Phenomenological Approach to a Unified Field Theory

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An attempt is made to give a phenomenological approach to a unified field theory by imposing four restrictions based on known experimental and theoretical considerations. The first two criteria, namely, that for weak electromagnetic fields the unified equations obey the flat space principle of conservation of energy and that the first-order corrections to Maxwell's equations not violate known experiments concerning the electron, lead to the requirement that a microscopic length appear in the Lagrangian. The remaining two conditions are a correspondence principle constraint for determining the form of the affinity and a gauge invariance condition. These four restrictions lead to a Lagrangian. The gauge invariance requirement forces the existence of a cosmological term. The field equations have been investigated for the spherically symmetric static solutions around a point electron. They lead to finite Coulomb energies, the microscopic length acting as the cut-off parameter.

1. INTRODUCTION

EVER since Einstein's fundamental work in general relativity appeared, attempts have been made to generalize it in order to include, within a single geometrical framework, both electromagnetic and gravitational phenomena. In recent years, approaches have tended towards broadening the geometrical foundation of space-time by assuming the existence of asymmetric affinities and metric tensors[1] in order that the antisymmetric Maxwell field be included in a natural fashion. This is, indeed, the next step in generalizing Riemannian geometry (though wider generalizations involving projective geometries may be considered) and we will restrict ourselves here to theories of this kind. Within this framework, then, the question arises as to what general considerations can be imposed to limit the possibilities available, in this fashion obtaining a phenomenological approach to unification.

Before turning to a detailed discussion of these points, it is perhaps important to ask why one should attempt to unify two theories which at first glance appear to have only one feature in common in that they both deal with macroscopic range (massless) fields. First, from the viewpoint of general relativity, it is well known that the general theory is incomplete as it stands. It derives a geometrical quantity representing the gravitational phenomena, $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$, which is proportional to the stress-energy tensor, $T_{\mu\nu}$. The latter is not determined by the theory and thus when distributed energy is present in space one does not have a complete set of equations. On the other hand, for point singularities, the dynamics is completely determined by the geometry.[2] It is natural (though, of course, not logically necessary) to try to extend the analysis so that both sides of Einstein's equation are obtained from a geometrical idea when the source term is a field. The inclusion of electromagnetic phenomena is thus a first step in this direction.

Second, it may be observed that electromagnetic effects have already entered into the gravitational theory in that a light ray is assumed to travel along the null geodesic. Thus the equations of motion for "ray optics" has already been included within the geometrical ideas of the general theory. In order to better understand this assumption, it would again seem reasonable to try to include "physical optics" within the same framework as the gravitational phenomena.

As a final point, it may be noted that if the above arguments suggesting a connection between the Maxwell field and general relativity are valid, the electromagnetic field is in the unique position of being coupled both to the macroscopic gravitational phenomena and the microscopic charged fields. It has recently been suggested,[3] also, that the inconsistencies discovered in quantum electrodynamics may be removed when gravitational effects are included. Since one of the roles of classical theory in quantum mechanics is to furnish one with a Lagrangian to be quantized, a deeper study as to what that Lagrangian is (and how it is modified by its interaction with the gravitational field) may prove fruitful.

2. PRELIMINARY THEORY

Recently Gupta and Kraichnan[4] have given an alternate derivation of general relativity based on Lorentz covariance rather than general covariance. Briefly, if one assumes that the gravitational field is represented by a spin-two particle, one might write down the following Lorentz covariant field equations for free space:

$$-\Box^2 h^{\mu\nu}=0;\quad \partial_\nu h^{\mu\nu}=0,\quad h^{\mu\nu}=h^{\nu\mu}. \tag{1}$$

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[1] A. Einstein and E. G. Straus, Ann. Math. **47**, 731 (1946); A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1953), fourth edition, Appendix; E. Schrödinger, Proc. Roy. Irish Acad. **A52**, 1 (1948).

[2] Einstein, Infeld, and Hoffman, Ann. Math. **39**, 65 (1938); L. Infeld and P. Wallace, Phys. Rev. **57**, 797 (1940).

[3] L. D. Landau, in *Niels Bohr and the Development of Physics*, edited by W. Pauli (McGraw-Hill Book Company, Inc., New York, 1955), pp. 60–61.

[4] S. N. Gupta, Phys. Rev. **96**, 1683 (1954); R. H. Kraichnan, Phys. Rev. **98**, 1118 (1955); see also A. Papapetrou, Proc. Roy. Irish Acad. **A52**, 11 (1948).

Since the only divergenceless symmetric tensor available is the stress-energy tensor, $T^{\mu\nu}$, the sources of the gravitational field must be $T^{\mu\nu}$. Further, since it is the total energy that must be conserved, any inhomogeneous term inserted on the right of Eq. (1) must include the stress-energy of any external fields present plus that carried by the gravitational field itself. Equation (1) can be generated by some Lagrangian L_0, from which one can derive the energy tensor $T_0^{\mu\nu}$ representing the gravitational field. Thus to next order, one amends Eq. (1) to read

$$-\tfrac{1}{2}\Box^2 h^{\mu\nu}=T_0^{\mu\nu}. \tag{2}$$

The procedure may now be continued [by finding an L_1 which generates Eq. (2)] and in this way one obtains an infinite series representation of the Einstein equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=0, \tag{3}$$

where[5]

$$R_{\mu\nu}=B^{\alpha}{}_{\mu\nu\alpha}=-\Gamma^{\alpha}{}_{\mu\nu,\alpha}+\Gamma^{\alpha}{}_{\mu\alpha,\nu}+\Gamma^{\beta}{}_{\alpha\nu}\Gamma^{\alpha}{}_{\mu\beta}-\Gamma^{\beta}{}_{\alpha\beta}\Gamma^{\alpha}{}_{\mu\nu},$$
$$R=g^{\mu\nu}R_{\mu\nu},\quad g^{\mu\nu}=\eta^{\mu\nu}+h^{\mu\nu}. \tag{4}$$

and $\eta^{\mu\nu}$ is the Lorentz metric. The basic ingredients that go into the derivation, thus, are the spin of the field and conservation of energy.

One may well ask what such an approach leads to when one considers the combined gravitational and electromagnetic equations, i.e., when one starts the analysis with Eq. (1) and

$$f^{\mu\nu}{}_{,\nu}=0;\quad f^{\mu\nu}=\eta^{\mu\alpha}\eta^{\nu\beta}f_{\alpha\beta},\quad f_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}. \tag{5}$$

One must now augment Eq. (2) with an extra term of the form $-\kappa[-f^{\mu\alpha}f^{\nu\beta}\eta_{\alpha\beta}+\frac{1}{4}\eta^{\mu\nu}f^{\sigma\tau}f^{\alpha\beta}\eta_{\sigma\alpha}\eta_{\tau\beta}]$ and eventually the entire Lagrangian with the term $-\frac{1}{2}\kappa(-g)^{\frac{1}{2}}f_{\mu\nu}\times f_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}$. One thus is lead to the field theory equations:

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-\kappa T_{\mu\nu}$$
$$=-\kappa[-f_{\mu\alpha}f_{\nu\beta}g^{\alpha\beta}+\tfrac{1}{4}g_{\mu\nu}f_{\sigma\tau}f_{\alpha\beta}g^{\sigma\alpha}g^{\tau\beta}], \tag{6a}$$
$$[(-g)^{\frac{1}{2}}g^{\mu\alpha}g^{\nu\beta}f_{\alpha\beta}]_{,\nu}=0. \tag{6b}$$

Equations (6) are, of course, quite well-known representing the simplest generalizations of Einstein's and Maxwell's theories according to the principle of equivalence. It is not a unified theory in that a geometrical interpretation of the electromagnetic field has not been made. However, one may easily be found in the following fashion. Even with a symmetric affinity, the contracted curvature tensor is not necessarily symmetric, the antisymmetric part being given by

$$\underset{\smile}{R_{\mu\nu}}=\Gamma^{\alpha}{}_{\mu\alpha,\nu}-\Gamma^{\alpha}{}_{\nu\alpha,\mu}. \tag{7}$$

[5] The comma denotes ordinary differentiation, $A_{,\alpha}=\partial A/\partial x^{\alpha}$, while $A_{;\alpha}$ or $A_{|\alpha}$ will be retained for covariant derivatives with respect to a specified affinity.

