (19), we introduce the variables

$$\nu\equiv E\omega/\mu \quad\text{and}\quad \Delta^2\equiv\mathbf{k}^2=E^2-\mu^2.$$

The energy in the center-of-mass system is

$$W=(2\mu\nu+M^2+\mu^2+2\Delta^2)^{\frac12},$$

and the scattering angle in this system is

$$\vartheta=2\sin^{-1}\{W\Delta[(\mu\nu+\Delta^2)^2-M^2\mu^2]^{-\frac12}\}.$$

In the "physical region" defined by $\omega\geqslant(E^2-\mu^2+M^2)^{\frac12}$ or

$$\nu\geqslant\mu^{-1}[(M^2+\Delta^2)(\mu^2+\Delta^2)]^{\frac12}\equiv\nu_{\min},$$

we obtain for $\mathrm{Im}\,\tilde{T}_{\omega,E}=\mathrm{Im}\,T_{p'k',pk}$, e.g., from (3), with an obvious generalization of (4), the equation

$$
\begin{aligned}
\mathrm{Im}\,T_{p'k',pk}&=\frac{1}{2}\int e^{+i\frac12(p+p')x}\langle k'|[O(x/2),O(-x/2)]|k\rangle dx\\
&=\tfrac12(2\pi)^4\sum\delta(k_\zeta-p-k)(T_{k\zeta,p'k'})^*T_{k\zeta,pk}
\end{aligned}
$$

and this, of course, coincides with the expression for the unitarity of the $S$ matrix which we obtain directly from (2):

$$
\begin{aligned}
\langle p'k'|S^\dagger S|pk\rangle=\langle p'k'|pk\rangle=\langle p'k'|pk\rangle+i(2\pi)^4\delta(p'+k'-p-k)[T_{p'k',pk}-(T_{pk,p'k'})^*\\
-i(2\pi)^4\textstyle\sum_\zeta\delta(k_\zeta-p-k)(T_{k\zeta,p'k'})^*T_{k\zeta,pk}],
\end{aligned}
$$

since the unitarity of the $S$ matrix is a direct consequence of the presupposed completeness of the states of both ingoing or outgoing particles.

In the "unphysical region" $\nu<\nu_{\min}$, where the $x$ integration cannot be carried out completely, there are contributions to the right-hand side of (19) from $k_{0\zeta}\geqslant M_\zeta$, this means

$$\nu\geqslant\mu^{-1}(\mu^2+\Delta^2)^{\frac12}M_\zeta-\mu-\mu^{-1}\Delta^2\equiv\tilde{\nu}_{\min}(M_\zeta),$$

whereas the extrapolation from the physical region would have given a $\delta$ function at

$$\bar{\nu}(M_\zeta)=\tfrac12\mu^{-1}(M_\zeta^2-M^2-\mu^2)-\mu^{-1}\Delta^2\leqslant\tilde{\nu}_{\min}(M_\zeta). \tag{20}$$

This function and the various regions are indicated in Fig. 2.

As one easily finds in a way entirely analogous to the forward scattering case, the contribution from the one-nucleon intermediate state is

$$-\frac{1}{2}\frac{[\mathfrak{M}(\mu/2)]^2}{(\frac12\mu+\Delta^2/\mu)^2-\nu_0^2}\left(1+2\frac{\Delta^2}{\mu^2}\right)\prod_\nu\frac{\nu_0^2-a_\nu^2}{(\frac12\mu+\Delta^2/\mu)^2-a_\nu^2},$$

and therefore can actually be described by a pair of poles at the locations we get from (20) for $M_\zeta = M$. Equation (19) now takes the form

$$\mathrm{Re}\,\tilde{T}_{\nu_0,\Delta^2} - \sum_\nu \mathrm{Re}\,\tilde{T}_{a_\nu',\Delta^2} \prod_{\zeta\neq\nu} \frac{\nu_0^2 - a_\zeta'^2}{a_\nu'^2 - a_\zeta'^2} + \frac{1}{2}\frac{g^2 M\mu}{(\frac{1}{2}\mu + \Delta^2/\mu)^2 - \nu^2}\left(1 + 2\frac{\Delta^2}{\mu^2}\right) - \frac{2}{\pi}\prod_\nu(\nu_0^2 - a_\nu'^2) P\int_{\nu_{\min}}^{\infty} \frac{\nu d\nu}{\nu^2 - \nu_0^2} \frac{\mathrm{Im}\,\tilde{T}_{\nu,\Delta^2}}{\prod_\nu(\nu^2 - a_\nu'^2)}$$

$$= 8\pi^2 \prod_\nu(\nu_0^2 - a_\nu'^2)\int_0^\infty rdr \sum_{M_\zeta \geqslant M+\mu} \int_{\tilde{\nu}_{\min}(M_\zeta)}^{\nu_{\min}} \frac{\nu d\nu}{\nu^2 - \nu_0^2} \frac{I_0[r\mu(\nu_{\min}^2 - \nu^2)^{\frac{1}{2}}(\mu^2+\Delta^2)^{-\frac{1}{2}}]}{\prod_\nu(\nu^2 - a_\nu'^2)} \delta[(\mu^2+\Delta^2)^{\frac{1}{2}} + \mu\nu(\mu^2+\Delta^2)^{-\frac{1}{2}} - k_{0\zeta}]$$