For the usual Christoffel affinity, $\Gamma^{\alpha}{}_{\mu\alpha}$ is a gradient and thus $\underset{\smile}{R_{\mu\nu}}$ does indeed vanish. In order to obtain an antisymmetric tensor, this suggests the adding of a contribution to the usual affinity such that $\Gamma^{\alpha}{}_{\mu\alpha}$ is no longer restricted to being a pure gradient. We therefore assume that the affinity is given by

$$\Gamma^{\mu}{}_{\alpha\beta}=C^{\mu}{}_{\alpha\beta}+\Gamma_{\beta}\delta^{\mu}{}_{\alpha}, \tag{8}$$

where

$$C^{\mu}{}_{\alpha\beta}=\tfrac{1}{2}g^{\mu\sigma}(g_{\alpha\sigma,\beta}+g_{\beta\sigma,\alpha}-g_{\alpha\beta,\sigma}). \tag{9}$$

In terms of the new affinity, the curvature tensor becomes

$$B^{\alpha}{}_{\mu\nu\beta}[\Gamma^{\sigma}{}_{\lambda\eta}]=B^{\alpha}{}_{\mu\nu\beta}[C^{\sigma}{}_{\lambda\eta}]+\delta^{\alpha}{}_{\mu}\bar{f}_{\nu\beta}, \tag{10}$$

where

$$\bar{f}_{\mu\nu}=\Gamma_{\nu,\mu}-\Gamma_{\mu,\nu}. \tag{11}$$

Thus space is flat only when both the gravitational field and the electromagnetic field, $\bar{f}_{\mu\nu}$, vanish. The contracted tensor becomes[6]

$$R_{\mu\nu}[\Gamma^{\sigma}{}_{\lambda\eta}]=R_{\mu\nu}[C^{\sigma}{}_{\lambda\eta}]+\bar{f}_{\nu\mu}=\underline{R_{\mu\nu}}+\bar{f}_{\nu\mu}. \tag{12}$$

Choosing a Lagrangian density of the form

$$\dot{\mathfrak{L}}=(-g)^{\frac{1}{2}}g^{\alpha\beta}R_{\alpha\beta}[\Gamma^{\sigma}{}_{\lambda\eta}]-\tfrac{1}{2}(-g)^{\frac{1}{2}}\underset{\smile}{R_{\mu\alpha}}\underset{\smile}{R_{\nu\beta}}g^{\mu\nu}g^{\alpha\beta} \tag{13}$$

to be varied with respect to $g_{\mu\nu}$ and Γ_{μ} one obtains Eqs. (6) (provided one measures the electromagnetic field, $f_{\mu\nu}$, in units such that $\bar{f}_{\mu\nu}=\kappa^{\frac{1}{2}}f_{\mu\nu}$).[7]

3. CONDITIONS ON THE CHOICE OF LAGRANGIAN

Though the theory discussed in the preceding section does not disagree with any of the well-known experimental tests of Maxwell's and Einstein's theories,[8] it is somewhat artificial in construction. In view of the fact, however, that Eqs. (6) depend mainly on the flat space concept of conservation of energy, they can act as a guide to the formulation of a more satisfactory theory. We thus adopt as our first assumption[9]:

A.—Any theory should reduce to the theory of Eqs. (6) (perhaps with a cosmological term) to a first approximation for weak electromagnetic fields.

We have required that Eqs. (6) be satisfied only for weak fields since the arguments leading to the form of the electromagnetic stress-energy tensor are based essentially upon the principle of equivalence and may not be valid for fields where space is more strongly curved.

[6] The second contraction gives nothing new since $B^{\alpha}{}_{\alpha\mu\nu}=-4\underset{\smile}{R_{\mu\nu}}$.

[7] While it has not been shown that the choice of affinity, Eq. (8), is the only one that will generate Eqs. (6), the simplicity of these equations make it hard to see how anything more complicated could be used.

[8] For electronic charge and mass, the warping of space causes deviations at $r\sim 10^{-34}$ cm.

[9] That the Einstein-Strauss theory does not satisfy this assumption has been first noted by A. Papapetrou, Proc. Roy. Irish Acad. **A52**, 69 (1948).

Electromagnetic theory has been quite successful, of course, within its domain of validity. One must therefore require that any new unified theory not disturb this. Coupled with the procedure of quantization this domain of acceptability of Maxwell's equations seems to be down to within at least $\sim 10^{-13}$ cm of the electron. We thus further assume

B.—First-order corrections to the Coulomb field of the electron should not become appreciable for $r \gtrsim 10^{-13}$ cm.

Conditions *A* and *B* do not at first sight seem to be very restrictive. However, we will try to indicate how the simultaneous validity of both *A* and *B* implies the necessity of a microscopic constant appearing in the Lagrangian while *B* alone implies the necessity of measuring the electromagnetic field in microscopic units for many schemes.

Let us consider a theory built from an asymmetric affinity and metric tensor. Presumably $\underset{\smile}{g}_{\mu\nu}$ will be related to the electromagnetic field while $\underline{g}_{\mu\nu}$ will be gravitational potential. One has, according to condition *A*, then, that for weak $\underset{\smile}{g}_{\mu\nu}$

$$R_{\mu\nu}(\underline{g}_{\alpha\beta})-\tfrac{1}{2}\underline{g}_{\mu\nu}R(\underline{g}_{\alpha\beta})=-T_{\mu\nu}(\underset{\smile}{g}^{\alpha\beta},\underline{g}^{\sigma\tau}), \qquad (14a)$$

$$[(-\det\underline{g}_{\sigma\tau})^{\frac{1}{2}}\underline{g}^{\mu\alpha}\underline{g}^{\nu\beta}\underset{\smile}{g}_{\alpha\beta}]_{,\nu}=0, \qquad (14b)$$

where $T_{\mu\nu}$ is a tensor of the same structure as the bracket on the right-hand side of Eq. (6a), i.e., quadratic in $\underset{\smile}{g}_{\alpha\beta}$. If one relates the electromagnetic field $f_{\mu\nu}$ to $\underset{\smile}{g}_{\mu\nu}$ via the equation[10]

$$\underset{\smile}{g}_{\mu\nu}=a^{\frac{1}{2}}f_{\mu\nu}, \qquad (15)$$

where a is a constant, then clearly $a=\kappa$. First-order corrections to Eq. (14b) will arise, perhaps, through replacing the determinant there by $\det(\underline{g}_{\alpha\beta}+\underset{\smile}{g}_{\alpha\beta})$. For an almost flat world ($\underline{g}_{\alpha\beta}\cong\eta_{\alpha\beta}$), this gives a correction term of $1+O(\underset{\smile}{g}_{\alpha\beta}{}^2)$, where $O(\underset{\smile}{g}_{\alpha\beta}{}^2)$ is of order $g_{\alpha\beta}{}^2$. Thus the change is roughly given by $\underset{\smile}{g}_{\mu\nu}{}^2=af_{\mu\nu}{}^2=\kappa f_{\mu\nu}{}^2$. But $\kappa f_{\mu\nu}{}^2$ is not dimensionless [having dimensions of $(\text{length})^{-2}$], indicating that either *A* is violated or a constant λ of dimension $(\text{length})^2$ explicitly enters into the formulas. Such a constant would have to appear in the Lagrangian. In this eventuality, the first-order change becomes $\lambda\kappa f_{\mu\nu}{}^2=\eta^{-1}f_{\mu\nu}{}^2$, where we have written $\lambda=1/\kappa\eta$. Condition *B*, however, requires that for $f_{\mu\nu}\sim e/r^2$ (where e is the electronic charge):

$$\eta^{-1}e^2/r_0{}^4<1, \quad r_0=e^2/mc^2, \quad m=\text{electronic mass}, \qquad (16)$$

which can be maintained only for $\eta>mc^2/r_0{}^3$; i.e., the constant appearing in the Lagrangian must be $\gtrsim$ electromagnetic energy density on the "surface" of the electron. If condition *A* were violated, one would still have to assume that $\underset{\smile}{g}_{\mu\nu}\sim\eta^{-\frac{1}{2}}f_{\mu\nu}$, i.e., the natural unit to measure field strengths is the microscopic one, η.

The question arises as to the choice of η (a lower limit only having been found). Assuming the mass, m, to be available, one must choose a length. If one does not wish to invent a new length, only the classical electronic radius or the Schwartzschild electronic radius is available. In view of the fact that the latter possibility does not seem to have anything to do with electromagnetic phenomena (and, further, is so small that the concept of distance becomes questionable), we shall provisionally assume that $\eta=mc^2/r_1{}^3$ where $r_1\sim r_0$.

Turning next to the question of affinity, it may be noted that even in general relativity the relation between the affinity and the Christoffel symbols is somewhat arbitrary. A more fundamental approach, perhaps, is to use the Palatini method and have the relation determined by one of the field equations. One thus has only to decide on the symmetry properties to be assigned to $\Gamma^\mu{}_{\alpha\beta}$ and $g_{\mu\nu}$. For this matter we again lean on the results of the preceding section and postulate

C.—The affinity has the form $\Gamma^\mu{}_{\alpha\beta}=C^\mu{}_{\alpha\beta}+\Gamma_\beta\delta^\mu{}_\alpha$, where $C^\mu{}_{\alpha\beta}=C^\mu{}_{\beta\alpha}$ and Γ_β is a vector. Also, the metric tensor obeys the relation $g_{\mu\nu}=g_{\nu\mu}$.

In order to justify, somewhat, the use of *C* we note that, according to assumption *A*, *C* is certainly the appropriate method of introducing the electromagnetic field when the fields are weak. Whether a more complicated form for the affinity is needed for stronger fields remains to be seen. In this respect condition *C* plays something of the role of the principle of equivalence in general relativity. For that case (via the elevator experiment, say) one learns that the metric tensor is related to gravitational fields while here (for weak fields) one learns that the Maxwell field is related to Γ_β (according to Sec. 2). As a further point it should be noted that it seems difficult to find a more general form of $\Gamma^\mu{}_{\alpha\beta}$ which will not also destroy condition *A*.

If Γ_β is to be associated with the vector potential, one must demand that when $\Gamma_\beta=\Lambda_{,\beta}$, where Λ is a scalar, there be no electromagnetic effects. According to Eq. (10), the curvature tensor is unchanged in this situation. However, aside from the field equations, there exists the geodesic equation for a neutral particle

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0. \qquad (17)$$

Equation (17) exists without any specification of the connection between $\Gamma^\mu{}_{\alpha\beta}$ and the metric tensor. For $\Gamma^\mu{}_{\alpha\beta}=C^\mu{}_{\alpha\beta}+\Lambda_{,\beta}\delta^\mu{}_\alpha$ one has

$$\frac{d^2x^\mu}{ds^2}+C^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}+\frac{d\Lambda}{ds}\frac{dx^\mu}{ds}=0. \qquad (18)$$

Equation (18) may be reduced to standard form by

[10] It is irrelevant to this argument whether $\underset{\smile}{g}_{\mu\nu}$ or its dual is the Maxwell field.

making the transformation $ds=[\exp\Lambda(x)]ds'$:

$$\frac{d^2x^\mu}{ds'^2}+C^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'}=0. \tag{19}$$

Thus when no electromagnetic field is present, a change of gauge $C^\mu{}_{\alpha\beta}\rightarrow C^\mu{}_{\alpha\beta}+\Lambda_{,\beta}\delta^\mu{}_\alpha$ implies a change of metric along the geodesic $ds\rightarrow[\exp\Lambda(x)]ds$, or equivalently $g_{\mu\nu}\rightarrow[\exp 2\Lambda(x)]g_{\mu\nu}$. Since one expects that the motion of a neutral particle will still depend upon $C^\mu{}_{\alpha\beta}$ only, even when a real electromagnetic field is present, one postulates[11]

D.—The Lagrangian must be invariant under the combined gauge transformation $\Gamma^\mu{}_{\alpha\beta}\rightarrow\Gamma^\mu{}_{\alpha\beta}+\Lambda_{,\beta}\delta^\mu{}_\alpha$ and $g_{\mu\nu}\rightarrow[\exp 2\Lambda(x)]g_{\mu\nu}$.

Conditions *C* and *D* may be viewed in a slightly different manner in terms of a "gauge-type" argument. In general, $ds^2=g_{\mu\nu}dx^\mu dx^\nu$ gives the readings between two events on the measuring rods and clocks. The freedom allowed in the transformation $ds\rightarrow[\exp\Lambda(x)]ds$ corresponds to the freedom of calibrating one's measuring devices differently at each point in space. When no electromagnetic field is present, a neutral particle will obey an equation of the form of (19), according to the general theory. This equation is invariant under a constant recalibration of one's rods ($\Lambda=$const), i.e., a constant change of measurement standards throughout all space-time is unobservable. Let us further assume that a variable recalibration $[\Lambda=\Lambda(x)]$ will also produce no observable effects. Now under

$$ds\rightarrow[\exp\Lambda(x)]ds,$$

Eq. (19) turns into Eq. (18), violating the above assumption unless, simultaneously, $C^\mu{}_{\alpha\beta}\rightarrow C^\mu{}_{\alpha\beta}+\Lambda_{,\beta}\delta^\mu{}_\alpha$. This last can be achieved if a new field Γ_β is introduced into the affinity $\Gamma^\mu{}_{\alpha\beta}=C^\mu{}_{\alpha\beta}+\Gamma_\beta\delta^\mu{}_\alpha$ such that $\Gamma_\beta\rightarrow\Gamma_\beta+\Lambda_{,\beta}$ under the recalibration transformation.

The invariance under the combined transformations of condition *D* assumes already that $C^\mu{}_{\alpha\beta}$ is not related to the metric tensor according to the usual Christoffel affinity,

$$C^\mu{}_{\alpha\beta}=\tfrac{1}{2}g^{\mu\sigma}g(_{\alpha\sigma,\beta}+g_{\beta\sigma,\alpha}-g_{\alpha\beta,\sigma}), \tag{20}$$

for if Eq. (20) held, it would not follow that $\Gamma^\mu{}_{\alpha\beta}\rightarrow\Gamma^\mu{}_{\alpha\beta}+\Lambda_{,\beta}\delta^\mu{}_\alpha$ when $g_{\mu\nu}\rightarrow[\exp 2\Lambda(x)]g_{\mu\nu}$.[12]

4. LAGRANGIAN AND THE FIELD EQUATIONS

We proceed next to the choice of a Lagrangian which does not violate assumptions *A* through *D*. One has available, in order to form scalars, the metric tensor, $g_{\alpha\beta}=g_{\beta\alpha}$, and the contracted curvature tensor

$$R_{\mu\nu}[\Gamma^\alpha{}_{\beta\gamma}]=R_{\mu\nu}[C^\alpha{}_{\beta\gamma}+\Gamma_\gamma\delta^\alpha{}_\beta].$$

Similarly there are two scalar densities: $(-g)^{\frac{1}{2}}$ and $(-\det R_{\mu\nu})^{\frac{1}{2}}$. The only Lagrangian that does not violate any of the conditions of the preceding section is

$$L=\int d^4x\,\mathfrak{L}(x)=\int d^4x\tfrac{1}{2}(-g)^{\frac{1}{2}}\times[\alpha_1 R_{\mu\alpha}R_{\nu\beta}g^{\mu\nu}g^{\alpha\beta}+\alpha_2(g^{\mu\nu}R_{\mu\nu})^2], \tag{21}$$

where α_1 and α_2 are two constants that will be determined below.[13] Equation (21) is also the most general bilinear Lagrangian that can be formed from $R_{\underline{\mu\nu}}$ and $R_{\underset{\smile}{\mu\nu}}$ as the units for measuring Γ_μ have not yet been decided upon.