$$\times \delta(|\mathbf{k}|^{-1}\mathbf{k}\cdot\mathbf{k}_\zeta) J_0[r(k_{0\zeta}^2 - M_\zeta^2)^{\frac{1}{2}}] \langle(\mu^2+\Delta^2)^{\frac{1}{2}}, -\mathbf{k}|O(0)|k_{0\zeta}, \mathbf{k}_\zeta\rangle\langle k_{0\zeta}, \mathbf{k}_\zeta|P(0)|(\mu^2+\Delta^2)^{\frac{1}{2}}, \mathbf{k}\rangle. \quad (21)$$

Since we do not yet know the analytic properties of the matrix elements that appear on the right-hand side of (21), we make use of the following device: The left-hand side of (21) being finite, the $r$ integral on the right-hand side must converge. So we may perform it in the Abelian sense without committing an error. Owing to the convergence factor $\exp(-\epsilon r)$, $\epsilon > 0$, we can interchange the $r$ and the $\nu$ integrations provided that $\mu(\nu_{\min} - \nu^2)^{\frac{1}{2}}(\mu^2+\Delta^2)^{-\frac{1}{2}} < \epsilon$. Since $\epsilon \rightarrow +0$ this means $\nu_{\min} - \tilde{\nu}_{\min}(M+\mu) < \epsilon^2/(2M)$ and therefore, as seen from Fig. 2, necessitates a restriction to the derivatives of $T_{\nu,\Delta^2}$ with respect to $\Delta^2$ at $\Delta^2 = 0$ or, in the center-of-mass system, with respect to angle at angle zero. If the limit $\epsilon \rightarrow 0$ exists, it necessarily is the correct result. Now from

$$\lim_{\epsilon\rightarrow+0}\int_0^\infty e^{-\epsilon r} J_0(ar) J_0(pr) rdr = 2\delta(a^2 - b^2),$$

we derive

$$\lim_{\epsilon\rightarrow+0}\int_0^\infty e^{-\epsilon r} I_0(ar) J_0(br) rdr$$
$$= 2\delta(b^2) + 2a^2\delta'(b^2) + a^4\delta''(b^2) + \cdots,$$

where $I_0(ar)$ is understood to be defined by the series

$$I_0(ar) = 1 + \tfrac{1}{4}a^2r^2 + 1/64 a^4 r^4 + \cdots,$$

and the required derivatives are supposed to exist. This means that we may perform the $r$ integration on the right-hand side of (21) as if $\nu > \nu_{\min}$ provided that we use the resulting formula only in the sense of the power series in $\Delta^2$ or, expressed geometrically, in the infinitesimal neighborhood of the forward direction. The integration clearly gives a similar integral as on the left-hand side of (21), extended from

$$\bar{\nu}(M+\mu) = M - \mu^{-1}\Delta^2$$

to $\nu_{\min}$ where in this unphysical region $\mathrm{Im}\,T_{\nu,\Delta^2}$ is

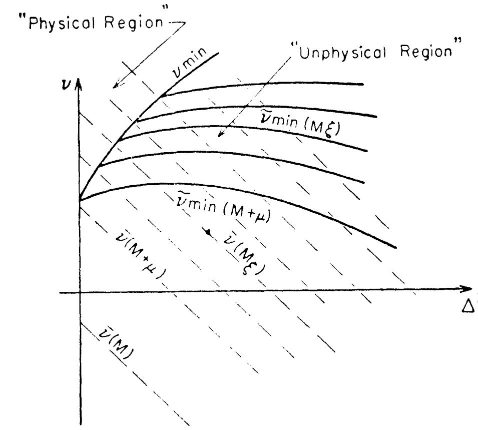


FIG. 2. "Physical" and "unphysical" regions of the variables $\Delta^2$, $\nu$ in (21).

defined by

$$\mathrm{Im}\,T_{\nu,\Delta^2} = \mathrm{Im}\,T_{\nu_{\min},\Delta^2} + (\nu - \nu_{\min})\frac{\partial}{\partial\nu}\mathrm{Im}\,T_{\nu,\Delta^2}\Big|_{\nu=\nu_{\min}} + \cdots,$$

or, more suitably, by

$$\mathrm{Im}\,T_{W,\vartheta} = \mathrm{Im}\,T_{W,0} + \vartheta\frac{\partial}{\partial\vartheta}\mathrm{Im}\,T_{W,\vartheta}\Big|_{\vartheta=0} + \cdots.$$

As long as these derivatives exist, the resulting formulas must hold. These are just the dispersion relations for the derivatives of the scattering amplitude with respect to angle as considered by various authors.[13,4] The remaining difficulty is that we do not yet know whether the needed derivatives at $\vartheta = 0$ do exist as a consequence of causality alone.

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[13] Gell-Mann, Goldberger, Thirring, Phys. Rev. **95**, 1612 (1954); G. F. Chew, reference 3.