Following Schrödinger[1] it is convenient to define the quantity[14]

$$\mathfrak{g}^{\mu\nu}=\delta\mathfrak{L}/\delta R_{\mu\nu}. \tag{22}$$

From Eq. (21) one obtains

$$\mathfrak{g}^{\underline{\mu\nu}}=(-g)^{\frac{1}{2}}[\alpha_1 g^{\mu\alpha}g^{\nu\beta}R_{\underline{\alpha\beta}}+\alpha_2 g^{\mu\nu}g^{\alpha\beta}R_{\underline{\alpha\beta}}], \tag{23a}$$

$$\mathfrak{g}^{\underset{\smile}{\mu\nu}}=\alpha_1(-g)^{\frac{1}{2}}g^{\mu\alpha}g^{\nu\beta}R_{\underset{\smile}{\alpha\beta}}. \tag{23b}$$

The first set of field equations may be found by varying Eq. (21) with respect to $C^\alpha{}_{\mu\nu}$ and Γ_μ. From Eq. (12) one has

$$\delta R_{\mu\nu}=-\delta C^\alpha{}_{\mu\nu;\alpha}+\delta C^\alpha{}_{\mu\alpha;\nu}+(\delta\Gamma_{\mu,\nu}-\delta\Gamma_{\nu,\mu}), \tag{24}$$

where the semicolon refers to covariant differentiation with respect to the symmetric affinity $C^\alpha{}_{\mu\nu}$. This leads to the equations

$$\mathfrak{g}^{\underline{\mu\nu}}{}_{;\alpha}-\tfrac{1}{2}(\mathfrak{g}^{\mu\beta}{}_{;\beta}\delta^\nu{}_\alpha+\mathfrak{g}^{\nu\beta}{}_{;\beta}\delta^\mu{}_\alpha)=0, \tag{25}$$

$$\mathfrak{g}^{\underset{\smile}{\mu\nu}}{}_{,\nu}=0. \tag{26}$$

Contracting Eq. (25) with respect to ν and α, one obtains directly [by imposing Eq. (26)]

$$\mathfrak{g}^{\underline{\mu\nu}}{}_{;\alpha}=0. \tag{27}$$

Hence, if one defines

$$\mathbf{g}^{\underline{\mu\nu}}=\mathfrak{g}^{\underline{\mu\nu}}(-\mathbf{g}_s)^{\frac{1}{2}},$$

where

$$\mathbf{g}_s=\det\mathbf{g}_{\underline{\alpha\beta}},\quad \mathbf{g}^{\underline{\mu\alpha}}\mathbf{g}_{\underline{\nu\alpha}}=\delta^\mu{}_\nu, \tag{28}$$

one obtains

$$C^\alpha{}_{\mu\nu}=\tfrac{1}{2}\mathbf{g}^{\underline{\alpha\sigma}}(\mathbf{g}_{\underline{\mu\sigma},\nu}+\mathbf{g}_{\underline{\nu\sigma},\mu}-\mathbf{g}_{\underline{\mu\nu},\sigma}), \tag{29}$$

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[11] The requirement of invariance under $\Gamma^\mu{}_{\alpha\beta}\rightarrow\Gamma^\mu{}_{\alpha\beta}+\Lambda_{,\beta}\delta^\mu{}_\alpha$ (the so-called λ transformation) has also been discussed by A. Einstein and B. Kaufman, Ann. Math. **62**, 128 (1955). See also P. G. Bergmann, Phys. Rev. **103**, 780 (1956).

[12] The assumption of Eq. (20) with invariance under $g_{\mu\nu}\rightarrow(\exp 2\Lambda)g_{\mu\nu}$ corresponds essentially to Weyl's theory. H. Weyl, *Space, Time, Matter* (Dover Publications, New York, 1950), pp. 282–312.

[13] Other possible terms that might appear, such as $\alpha_3(-\det R_{\mu\nu})^{\frac{1}{2}}$, violates condition *B* unless, of course, α_3 itself is very small.

[14] German symbols will denote tensor densities.

and $R_{\underline{\mu\nu}}$ is the usual contracted curvature tensor (as a function of $\mathbf{g}^{\underline{\alpha\beta}}$). One also has then $R_{\underline{\mu\nu}}=-\bar{f}_{\mu\nu}$.

Setting $\bar{f}_{\mu\nu}=a^{\frac{1}{2}}f_{\mu\nu}$ in Eq. (23b), where a is a constant and $f_{\mu\nu}$ is the electromagnetic field, (26) becomes the source-free Maxwell equations:

$$[(-g)^{\frac{1}{2}}g^{\mu\alpha}g^{\nu\beta}f_{\alpha\beta}]_{,\nu}=0. \tag{30}$$

Equation (30) may alternately be written in the form

$$f^{\mu\nu}{}_{|\nu}=0, \quad f^{\mu\nu}=g^{\mu\alpha}g^{\nu\beta}f_{\alpha\beta}, \tag{31}$$

where $|\nu$ means covariant differentiation with respect to a Christoffel affinity defined in terms of $g^{\alpha\beta}$. One thus has two "metric tensors" occurring in the theory: a "gravitational" tensor $\mathbf{g}_{\underline{\alpha\beta}}$ and an "electromagnetic" one $g_{\alpha\beta}$. This analogy follows through the rest of the equations. However, the original assumptions imply that $g_{\alpha\beta}$ defines the metric as measured by rods and clocks while $\mathbf{g}_{\underline{\alpha\beta}}$ is a derived quantity.

The second set of field equations are obtained by varying with respect to $g^{\mu\nu}$:

$$\begin{aligned}0=(-g)^{\frac{1}{2}}[\tfrac{1}{2}\alpha_1\{&R_{\mu\alpha}g^{\alpha\beta}R_{\nu\beta}\\ &+R_{\alpha\mu}g^{\alpha\beta}R_{\beta\nu}-\tfrac{1}{2}g_{\mu\nu}R_{\sigma\alpha}g^{\sigma\tau}g^{\alpha\beta}R_{\tau\beta}\}\\ &+\tfrac{1}{2}\alpha_2\{2R_{\underline{\mu\nu}}g^{\alpha\beta}R_{\underline{\alpha\beta}}-\tfrac{1}{2}g_{\mu\nu}(g^{\alpha\beta}R_{\underline{\alpha\beta}})^2\}].\end{aligned} \tag{32}$$

Equation (32) may easily be recast into the form

$$\begin{aligned}0=(-g)^{\frac{1}{2}}[\alpha_1\{&R_{\underline{\nu\alpha}}g^{\mu\lambda}g^{\alpha\beta}R_{\underline{\lambda\beta}}-\tfrac{1}{4}\delta^{\mu}{}_{\nu}R_{\underline{\sigma\alpha}}g^{\sigma\tau}g^{\alpha\beta}R_{\underline{\tau\beta}}\}\\ &+\alpha_2\{R_{\underline{\nu\lambda}}g^{\mu\lambda}g^{\alpha\beta}R_{\underline{\alpha\beta}}-\tfrac{1}{4}\delta^{\mu}{}_{\nu}(g^{\alpha\beta}R_{\underline{\alpha\beta}})^2\}]\\ &-\alpha_1a(-g)^{\frac{1}{2}}T^{\mu}{}_{\nu},\end{aligned} \tag{33}$$

where $T^{\mu}{}_{\nu}$ is the electromagnetic stress-energy tensor

$$T^{\mu}{}_{\nu}=-g^{\mu\lambda}g^{\alpha\beta}f_{\nu\alpha}f_{\lambda\beta}+\tfrac{1}{4}\delta^{\mu}{}_{\nu}g^{\sigma\tau}g^{\alpha\beta}f_{\sigma\alpha}f_{\tau\beta}. \tag{34}$$

Equation (33) is quadratic in the curvatures $R_{\underline{\mu\nu}}$. It may, however, be linearized in these quantities by eliminating one of the R's in favor of $\mathbf{g}^{\underline{\alpha\beta}}$ in the first brace by Eq. (23a). One obtains in this fashion

$$\mathfrak{g}^{\underline{\mu\alpha}}R_{\underline{\nu\alpha}}-\tfrac{1}{4}\delta^{\mu}{}_{\nu}\mathfrak{g}^{\underline{\alpha\beta}}R_{\underline{\alpha\beta}}=\alpha_1a(-g)^{\frac{1}{2}}T^{\mu}{}_{\nu}. \tag{35}$$

Equation (35) resembles Einstein's equations. However, while $\mathbf{g}^{\underline{\alpha\beta}}$ appears on the left-hand side, $g^{\alpha\beta}$ occurs in $T^{\mu}{}_{\nu}$ and $(-g)^{\frac{1}{2}}$ indicating effects to be found in this theory that are not contained in Eqs. (6). Also, the condition $T^{\mu}{}_{\mu}=0$ is satisfied identically by the left hand side (a phenomena resulting from the gauge invariance of $\mathfrak{L}$).

We turn next to the problem of obtaining a relation between $\mathbf{g}^{\underline{\alpha\beta}}$ and $g^{\alpha\beta}$. This may easily be done by again eliminating $R_{\underline{\mu\nu}}$ from Eq. (35) with the aid of (23a). From the latter equation, one has

$$R_{\underline{\mu\nu}}=\alpha_1^{-1}(-g)^{-\frac{1}{2}}[\mathfrak{g}^{\underline{\sigma\tau}}g_{\sigma\mu}g_{\tau\nu}-\alpha_2(-g)^{\frac{1}{2}}g_{\mu\nu}g^{\alpha\beta}R_{\underline{\alpha\beta}}]. \tag{36}$$

Hence

$$g^{\alpha\beta}R_{\underline{\alpha\beta}}=(\alpha_1+4\alpha_2)^{-1}(-g)^{-\frac{1}{2}}\mathfrak{g}^{\underline{\sigma\tau}}g_{\sigma\tau},$$

or

$$R_{\underline{\mu\nu}}=\alpha_1^{-1}(-g)^{-\frac{1}{2}}[\mathfrak{g}^{\underline{\sigma\tau}}g_{\sigma\mu}g_{\tau\nu}-\alpha_2(\alpha_1+4\alpha_2)^{-1}g_{\mu\nu}\mathfrak{g}^{\underline{\sigma\tau}}g_{\sigma\tau}]. \tag{37}$$

Substituting into Eq. (35), one obtains

$$\begin{aligned}\{\mathfrak{g}^{\underline{\mu\beta}}\mathfrak{g}^{\underline{\sigma\tau}}g_{\sigma\nu}g_{\tau\beta}-\alpha_2(\alpha_1+4\alpha_2)^{-1}\mathfrak{g}^{\underline{\mu\beta}}g_{\nu\beta}\mathfrak{g}^{\underline{\sigma\tau}}g_{\sigma\tau}\}&\\ -\tfrac{1}{4}\delta^{\mu}{}_{\nu}\{\mathfrak{g}^{\underline{\alpha\beta}}\mathfrak{g}^{\underline{\sigma\tau}}g_{\sigma\alpha}g_{\tau\beta}-\alpha_2(\alpha_1+4\alpha_2)^{-1}(\mathfrak{g}^{\alpha\beta}g_{\alpha\beta})^2\}&\\ =\alpha_1^2a(-g)T^{\mu}{}_{\nu}.&\end{aligned} \tag{38}$$

Equation (38) does not uniquely determine $g_{\alpha\beta}$ as a function of $\mathbf{g}_{\underline{\alpha\beta}}$ and $f_{\mu\nu}$ since the condition $T^{\mu}{}_{\mu}=0$ is identically satisfied by the left member. In fact, if one has a solution to Eq. (38) of the form $g_{\alpha\beta}=s_{\alpha\beta}(\mathbf{g}_{\underline{\mu\nu}})$, then another solution is easily seen to be

$$g_{\alpha\beta}=[\exp\Lambda(x)]s_{\alpha\beta},$$

where $\Lambda(x)$ is any scalar. Since Eqs. (23a), (30), and (32) are all invariant to the scale of $g_{\alpha\beta}$, the scale is not determined by the field equations, this being, of course, just the statement of the gauge invariance of the theory.

We now determine the constants α_1 and α_2 so that the theory conforms with conditions A and B. When no electromagnetic field is present, Eq. (38) has only the solution

$$g_{\alpha\beta}=[\exp\Lambda(x)]\mathbf{g}_{\underline{\alpha\beta}}, \tag{39}$$

where $\Lambda(x)$ is an arbitrary scalar. Equations (37) then becomes

$$R_{\underline{\mu\nu}}=(\alpha_1+4\alpha_2)^{-1}\mathbf{g}_{\underline{\mu\nu}}, \tag{40}$$

which are Einstein's equations with the cosmological term

$$\lambda=(\alpha_1+4\alpha_2)^{-1}. \tag{41}$$

Equation (35) leads to a consistent result.[15] The solution of Eq. (40) produces a unique determination of $\mathbf{g}_{\underline{\mu\nu}}$ (when some set of coordinate conditions have been imposed). However, the metric is not determined uniquely from Eq. (39). Hence $C^{\mu}{}_{\alpha\beta}$ is determined, not in terms of $g_{\alpha\beta}$, but in terms of $\mathbf{g}_{\underline{\alpha\beta}}$, a result in accordance with that obtained at the end of Sec. 3.

We turn next to the situation of a weak electromagnetic field. Rewriting Eq. (38) as

$$\begin{aligned}\{\mathbf{g}^{\underline{\mu\beta}}\mathbf{g}^{\sigma\tau}g_{\sigma\nu}g_{\tau\beta}-\alpha_2(\alpha_1+4\alpha_2)^{-1}\mathbf{g}^{\underline{\mu\beta}}g_{\nu\beta}\mathbf{g}^{\underline{\sigma\tau}}g_{\sigma\tau}\}&\\ -\tfrac{1}{4}\delta^{\mu}{}_{\nu}\{\mathbf{g}^{\underline{\alpha\beta}}\mathbf{g}^{\underline{\sigma\tau}}g_{\sigma\alpha}g_{\tau\beta}-\alpha_2(\alpha_1+4\alpha_2)^{-1}(\mathbf{g}^{\underline{\alpha\beta}}g_{\alpha\beta})^2\}&\\ =\alpha_1^2a(-g/-\mathbf{g}_s)T^{\mu}{}_{\nu}(g_{\alpha\beta},f_{\sigma\tau}),&\end{aligned} \tag{42}$$

[15] If one continues with the association of anything on the right-hand side of Einstein's equations with a stress-energy tensor (to first order), one is led to equating λ with $\kappa\rho_{\mu}c^2$ (where ρ_{μ} is the mean density of matter in the universe) and in this way to the steady state cosmology. However, such an association in the cosmological domain does not seem to be necessary.

one may assume a solution of the form

$$g_{\alpha\beta}=\mathbf{g}_{\underline{\alpha\beta}}+h_{\alpha\beta}, \tag{43}$$

where $h_{\alpha\beta}$ is considered small. The gauge condition may conveniently be set by

$$h=0;\quad h=h^{\alpha}{}_{\alpha}=\overset{\alpha\beta}{\mathbf{g}}h_{\alpha\beta}. \tag{44}$$

To first order in $h_{\alpha\beta}$, one easily obtains

$$h_{\mu\nu}=\tfrac{1}{2}\alpha_1{}^2a(\alpha_1+4\alpha_2)(\alpha_1+2\alpha_2)^{-1}T_{\mu\nu}(\mathbf{g}_{\underline{\alpha\beta}},f_{\sigma\tau}). \tag{45}$$

From Eq. (37) one may show that, to within quartic terms in $f_{\mu\nu}$,

$$\tfrac{1}{4}\overset{\mu\nu}{\mathbf{g}}R_{\underline{\mu\nu}}=(\alpha_1+4\alpha_2)^{-1}(-\mathbf{g}_s)^{\frac{1}{2}}=\lambda(-\mathbf{g}_s)^{\frac{1}{2}}.$$

Hence Eq. (35) becomes, to the lowest order,

$$\overset{\mu\alpha}{\mathbf{g}}R_{\underline{\nu\alpha}}-\tfrac{1}{2}\delta^{\mu}{}_{\nu}\overset{\alpha\beta}{\mathbf{g}}R_{\underline{\alpha\beta}}+\lambda\delta^{\mu}{}_{\nu}=\alpha_1aT^{\mu}{}_{\nu}(\mathbf{g}_{\underline{\alpha\beta}},f_{\sigma\tau}) \tag{46}$$

indicating that assumption A has been verified and that

$$\alpha_1a=-\kappa. \tag{47}$$

One may, in general, rewrite Eq. (35) as

$$(-\mathbf{g}_s)^{\frac{1}{2}}\Big\{\overset{\mu\alpha}{\mathbf{g}}R_{\underline{\nu\alpha}}-\tfrac{1}{4}\delta^{\mu}{}_{\nu}\Big[\alpha_1{}^{-1}\Big(\frac{-\mathbf{g}_s}{-g}\Big)^{\frac{1}{2}}\times[\overset{\alpha\beta}{\mathbf{g}}\overset{\sigma\tau}{\mathbf{g}}g_{\sigma\alpha}g_{\tau\beta}-\alpha_2(\alpha_1+4\alpha_2)^{-1}(\overset{\alpha\beta}{\mathbf{g}}g_{\alpha\beta})^2]\Big]\Big\}=-\kappa(-g)^{\frac{1}{2}}T^{\mu}{}_{\nu}(g_{\alpha\beta},f_{\sigma\tau}), \tag{48}$$

where the quantity in the square brackets will differ from 4λ only for strong electromagnetic fields. Equations (29), (30), (38), and (48) then become the field equations defining the theory.

Returning to Eq. (45), condition B requires that $h_{\mu\nu}\sim\mathbf{g}_{\underline{\mu\nu}}\sim1$ only for $r\lesssim10^{-13}$ cm. Since for an electron $T_{\mu\nu}\sim e^2/r^4$, one has

$$\begin{aligned}1&=\tfrac{1}{2}\kappa\alpha_1(\alpha_1+4\alpha_2)/(\alpha_1+2\alpha_2)e^2/r_1{}^4\\&=\tfrac{1}{2}\kappa\rho_ec^2\alpha_1(\alpha_1+4\alpha_2)/(\alpha_1+2\alpha_2),\end{aligned} \tag{49}$$

where $r_1\sim e^2/mc^2$ and $\rho_ec^2\equiv e^2/r_1{}^4$ is the order of the energy density on the "surface" of the electron.[16] Equations (49) and (41) may be solved for α_1 and α_2 yielding

$$\alpha_1=\frac{1}{\kappa\rho_ec^2}\frac{1}{1-(1/N)},\quad \alpha_2=\frac{1}{4\lambda}\frac{1-(2/N)}{1-(1/N)},\qquad N=\frac{\kappa\rho_ec^2}{\lambda}\sim10^{38}. \tag{50}$$

Hence

$$\frac{\alpha_2}{\alpha_1+4\alpha_2}=\frac{1}{4}\frac{1-(2/N)}{1-(1/n)}\cong\tfrac{1}{4},\quad a\cong-\kappa^2\rho_ec^2,\qquad \alpha_1{}^{-1}\cong\kappa\rho_ec^2,\quad \alpha_2{}^{-1}\cong4\lambda. \tag{51}$$

For all practical considerations, terms of order $1/N$ may be neglected.[17]

One might note that if condition D is dropped, there appears to be one more Lagrangian available:

$$\mathfrak{L}=(-g)^{\frac{1}{2}}(g^{\alpha\beta}R_{\alpha\beta}+\tfrac{1}{2}\alpha_1R_{\mu\alpha}g^{\mu\nu}g^{\alpha\beta}R_{\nu\beta}). \tag{52}$$

Equation (52) leads to results similar to those obtained here and in the next section (the spherically symmetric solutions are slightly more complicated, though qualitatively the same). The main distinction seems to involve the lack of a cosmological term. While, of course, one might add the term $\lambda(-g)^{\frac{1}{2}}$ to Eq. (52), it appears that in this theory (as in Weyl's) the only natural way of introducing a cosmological term is involved in the gauge invariance of the metric ds.

5. SPHERICALLY SYMMETRIC SOLUTION

We consider next the static spherically symmetric solution corresponding to the electric field around a point charged particle. We assume a coordinate frame such that the "gravitational" metric takes the form $\mathbf{g}_{\underline{\mu\nu}}=\mathbf{g}_{\mu}\delta_{\mu\nu}$ where

$$\mathbf{g}_1(r)=-\eta(r),\quad \mathbf{g}_2(r)=-r^2,\quad \mathbf{g}_3(r)=-r^2\sin^2\theta,\quad \mathbf{g}_4(r)=\lambda(r). \tag{53}$$

In general it is not possible for both $\mathbf{g}_{\underline{\mu\nu}}$ and $g_{\mu\nu}$ to have an r^2 dependence for the angular part of the line element. However, for this case it will be seen that a gauge transformation can be made to insure this. We assume first the general form $g_{\mu\nu}=g_{\mu}\delta_{\mu\nu}$, where

$$\begin{aligned}g_1&=-\eta(r)f_1(r)f^{-\frac{1}{4}},\quad g_2(r)=-r^2f_2(r)f^{-\frac{1}{4}},\\ g_3(r)&=-r^2\sin^2\theta f_3(r)f^{-\frac{1}{4}},\quad g_4(r)=\lambda(r)f_4(r)f^{-\frac{1}{4}},\\ f^{\frac{1}{4}}&=(f_1f_2f_3f_4)^{\frac{1}{4}}.\end{aligned} \tag{54}$$

The only surviving Maxwell equation reads:

$$\frac{\partial}{\partial r}[-g^4g^1(-g)^{\frac{1}{2}}f_{14}]=\frac{\partial}{\partial r}\Big[f_{14}r^2\sin\theta\Big(\frac{f_2f_3}{f_1f_4\eta\lambda}\Big)^{\frac{1}{2}}\Big]=0, \tag{55}$$

or

$$f_{14}=f_{41}=(e/r^2)(f_1f_4\eta\lambda/f_2f_3)^{\frac{1}{2}}, \tag{56}$$

[16] The sign of Eq. (49) is *a priori* arbitrary and one is in a position somewhat analogous to the problem of the choice of sign of κ in general relativity. However, with the sign as written, the spherically symmetric solutions of the next section are nonsingular (except at the origin). Unfortunately, α_1 now turns out to be positive and hence $\bar{f}_{\mu\nu}=a^{\frac{1}{2}}f_{\mu\nu}$ imaginary. This difficulty can be removed, however, by slightly modifying the Lagrangian to read $\mathfrak{L}=\frac{1}{2}(-g)^{\frac{1}{2}}[\alpha_1(R_{\underline{\mu\alpha}}R_{\underline{\nu\beta}}g^{\mu\nu}g^{\alpha\beta}-R_{\mu\alpha}R_{\nu\beta}g^{\mu\nu}g^{\alpha\beta}+\alpha_2(g^{\mu\nu}R_{\mu\nu})^2]$, the net effect being only to change the sign in Eq. (47).

[17] In fact, one could always add terms to the left member of Eq. (49) of order $1/N$ (and hence without violating postulate B) such that the approximate results of Eq. (51) are exact for α_1.

where e is a constant of integration. Substituting Eqs. (54) and (56) into Eq. (34), one finds that $T^{\mu}{}_{\nu}(g,f) = T_{\mu}\delta^{\mu}{}_{\nu}$ and

$$T_4 = T_1 = -T_2 = -T_3 = \tfrac{1}{2}(e^2/r^4)f_1f_4. \quad (57)$$

Thus Eq. (42), which determines the f_μ, is independent of $\eta(r)$ and $\lambda(r)$:

$$\{f_\mu{}^2 - \tfrac{1}{4}f_\mu\Sigma_\alpha f_\alpha\} - \tfrac{1}{4}\{\Sigma_\alpha f_\alpha{}^2 - \tfrac{1}{4}(\Sigma_\alpha f_\alpha)^2\} = -(\rho_e c^2)^{-1}T_\mu. \quad (58)$$

Since $T_1 = T_4$ and $T_2 = T_3$, the nonsingular spherically symmetric solutions require that $f_1 = f_4$, $f_2 = f_3$. Hence (58) reduces to

$$\{f_1{}^2 - \tfrac{1}{2}f_1(f_1+f_2)\} - \tfrac{1}{2}\{f_1{}^2 + f_2{}^2 - \tfrac{1}{2}(f_1+f_2)^2\} + \tfrac{1}{4}\mu(r)f_1{}^2 = 0, \quad (59a)$$

$$\{f_2{}^2 - \tfrac{1}{4}f_2(f_1+f_2)\} - \tfrac{1}{2}\{f_1{}^2 + f_2{}^2 - \tfrac{1}{2}(f_1+f_2)^2\} - \tfrac{1}{4}\mu(r)f_1{}^2 = 0, \quad (59b)$$

where

$$\mu(r) = (2/\rho_e c^2)(e^2/r^4). \quad (60)$$

Equation (59b) is equivalent to (59a), the latter yielding[18]

$$f_1/f_2 = [1+\mu(r)]^{-\frac{1}{2}}. \quad (61)$$

Turning to the gravitational equations, Eq. (48) may be rewritten as

$$\overset{\mu\alpha}{\mathbf{g}} \underline{R}_{\nu\alpha} - \tfrac{1}{2}\delta^{\mu}{}_{\nu}\overset{\alpha\beta}{\mathbf{g}} \underline{R}_{\alpha\beta} + \tfrac{1}{4}\delta^{\mu}{}_{\nu}[\ ] = -\kappa(-g/-\mathbf{g}_s)^{\frac{1}{2}}T^{\mu}{}_{\nu} = -\kappa f^{-\frac{1}{2}}T^{\mu}{}_{\nu}, \quad (62)$$

where the expression in square brackets is given by

$$[\ ] = \alpha_1{}^{-1}(-\mathbf{g}_s/-g)^{\frac{1}{2}}\{\overset{\alpha\beta}{\mathbf{g}}\overset{\sigma\tau}{\mathbf{g}} g_{\sigma\alpha}g_{\tau\beta} - \tfrac{1}{4}(\overset{\alpha\beta}{\mathbf{g}} g_{\alpha\beta})^2\} = \kappa\rho_e c^2(f_1/f_2 - 1)^2 f_2/f_1, \quad (63)$$

from Eq. (54) (neglecting terms proportional to λ). When one writes $\eta(r) = \exp\sigma(r)$, $\lambda(r) = \exp\nu(r)$ the nonvanishing components of Eq. (62) are easily seen to be

$$e^{-\sigma}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \frac{1}{4}\kappa\rho_e c^2\left(\frac{f_1}{f_2} - 1\right)^2\frac{f_2}{f_1} = -\frac{\kappa}{2}\frac{e^2}{r^4}\frac{f_1}{f_2},$$

$$e^{-\sigma}\left(\frac{\nu''}{2} - \frac{\lambda'\nu'}{4} + \frac{\nu'^2}{4} + \frac{\nu' - \lambda'}{\alpha r}\right) + \frac{\kappa\rho_e c^2}{4}\left(\frac{f_1}{f_2} - 1\right)^2\frac{f_2}{f_1} = \frac{\kappa}{2}\frac{e^2}{r^4}\frac{f_1}{f_2}, \quad (64)$$

$$-e^{-\sigma}\left(\frac{\sigma'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} + \frac{\kappa\rho_e c^2}{4}\left(\frac{f_1}{f_2} - 1\right)^2\frac{f_2}{f_1} = -\frac{\kappa}{2}\frac{e^2}{r^4}\frac{f_1}{f_2},$$

[18] We take the positive root in order to preserve the signature of the metric.

where $\nu' = d\nu(r)/dr$, etc. The solution to (64) is given by

$$\eta^{-1}(r) = \lambda(r) = 1 - \frac{2m}{r} - \frac{\kappa e^2}{2r}\int_0^r \frac{f_1}{f_2}\frac{dr}{r^2} - \frac{\kappa\rho_e c^2}{4r}\int_0^r \left(\frac{f_1}{f_2} - 1\right)^2\frac{f_2}{f_1}r^2 dr, \quad (65)$$

where m is the second constant of integration. Since $f_1/f_2 \sim r^2$ near the origin, the electrical term of Eq. (65) approach a finite value for small r. Hence a Schwarzschild type singularity still occurs though $(-\mathbf{g}_s)^{\frac{1}{2}}$ is always finite.

We now choose the gauge such that the angular components of $g_{\mu\nu}$ have only an r^2 radial dependence, i.e., set $f_1 = 1$. One has then

$$g_1(r) = -\lambda^{-1}(r)[1+\mu(r)]^{-\frac{1}{4}}, \quad g_2(r) = -r^2, \quad g_3(r) = -r^2\sin^2\theta, \quad g_4(r) = \lambda(r)[1+\mu(r)]^{-\frac{1}{4}}. \quad (66)$$

Similarly, from Eqs. (56) and (57),

$$f_{14} = (e/r^2)[1+\mu(r)]^{-\frac{1}{4}}, \quad T_4(-g)^{\frac{1}{2}} = \tfrac{1}{2}(e^2/r^4)[1+\mu(r)]^{-\frac{1}{2}}r^2\sin\theta. \quad (67)$$

Both the electric field, f_{14}, and the energy density, $T_4(-g)^{\frac{1}{2}}$, are finite at the origin. Thus the point electron produces finite electromagnetic quantities in the theory. The fact that the integral of $T_4(-g)^{\frac{1}{2}}$ exists, suggests the possibility of equating it to mc^2 and thus determining ρ_e. One has then

$$mc^2 = \int_0^\infty T_4(-g)^{\frac{1}{2}}drd\theta d\varphi = \left(\frac{e^2}{8\pi}\right)\int_0^\infty drd\Omega r^{-2}[1+r_1{}^4/r^4]^{-\frac{1}{2}}, \quad (68)$$
$$r_1{}^4 = e^2/(2\pi\rho_e c^2),$$

where we have re-expressed the charge in unrationalized units. Performing the integral, one obtains $r_1 = 0.93r_0 \equiv 0.93e^2/mc^2$, or

$$\rho_e c^2 = 1.4(1/2\pi)mc^2/r_0{}^3. \quad (69)$$

It should, of course, be mentioned that even if the hypothesis that all the electronic mass is electromagnetic be valid, the above figures will be changed when one more completely includes the matter field into the coupling as well as quantum effects. Thus the result given in Eq. (69) must be viewed at best as provisional.[19]

6. CONCLUSIONS

In the preceding sections, an attempt has been made to build up a unified field theory based on certain

[19] It is interesting to note that an exact solution of the equations for a plane monochromatic wave also exists. Of course the superposition theorem ceases to hold for frequencies such that $T^{\mu}{}_{\nu} \sim \rho_e c^2$. Similarly, for high frequencies, the wave does not travel along the null geodesic.

general principles (of both experimental and theoretical origin). Before discussing the results obtained, one might ask whether a less general Lagrangian than (21) is compatable with these assumptions. As pointed out earlier, Eq. (21) is the most general structure quadratic in $R_{\underline{\mu\nu}}$ and $R_{\underset{\smile}{\mu\nu}}$:

$$\mathfrak{L}(x)=\tfrac{1}{2}\alpha_1(-g)^{\frac{1}{2}}R_{\underline{\mu\alpha}}R_{\underline{\nu\beta}}g^{\mu\nu}g^{\alpha\beta}+\tfrac{1}{2}\alpha_1 a(-g)^{\frac{1}{2}}f_{\mu\alpha}f_{\nu\beta}g^{\mu\nu}g^{\alpha\beta}+\tfrac{1}{2}\alpha_2(-g)^{\frac{1}{2}}(g^{\mu\nu}R_{\underline{\mu\nu}})^2. \quad (70)$$

If one drops the third term (sets $\alpha_2=0$), Eq. (41) shows that $\lambda=1/\alpha_1$ and hence Eq. (49) (with $\alpha_2=0$) will violate condition B. The other possibility of a more restrictive choice for $\mathfrak{L}(x)$ stems from limiting α_1 to zero (but keeping $\alpha_1 a$ finite). From Eq. (51) this corresponds to limiting $\rho_e c^2$ to infinity, the theory resulting being precisely that of Eqs. (6). Thus, aside from this limiting case (and an issue as to a sign[16]), the conditions laid down in Sec. 3 force a finite-electron theory.

As was seen explicitly in Sec. 5, it is not possible by a coordinate or gauge transformation to make the "gravitational" metric tensor $\mathbf{g}_{\underline{\mu\nu}}$ equal to the true metric tensor $g_{\mu\nu}$ when an electromagnetic field is present. One would expect, according to some Einstein, Infeld, and Hoffman type analysis, that $\mathbf{g}_{\underline{\mu\nu}}$ governs the motion of a neutral particle since this is the tensor that appears in $C^{\mu}{}_{\alpha\beta}$ and hence in the gravitational side of the equations. Thus the measure of length along a neutral particle's geodesic is given by $ds'^2=\mathbf{g}_{\mu\nu}dx^\mu dx^\nu$. On the other hand, it is assumed that the rods and clocks measure length according to $ds^2=g_{\mu\nu}dx^\mu dx^\nu$. These two quantities will, of course, agree (to within a gauge recalibration) except at very small distances. Since the neutral geodesic will not be altered significantly by the presence of the electromagnetic field, the fact that $ds\neq ds'$ under these circumstances implies that the rods and clocks have been influenced by the presence of electromagnetic energy. In general relativity, it is always possible to set up a local Lorentz coordinate system $ds^2=-dr^2+c^2dt^2$, over a small space-time region, where dr and dt measure directly the readings on rods and clocks. The presence of c in the metric indicates that a consistent electromagnetic definition of time is available in terms of the distance, cdt, that an electromagnetic wave travels between two events.[20] It would appear from the above discussion, that the theory presented here suggests that all rods and clocks are electromagnetic in nature. This conclusion is, of course, quite preliminary since the neutral nuclear fields have not been included into the formalism.

From the spherically symmetric solution of Sec. 5 it is clear that $\mathbf{g}_{\underline{\mu\nu}}$ deviates from the Lorentz metric, $\eta_{\mu\nu}$, only at extremely small distances. Hence, for all practical electrodynamical problems, Eqs. (30) and (38) can be replaced by

$$[(-g)^{\frac{1}{2}}g^{\mu\alpha}g^{\nu\beta}f_{\alpha\beta}]_{,\nu}=0, \quad (71)$$

$$\{\eta^{\mu\beta}\eta^{\sigma\tau}g_{\sigma\nu}g_{\tau\beta}-\tfrac{1}{4}\eta^{\mu\beta}g_{\nu\beta}\eta^{\sigma\tau}g_{\sigma\tau}\} -\tfrac{1}{4}\delta^{\mu}{}_{\nu}\{\eta^{\alpha\beta}\eta^{\sigma\tau}g_{\sigma\alpha}g_{\tau\beta}-\tfrac{1}{4}(\eta^{\alpha\beta}g_{\alpha\beta})^2\} =-(1/\rho_e c^2)T^{\mu}{}_{\nu}(g,f). \quad (72)$$

The major effect of the unification, thus, is the kinematical constraint (72) on Maxwell's equations (71). Since $\eta^{\mu\nu}$ is a Lorentz tensor, these equations are meant to replace the usual Lorentz-invariant Maxwell equations.

The approach being followed here is considerably different from that of Einstein. Einstein hoped to obtain completely regular solutions of the field equations to represent particles.[21] As mentioned earlier, the viewpoint being adopted in this paper eventually requires the introduction of extra terms to represent the matter stress-energy tensor. Thus to describe electrons, one will need an added structure similar to the Dirac Lagrangian. The question as to whether these nonlinear interactions can be quantized remains unanswered.

A preliminary investigation has been unable to discover any invariances of the Lagrangian aside from those following from the usual freedom of coordinate and gauge transformations. Thus the solutions to the field equations seem to be constrained only to the amount that they are in general relativity and electromagnetic theory separately. This was seen explicitly in the spherically symmetric solutions of the preceding section.

As a final point, it might be mentioned that Infeld and Wallace[2] have shown that for a theory obeying Eqs. (6), a charged point particle travels according to the Dirac equations of motion.[22] Hence to a first approximation, the electron will obey the usual classical equations with radiation reaction. At high energies and small distances, one would expect factors of the type appearing in Eq. (67) to enter, a result which might damp the runaway solutions.

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[20] This definition of a clock is also consistent with the requirement that the stress-energy carried by the measuring devices postulated not disturb the local flatness of the space, since the velocity of light remains constant as the amplitude of the wave tends to zero.

[21] This hope does not seem to be born out in the spherically symmetric static solutions found by W. B. Bonnor, Proc. Roy. Soc. (London) **A210**, 427 (1952).

[22] P. A. M. Dirac, Proc. Roy. Soc. (London) **167**, 148 (1938). One must also assume retarded potentials.